



How can procedural flowcharts support the development of mathematics problem-solving skills?

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Abstract

Supporting students' problem-solving skills, solution planning and sequencing of different stages that are involved in successfully developing a meaningful solution to a problem has been a challenge for teachers. This case study was informed by reflective investigation methodology which explored how procedural flowcharts can support student mathematics problem solving in a senior Mathematical Methods subject in Queensland. The paper used thematic analysis to analyse and report on teachers' perceptions of the utility of procedural flowcharts during problem solving as well as content analysis on how student-developed flowcharts can support their problem-solving skills. Results show that development of procedural flowcharts can support problem solving as it helps with integration of problem-solving stages.

Keywords Problem solving · Procedural flowcharts · Problem-solving stages · Solution planning · Visual representation

Introduction

Problem solving is central to teaching and learning of mathematics (see Cai, 2010; Lester, 2013; Schoenfeld et al., 2014). For decades, research in mathematics problem solving, including special issues from leading mathematics education journals (see, *Educational Studies in Mathematics*, (Vol. 83, no. 2013); *The Mathematics Enthusiast*, (Vol. 10, nos. 1–2); *ZDM*, (Vol. 39, nos. 5–6)), have offered significant

A research report which is part of a PhD study by the first author who is an experienced high school mathematics teacher in Queensland, Australia. The second and third authors are primary and secondary advisors respectively. Correspondence concerning this article should be addressed to David Chinofunga, College of Arts, Society and Education, Nguma-bada Campus, Smithfield, Building A4, Cairns, PO Box 6811 Cairns QLD 4870, Australia.

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insights but struggled to produce well-articulated guidelines for educational practice (English & Gainsburg, 2016). This could possibly be the reason why mathematics teachers' efforts to improve students' problem-solving skills have not produced the desired results (Anderson, 2014; English & Gainsburg, 2016). Despite Polya's (1945) heuristics being so valuable in problem solving, there appears to be limited success when translated into the classroom environment (English & Gainsburg, 2016). English and Gainsburg went further to posit that one of the issues to be addressed is how to support problem-solving competency in students during the process of problem solving. Thus, teachers' perceptions in this study are a valuable part in evaluating how procedural flowcharts can support problem solving.

The problem-solving process is a dialogue between the prior knowledge the problem solver possesses, the tentative plan of solving the problem and other relevant thoughts and facts (Schoenfeld, 1983). However, research is still needed on tools that teachers can use to support students during problem solving (Lester & Cai, 2016). Although research in mathematics problem solving has been progressing, it has remained largely theoretical (Lester, 2013). Schoenfeld (2013) suggests that research focus should now advance from the framework for examining problem solving to explore how ideas grow and are presented and shared during the problem-solving process. Recently, Kaitera and Harmoinen (2022) emphasised the need to support teachers through resources that can help students develop problem solving skills. They went on to posit that resources that can assist students in presenting different approaches to a solution and displaying their understanding are critical to build their problem-solving skills.

The study by Kaitera and Harmoinen (2022) introduced mathematics students to 'problem-solving keys' which are heuristics for problem solving that students are to follow as they engage with tasks. Their conclusion was also noted by Vale and Barbosa (2018) who observed that a key area that would benefit from further research is the identification of strategies or plan that support students' ability to construct and present their mathematical knowledge effectively during problem solving, particularly if complex processes such as integration and modification of several procedures are involved. Similarly, students face challenges in connecting or bringing all the ideas together and showing how they relate as they work towards the solution (Reinholz, 2020). Problem solving in mathematics is challenging for students (Ahmad et al., 2010), and therefore, supporting students' problem-solving skills needs urgent attention (Schoenfeld, 2016). Furthermore, Mason (2016) posits that the crucial yet not significantly understood issue for adopting a problem-solving approach to teaching is the issue of "when to introduce explanatory tasks, when to intervene and in what way" (p. 263). Therefore, teachers also need resources to support the teaching of problem-solving skills, often because they were not taught these approaches when they were school students (Kaitera & Harmoinen, 2022; Sakshaug & Wohlhuter, 2010).

Flowcharts have been widely used in problem solving across different fields. In a technology-rich learning environment such as Lego Robotics, creating flowcharts to explain processes was observed to facilitate understanding, thinking, making sense of how procedures relate, investigating and communicating the solution (Norton et al., 2007). They are effective in guiding students during problem solving (Gencer,

2023), enhancing achievement and improving problem-solving skills in game-based intelligent tutoring (Hooshyar et al., 2016). Flowcharts have been identified as an effective problem-solving tool in health administration (McGowan & Boscia, 2016). In mathematics education, heuristic trees and flowcharts were observed to supplement each other in influencing students' problem solving behaviour (Bos & van den Bogaart, 2022). Importantly, McGowan and Boscia emphasised that "one of the greatest advantages of a flowchart is its ability to provide for the visualisation of complex processes, aiding in the understanding of the flow of work, identifying nonvalue-adding activities and areas of concern, and leading to improved problem-solving and decision-making" (p. 213). Identifying the most appropriate strategy and making the correct decision at the right stage are keys to problem solving. Teaching students to use visual representations like flowcharts as part of problem solving supports the ability to easily identify new relationships among different procedures and assess the solution being communicated faster as visual representations are more understandable (Vale et al., 2018).

The purpose of this case study was to explore, through an in-depth teacher's interview, and student-developed artefacts, the utility of procedural flowcharts in supporting the development of students' problem-solving skills in mathematics. The study will focus on problem solving in Mathematical Methods which is one of the calculus-based mathematics subjects at senior school in Queensland. The aim was to investigate if the development of procedural flowcharts supported students in planning, logically connecting and integrating mathematical procedures (knowledge) and to communicate the solution effectively during problem solving. The use of flowcharts in this study was underpinned by the understanding that visual aids that support cognitive processes and interlinking of ideas and procedures influence decision-making, which is vital in problem-based learning (McGowan & Boscia, 2016). Moreover, flowcharts are effective tools for communicating the processes that need to be followed in problem solving (Krohn, 1983).

Problem-solving learning in mathematics education

The drive to embrace a problem-solving approach to develop and deepen students' mathematics knowledge has been a priority in mathematics education (Koellner et al., 2011; Sztajn et al., 2017). In the problem-solving approach, the teacher provides the problem to be investigated by students who then design ways to solve it (Colburn, 2000). To engage in problem solving, students are expected to use concepts and procedures that they have learnt (prior knowledge) and apply them in unfamiliar situations (Matty, 2016). Teachers are encouraged to promote problem-solving activities as they involve students engaging with a mathematics task where the procedure or method to the solution is not known in advance (National Council of Teachers of Mathematics [NCTM], 2000), thus providing opportunities for deep understanding as well as providing students with the opportunity to develop a unique solution (Queensland Curriculum and Assessment Authority [QCAA], 2018). Using this approach, students are given a more active role through applying and adapting procedures to solve a non-routine problem and then communicating the method

(Karp & Wasserman, 2015). The central role problem solving plays in developing students' mathematical understanding has resulted in the development of different problem-solving models over the years.

The process of problem solving in mathematics requires knowledge to be organised as the solution is developed and then communicated. Polya is among the first to systematise problem solving in mathematics (Voskoglou, 2021). Students need to understand the problem, plan the solution, execute the plan and reflect on the solution and process (Polya, 1971). Voskoglou's (2021) problem-solving model emphasised that the process of modelling involves analysis of problem, mathematisation, solution development, validation and implementation. Similarly, problem solving is guided by four phases: discover, devise, develop and defend (Makar, 2012). During problem solving, students engage with an unfamiliar real-world problem, develop plans in response, justify mathematically through representation, then evaluate and communicate the solution (Artigue & Blomhøj, 2013). Furthermore, Schoenfeld (1980) posited that problem solving involves problem analysis, exploration, design, implementation and verification of the solution. When using a problem-solving approach, students can pose questions, develop way(s) to answer problems (which might include drawing diagrams, carrying out calculations, defining relationships and making conclusions), interpreting, evaluating and communicating the solution (Artigue et al., 2020; Dorier & Maass, 2020). Problem solving involves understanding the problem, devising and executing the plan and evaluating (Nieuwoudt, 2015). Likewise, Blum and Leiß (2007) developed a modelling approach that was informed by these stages, understanding, simplifying, mathematising, working mathematically, interpreting and validating.

Similarly, mathematical modelling involves problem identification from a contextualised real-world problem, linking the solution to mathematics concepts, carrying out mathematic manipulations, justifying and evaluating the solution in relation to the problem and communicating findings (Geiger et al., 2021). Likewise, in modelling, Galbraith and Stillman (2006) suggested that further research is needed in fostering students' ability to transition effectively from one phase to the next. "Mathematical modelling is a special kind of problem solving which formulates and solves mathematically real-world problems connected to science and everyday life situations" (Voskoglou, 2021 p. 85). As part of problem solving, mathematical modelling requires students to interpret information from a variety of narrative, expository and graphic texts that reflect authentic real-life situations (Doyle, 2005).

There are different approaches to problem solving and modelling, but all of them focus on the solving of real-world problems using mathematical procedures and strategies (Hankeln, 2020). A literature synthesis is critical where several models exist as it can be used to develop an overarching conceptual model (Snyder, 2019). Torraco (2005) noted that literature synthesis can be used to integrate different models that address the same phenomenon. For example, in this study, it was used to integrate problem solving models cited in the literature. Moreover, the review was necessitated by the need to reconceptualise the problem-solving model by Polya (1971) to include the understanding that the definition of problem solving has now broadened to include modelling. Torraco went further to suggest that as literature grows, and knowledge expands on a topic which might accommodate new

insights, there is a need for literature synthesis with the aim to reflect the changes. Thus, the model in Fig. 1 took into consideration the key stages broadly identified by the researchers and the understanding that modelling is part of problem solving. Problem solving and modelling is generally a linear process that can include loops depending on how the problem identification, mathematisation and implementation effectively address the problem (Blum & Leiß, 2007; Polya, 1957).

Figure 1 identifies the main stages that inform mathematics problem solving from the literature.

Problem identification and the design to solve the problem might be revisited if the procedures that were identified and their mathematical justification do not address the problem. Likewise, justification and evaluation after implementation might prompt the problem solver to realise that the problem was incorrectly identified. The loop is identified by the backward arrow, and the main problem-solving stages are identified by the linear arrows. The Australian Curriculum, Assessment and Reporting Authority notes that during problem solving:

Students solve problems when they use mathematics to represent unfamiliar situations, when they design investigations and plan their approaches, when they apply their existing knowledge to seek solutions, and when they verify that their answers are reasonable. Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. (Australia Curriculum and Reporting Authority, 2014, p. 5)

Therefore, during problem solving, students have to plan the solution to the problem and be able to communicate all the key processes involved. However, although problem solving is highly recommended in mathematics education, it presents several challenges for teachers in terms of how they can best support students to connect the processes and mathematics concepts into something coherent that can lead to a meaningful solution (Hacker, 1998). Therefore, relevant tools that support problem solving and decision-making can make a difference for both mathematics teachers and students (McGowan & Boscia, 2016).

Students can solve problems better if they can think critically (Kules, 2016). Problem solving requires their active engagement in analysing, conceptualising, applying concepts, evaluating, comparing, sequencing, synthesising, reasoning, reflecting and communicating, which are skills that are said to promote critical thinking (Kim et al., 2012; King, 1995; Moon, 2008; QCAA, 2018). Similarly,

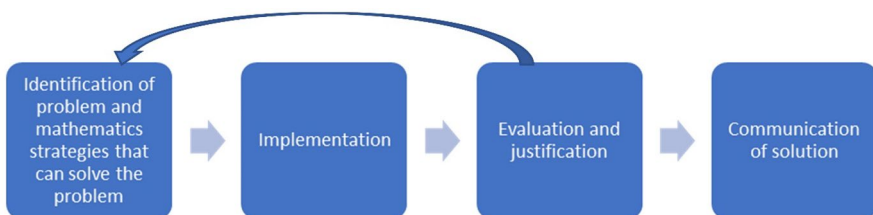


Fig. 1 Stages of mathematics problem solving

the ability to undertake problem solving is supported when students are provided with the opportunity to sequence ideas logically and evaluate the optimal strategy to solve the problem (Parvaneh & Duncan, 2021). However, finding tools that can support problem solving has been a focus for researchers for a long time but with very limited breakthroughs (McCormick et al., 2015). This study explored how procedural flowcharts as visual representations can support students in organising ideas, execute procedures, justify solutions and communicate their solution.

Importance of visual representations in mathematics problem-solving

Research on how visual representations support mathematics discovery and structural thinking in problem solving has come a long way (see Hadamard, 1945; Krutetskii, 1976; Polya, 1957). Visual representations are classified as graphs, tables, maps, diagrams, networks and icons and are widely used to convey information in a recognisable form that can be easily interpreted without resorting to tedious computations (Lohse et al., 1994). Visual representations can be used as a tool to capture mathematics relations and processes (van Garderen et al., 2021) and used in many cognitive tasks such as problem solving, reasoning and decision making (Zhang, 1997). Indeed, representations can be modes of communicating during concepts exploration and problem solving (Roth & McGinn, 1998). Likewise, visual representations can be a powerful way of presenting the solution to a problem, including self-monitoring on how the problem is being solved (Kingsdorf & Krawec, 2014; Krawec, 2014). Using visualisations created by teachers or students in mathematics can support students' problem-solving abilities (Csíkos et al., 2012).

Visual representations show thoughts in non-linguistic format, which is effective for communication and reflection. "Visual representations serve as tools for thinking about and solving problems. They also help students communicate their thinking to others" (NCTM, 2000, p. 206). In mathematics, visual representation plays a significant role in showing the cognitive constructs of the solution (Owens & Clements, 1998), a view echoed by Arcavi (2003), who said that visual representations can be appreciated as a central part of reasoning and as a resource to use in problem solving. More importantly, they can be used to represent the logical progression of ideas and reasoning during problem solving (Roam, 2009). However, there is need to explore how visual representations can be used to support and illustrate the problem-solving process and to create connections among concepts (Stylianou, 2010). Importantly, developing diagrams is often a recommended strategy for solving mathematics problems (Pape & Tchoshanov, 2001; Jitendra et al., 2013; Zahner & Corter, 2010). Therefore, this study will explore the utility of procedural flowcharts as a visual representation and resource in supporting problem analysis, problem understanding, solution development and evaluation, while communicating the whole problem-solving process effectively. It will go further to explore how development of procedural flowcharts can support educational practice in Mathematical Methods subject.

Procedural flowcharts are a visual representation of procedures, corresponding steps and stages of evaluation of a solution to a problem (Chinofunga et al.,

2022). These authors noted that procedural flowcharts developed by the teacher can guide students during the inquiry process and highlight key procedures and stages for decision-making during the process of problem solving. This is because “a procedural flowchart graphically displays the information–decision–action sequences in the proposed order” (Krohn, 1983, p. 573). Similarly, Chinofunga and colleagues (2022) emphasised that procedural flowcharts can be used to visually represent procedural flexibility as more than one procedure can be accommodated, making it easier to compare the effectiveness of different procedures as they are being applied. They further posited that student-developed procedural flowcharts provide students with the opportunity to comprehensively engage with the problem and brainstorm different ways of solving it, thus deepening their mathematics knowledge. Moreover, a procedural flowchart can be a visual presentation of an individual or group solution during problem solving.

Research has identified extended benefits of problem solving in small groups (Laughlin et al., 2006). Giving groups an opportunity to present a solution visually can be a quicker way to evaluate a group solution because visual representations can represent large amounts of information (even from different sources) in a simple way (Raiyn, 2016). Equally, Vale and colleagues encouraged visual representation of solutions with multisolutions as a tool to teach students problem solving (2018). Therefore, students can be asked to develop procedural flowcharts individually then come together to synthesise different procedural flowcharts.

Similarly, flowcharts are a visual aid used to represent how procedures interrelate and function together. “They are tools to visually break down complex information into individual building blocks and how the blocks are connected” (Grosskinsky et al., 2019, p. 24). They outlay steps in a procedure and show how they can be applied, thus helping to visualise the process (Ledin & Machin, 2020; Reingewertz, 2013). Flowcharts can also be used when a logical and sequenced approach is needed to address a problem (Cantatore & Stevens, 2016). Importantly, in schools, Norton and colleagues (2007) noted that “planning facilitated through the use of flow charts should be actively encouraged and scaffolded so that students can appreciate the potential of flow charts to facilitate problem-solving capabilities” (p. 15). This was because the use of flowcharts in problem solving provided a mental representation of a proposed approach to solve a task (Jonassen, 2012). The success of flowcharts in problem solving in different fields can be attributed to their ability to facilitate deep engagement in planning the solution to the problem.

Flowcharts use has distinct advantages that can benefit problem solving. Norton and colleagues (2007) posited that using a well-planned and well-constructed flowchart in problem solving results in a good-quality solution. Furthermore, flowcharts can also be a two-way communication resource between a teacher and students or among students (Grosskinsky et al., 2019). These authors further noted that flowcharts can help in checking students’ progress, tracking their progress and guide them. They can also be used to highlight important procedures that students can follow during the process of problem solving.

Similarly, flowcharts can be used to provide a bigger picture of the solution to a problem (Davidowitz & Rollnick, 2001). Flowcharts help students gain an overall

and coherent understanding of the procedures involved in solving the problem as they promote conceptual chunking (Norton et al., 2007). Importantly, “they may function to amplify the zone of proximal development for students by simplifying tasks in the zone” (Davidowitz & Rollnick, 2001, p. 22). Use of flowcharts by students reduces the cognitive load which then may help them focus on more complex tasks (Berger, 1998; Sweller et al., 2019). Indeed, development of problem-solving skills can be supported when teachers introduce learning tools such as flowcharts, because they can help structure how the solution is organised (Santoso & Syarifuddin, 2020). Therefore, the use of procedural flowcharts in mathematics problem solving has the potential to transform the process.

The research question in this study was informed by the understanding that limited resources are available to teachers to support students’ problem-solving abilities. In addition, the literature indicates that visual representation such as procedural flowcharts can support students’ potential in problem solving. Therefore, the research described in this study addressed the following research question: What are teachers’ perceptions of how procedural flowcharts can support the development of students’ problem-solving skills in the Mathematical Methods subject?

Methodology

The case study draws from the reflective investigation methodology (Trouche et al., 2018, 2020). The methodology explores how teaching and learning was supported by facilitating a teacher’s reflection on the unexpected use of a resource, in this case procedural flowcharts. The reflective methodology emphasises a teacher’s active participation through soliciting views on the current practice and recollection on previous work (Trouche et al., 2020). Using the methodology, a teacher is asked to reflect on and describe the resource used, the structure (related to the activity), the implementation and the outcomes (Huang et al., 2023).

This case study focuses on phases three and four of a broad PhD study that involved four phases. The broad study was informed by constructivism. Firstly, phase one investigated Queensland senior students’ mathematics enrolment in different mathematics curricula options from 2010 to 2020. Secondly, phase two developed and introduced pedagogical resources that could support planning, teaching and learning of calculus-based mathematics with a special focus on functions in mathematical methods. The pedagogical resources included a framework on mathematics content sequencing which was developed through literature synthesis to guide teachers on how to sequence mathematics content during planning. Furthermore, the phase also introduced concept maps as a resource for linking prior knowledge to new knowledge in a constructivist setting. Procedural flowcharts were also introduced to teachers in this phase as a resource to support development of procedural fluency in mathematics. Importantly, a conference workshop organised by the Queensland Association of Mathematics Teachers (Cairns Region) provided an opportunity for teachers to contribute their observations on ways that concept maps and procedural flowcharts can be used to support teaching. Thirdly, phase three was a mixed-method study that focused on evaluating the pedagogical resources

that were developed or introduced in phase two with 16 purposively sampled senior mathematics teachers in Queensland who had been given a full school term to use the resources in their practice. Some qualitative data collected through semistructured interviews from phase three were included in the results of the study reported here. During the analysis of the qualitative data, a new theme emerged which pointed to the unexpected use of procedural flowcharts during teaching and learning beyond developing procedural fluency. As a result, the researchers decided to explore how development of procedural flowcharts supported teaching and learning of mathematics as an additional phase. Phase four involved an in-depth interview with Ms. Simon (pseudonym) a teacher who had unexpectedly applied procedural flowcharts in a problem-solving task, which warranted further investigation. Ms. Simon's use of procedural flowcharts was unexpected as she had used them outside the context and original focus of the broader study. Importantly, in phase four, artefacts created by the teacher and her four students in the problem-solving task were also collected.

Ms. Simon (pseudonym) had explored the use of procedural flowcharts in a problem-solving and modelling task (PSMT) in her year 11 Mathematical Methods class. This included an introduction to procedural flowcharts, followed by setting the students a task whereby they were asked to develop a procedural flowchart as an overview on how they would approach a problem-solving task. The students were expected to first develop the procedural flowcharts independently then to work collaboratively to develop and structure an alternative solution to the same task. The student-developed procedural flowcharts (artefacts) and the in-depth interview with Ms. Simon were included in the analysis. As this was an additional study, an ethics amendment was applied for and granted by the James Cook University Ethics committee, approval Number H8201, as the collection of students' artefacts was not covered by the main study ethics approval for teachers.

Research context of phase four of the study

In the state of Queensland, senior mathematics students engage with three formal assessments (set by schools but endorsed by QCAA) in year 12 before the end of year external examination. The formal internal assessments consist of two written examinations and a problem-solving and modelling task (PSMT). The PSMT is expected to cover content from Unit 3 (Further Calculus). The summative external examination contributes 50% and the PSMT 20% of the overall final mark, demonstrating that the PSMT carries the highest weight among the three formal internal assessments.

The PSMT is the first assessment in the first term of year 12 and is set to be completed in 4 weeks. Students are given 3 h of class time to work on the task within the 4 weeks and write a report of up to 10 pages or 2000 words. The 4 weeks are divided into four check points, one per week with the fourth being the submission date. On the other three checkpoints, students are expected to email their progress to the teacher. At checkpoint one, the student will formulate a general plan on how to solve the problem which is detailed enough for the teacher to provide meaningful

feedback. Checkpoint one is where this study expects teachers to provide students with the opportunity to develop a procedural flowchart of the plan to reach the solution. Importantly at checkpoint one, teachers are interested in understanding which mathematics concepts students will select and apply to try and solve the problem and how the concepts integrate or complement each other to develop a mathematically coherent, valid and appropriate solution. Moreover, teachers are expected to have provided students with opportunities to develop skills in undertaking problem-solving and modelling task before they engage with this formal internal assessment. The QCAA has provided a flowchart to guide teachers and students on how to approach a PSMT ([Appendix 1](#))

Participants in phase four of the study

Ms. Simon and a group of four students were the participants in this study. Ms. Simon had studied mathematics as part of her undergraduate education degree, which set her as a highly qualified mathematics teacher. At the time of this study, she was the Head of Science and Mathematics and a senior mathematics teacher at one of the state high schools in Queensland. She had 35 years' experience in teaching mathematics across Australia in both private and state schools, 15 of which were as a curriculum leader. She was also part of the science, technology, engineering and mathematics (STEM) state-wide professional working group. Since the inception of the external examination in Queensland in 2020, she had been an external examination marker and an assessment endorser for Mathematical Methods with QCAA. The students who were part of this study were aged between 17 and 18 years and were from Ms. Simon's Mathematical Methods senior class. Two artefacts were from individual students, and the third was a collaborative work from the two students.

Phase four data collection

First, data were collected through an in-depth interview between the researcher and Ms. Simon. The researcher used pre-prepared questions and incidental questions arising from the interview. The questions focused on exploring how she had used procedural flowcharts in a PSMT with her students. The interview also focused on her experiences, observations, opinions, perceptions and results, comparing the new experience with how she had previously engaged her students in such tasks. The interview lasted 40 min, was transcribed and coded so as to provide evidence of the processes involved in the problem solving. Some of the pre-prepared questions were as follows:

1. What made you consider procedural flowcharts as a resource that can be used in a PSMT?
2. How have you used procedural flowcharts in PSMT?
3. How has the use of procedural flowcharts transformed students' problem-solving skills?

4. How have you integrated procedural flowcharts to complement the QCAA flow-chart on PSMT in mathematics?
5. What was your experience of using procedural flowcharts in a collaborative setting?
6. How can procedural flowcharts aid scaffolding of problem-solving tasks?

Second, Ms. Simon shared her formative practice PSMT task (described in detail below), and three of her students' artefacts. The artefacts that she shared (with the students' permission) were a critical source of data as they were a demonstration of how procedural flowcharts produced by students can support the development of problem solving and provided an insight into the use of procedural flowcharts in a PSMT.

Problem-solving and assessment task

The formative practice PSMT that Ms. Simon shared is summarised below under the subheadings: Scenario, Task, Checkpoints and Scaffolding.

Scenario

You are part of a team that is working on opening a new upmarket Coffee Café. Your team has decided to cater for mainly three different types of customers. Those who:

1. Consume their coffee fast.
2. Have a fairly good amount of time to finish their coffee.
3. Want to drink their coffee very slowly as they may be reading a book or chatting.

The team has tasked you to come up with a mode or models that can be used to understand the cooling of coffee in relation to the material the cup is made from and the temperature of the surroundings.

Task

Write a mathematical report of at most 2000 words or up to 10 pages that explains how you developed the cooling model/s and took into consideration the open cup, the material the cup was made from, the cooling time, the initial temperature of the coffee and the temperature of the surroundings.

- Design an experiment that investigates the differences in the time of cooling of a liquid in open cups made from different materials. Record your data in a table.
- Develop a procedural flowchart that shows the steps that you used to arrive at a solution for the problem.
- Justify your procedures and decisions by explaining mathematical reasoning.
- Provide a mathematical analysis of formulating and evaluating models using both mathematical manipulation and technology.

- Provide a mathematical analysis that involves differentiation (rate of change) and/or anti-differentiation (area under a curve) to satisfy the needs of each category of customers.
- Evaluate the reasonableness of solutions.

You must consider Newton's Law of Cooling which states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings. For a body that has a higher temperature than its surroundings, Newton's Law of Cooling can model the rate at which the object is cooling in its surroundings through an exponential equation. This equation can be used to model any object cooling in its surroundings:

$$y = A_0 e^{kt}$$

where:

- y is the difference between the temperature of the body and its surroundings after t minutes,
- A_0 is the difference between the initial temperature of the body and its surroundings,
- k is the cooling constant.

Checkpoints

- Week 1 Students provide individual data from the experiment and create a procedural flowchart showing the proposed solution to the problem. Teacher provides individual feedback.
- Week 2 Students provide a consolidated group procedural flowchart. Teacher provides group feedback
- Week 3 Students email a copy of their individually developed draft report for feedback.
- Week 4 Students submit individual final response in digital (PDF format) by emailing a copy to their teacher, providing a printed copy to their teacher and saving a copy in their Maths folder.

Additional requirements/instructions

- The response must be presented using an appropriate mathematical genre (i.e., a mathematical report).
- The approach to problem-solving and mathematical modelling must be used.
- All sources must be referenced.

Data analysis

Interviews

The analysis of data includes some observations and perceptions of mathematics teachers which were collected through surveys and interviews from phase three of the broader PhD study. The survey and interviews data in the broader study including phase four in-depth interview with Ms. Simon were transcribed and coded using thematic analysis (TA). TA is widely used in qualitative research to identify and describe patterns of meaning within data (Braun & Clarke, 2006; Ozuem et al., 2022). The thematic validity was ensured using theory triangulation. It involves sharing qualitative responses among colleagues at different status positions in the field and then comparing findings and conclusions (Guion et al., 2011). The study adopted the inductive approach which produces codes that are solely reflective of the contents of the data (Byrne, 2022).

Coding was done with no pre-set codes, and line-by-line coding was used as this was mainly an inductive analysis. The research team comprising of the researcher and two advisors/supervisors met to set the initial coding mechanism and code part of the data for consistency before independent coding of all the data. This is supported by King (2004) who suggested that when searching for themes, it is best to start with a few codes to help guide analysis. The data covered a wide variety of concepts, so initially the different concepts that grouped the research questions as 'conceptual themes' were utilised to organise the data. The research team examined the codes, checking their meaning and relationships between them to determine which ones were underpinned by a central concept. In Excel, codes that shared a core idea from the initial phase that used data from the open-ended responses and interview transcripts were colour coded. After the independent thematic analysis, the filter function in Excel was used to sort the codes using cell colour. Moreover, Excel provided the opportunity to identify duplicates as codes were collated from the three researchers. Same coloured codes were synthesised to develop a general pattern of meaning, which we referred to as candidate themes. The sorting and collation approach would bring together all codes under each theme which then would facilitate further analysis and review (Bree et al., 2014).

The research team went on to review the relationship of the data and the codes that informed the themes. This is supported by Braun and Clarke (2012, 2021) who posited that researchers should conduct a recursive review of the candidate themes in relation to the coded data items and the entire dataset. During the review, whenever themes were integrated or codes were moved to another theme, a new spreadsheet was created so that if further review was necessary, the old data and layout would still be available. Importantly, if the codes form a coherent and meaningful pattern, the theme makes a logical argument and may be representative of the data (Nowell et al., 2017). Furthermore, the team also reviewed the themes in relation to the data. This is because Nowell and others posited that themes should provide the most accurate interpretation of the data. The research team also discussed and wrote detailed analysis for each candidate theme identifying the main story behind each

theme and how each one fit into the overall story about the data through the lens of the research questions. Finally, the researchers also linked quotes to final themes reached during the analysis. Illustrating findings with direct quotations from the participants strengthen the face validity and credibility of the research (Bryne, 2022; Patton, 2002; Nowell et al., 2017).

Student artefacts

The students' artefacts (procedural flowcharts) in Figs. 5, 6 and 7 were analysed using content analysis. Content analysis can be used to analyse written, verbal or visual representations (Cole, 1988; Elo & Kyngäs, 2008). Content analysis is ideal when there is a greater need to identify critical processes (Lederman, 1991). Unlike interviews, documents that are ideal for qualitative analysis should be developed independently without the researcher's involvement (Merriam & Tisdell, 2015). In fact, the documents should not have been prepared for the purpose of research (Hughes & Goodwin, 2014), hence they are a stable and discrete data source (De Massis & Kotlar, 2014; Merriam & Tisdell, 2015). The students' artefacts used in this study were not prepared for the purpose of the study but as a mathematics task. Deductive content analysis is used when the structure of analysis is implemented on the basis of previous knowledge and the purpose of the study is model testing or confirmation (Burns & Grove, 2009). Similarly, it is an analytical method that aims to test existing concepts, models or hypotheses in a new context (Kyngäs et al., 2020). They went further to note that researchers can use deductive analysis to determine how a model fit a new context.

Deductive content analysis follows three main stages: preparation, organising and reporting (Elo et al., 2014; Elo & Kyngäs, 2008). Firstly, preparation involves identifying the unit of analysis (Guthrie et al., 2004). In this study, the unit of analysis are the artefacts developed by the students. Furthermore, the phase requires the researcher to be immersed in the data reading and digesting to make sense of the whole set of data through reflexivity, open-mindedness and following the rationale of what guided participants' narratives or in developing the artefact (Azungah, 2018). Secondly, a categorisation matrix based on existing knowledge should be developed or identified to facilitate the coding of the data according to categories (Hsieh & Shannon, 2005) (Table 1). Importantly, when using deductive content analysis, researchers require a theoretical structure or model from which they can build an analysis matrix (Kyngäs et al., 2020). Finally, the analysis results should be reported in ways that promote interpretation of the data and the results, for example, in tabular form (Elo & Kyngäs, 2008) (Fig. 2).

The students' procedural flowcharts were coded and interpreted on how they respond to different stages of problem solving. The researcher's codes, interpretations and findings should be clearly derived and justified using the available data and then inform conclusions and interpretations for confirmability (Tobin & Begley, 2004). The artefacts were shared between the researcher and his supervisors; the analysis was done independently then reviewed by the researcher and his supervisors. Schreier (2012) recommended that analysis should be done by more than one person to promote thoroughness and broaden the interpretation of the data. Schreier went further to note that if the categorisation matrix is clear and of high quality,

Table 1 Categorisation matrix, codes and interpretation of students' developed procedural flowcharts using problem-solving stages

Problem-solving stages	Identify problem and design solution plan	Implementation	Evaluate and justify	Communicate
What is involved?	Involves problem identification and interpreting the problem mathematically. At this stage, task analysis and planning are keys.	Involves application of mathematics procedures in solving the problem.	Evaluate if findings can address the problem. This is done by referring the solution(s) to the problem that need to be solved. Sometimes adjustments should be made to better respond to the problem.	Sharing the problem solution.
Student 1	Differentiation-rate of change of temperature against time How to collect data-experiment How to formulate models-technology How to apply concept of rate of change-rate of change at a point. Average rate of change (intervals).	Rate at 5, 15 and 30 min for each cup. Average rate at 5–15, 5–30 and 5–60 min for each cup. Taking note, it is a decreasing graph (cooling).	Checking if rate at a point can solve the problem Checking if average rate of change of an interval can solve the problem Checking if the procedures converge to the same conclusion. If not, then which one is the best (help determining strengths and limitations)?	Flowchart shows a reasonable development of steps to solving the problem that includes how mathematics concepts will be used and how different solutions will inform key results and justify the solution Final report.
Student 2	Integration-area under a curve How to collect data-experiment How to formulate models-algebra and technology How to apply concept of integration-definite integration.	Definite integration intervals 0–10, 0–20 and 0–30 min for each cup. Area size determines cooling rate.	Check if definite integration addresses the problem.	Flowchart shows a reasonable development of steps to solving the problem that includes how mathematics concepts will be used and how different solutions will inform key results and justify the solution Final report.

Table 1 (continued)

Problem-solving stages	Identify problem and design solution plan	Implementation	Evaluate and justify	Communicate
Students 1 and 2 (collaboration)	<p>Differentiation (rate of change of temperature against time) and integration (area under a curve)</p> <p>How to collect data-experiment</p> <p>How to formulate models-technology and algebra</p> <p>How to apply concept of rate of change-rate of change at a point. Average rate of change (intervals)</p> <p>How to apply concept of integration (area under a curve)-trapezoidal rule and definite integration.</p>	<p>-Rate at 5, 10 and 15 min for each cup. Average rate at 0–10, 0–20 and 0–30 min for each cup. Taking note, it is a decreasing graph (cooling)</p> <p>-Trapezoidal rule applied between 0 and 30 min</p> <p>-Definite integration intervals 0–10, 0–20 and 0–30 min for each cup. Area size determine cooling rate.</p>	<p>Checking if rate at a point can solve the problem. Checking if average rate of change of an interval can solve the problem</p> <p>Checking if trapezoidal rule can solve the problem</p> <p>Checking if definite integration can resolve the problem.</p> <p>Checking if the procedures converge to the same conclusion. If not then which one is the best (help determining strengths and limitations).</p>	<p>Flowchart shows a reasonable development of steps to solving the problem that include how mathematics concepts will be used and how different solutions will inform key results and justify the solution</p> <p>Final report.</p>

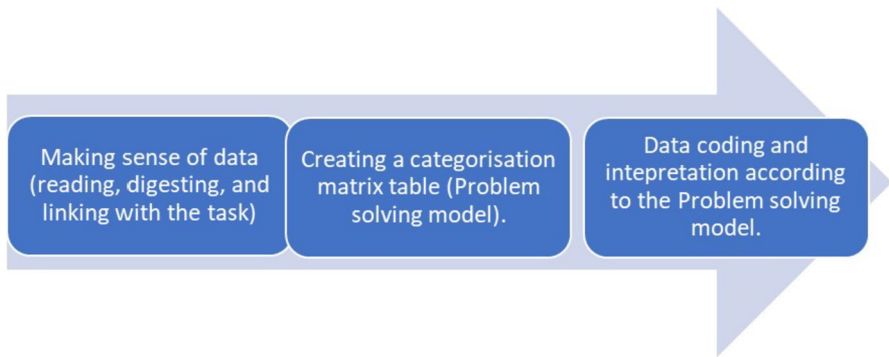


Fig. 2 Stages followed during analysis of procedural flowcharts

the coding should produce very little discrepancies. Very little discrepancies were observed except that some stages on the students' procedural flowcharts overlapped between skills.

Results

This section presents results from the analysis of the interviews data and student artefacts.

Semi-structured interviews

The thematic analysis of interviews resulted in two themes:

- The utility of procedural flowcharts in supporting mathematics problem solving.
- The utility of procedural flowcharts in supporting the integration of the four stages of mathematics problem solving.

In phase three, which prompted the targeted phase four study described in this study, teachers were asked the question, "How have you used procedural flowcharts to enhance teaching and learning of mathematics?" The question was not specific to problem solving but the teachers' observations and perceptions strongly related to problem-solving and student-centred learning.

Theme 1 The utility of procedural flowcharts generally supports mathematics problem solving

The visual nature of procedural flowcharts was seen as an advantage to both teachers and students. For students, drawing a flowchart was easier than writing paragraphs

to explain how they had arrived at the intended solution. For teachers, the flowchart was easier to process for timely feedback to students. Developing a procedural flowchart at the first checkpoint in the PSMT allows teachers to provide valuable feedback as the procedural flowchart can be used to represent several processes compared to written because of its visual nature. Engagement can be promoted because students can use the targeted feedback to improve their solutions as they will have provided a detailed overview of how they propose to solve the problem.

They present steps in diagrammatic form which is easy to process and easy to understand and process... students prefer them more as its in diagrammatic form and I have witnessed more students engaging. (Participant 8, phase three study)
I find it (visual) a really efficient way for me to look at the proposed individual students processes and provide relevant feedback to the student or for the student to consider. And, you know, once the students are comfortable with using these procedural flowcharts you know, I find it much easier for me to give them relevant feedback, and I actually find that feedback more worthwhile than feedback we used to give them, you know, that was just based on what they wrote in paragraphs,...students get to practice in creating their own visual display, which communicates their intended strategies to solve the problem, then they have opportunities to use it, and fine tune it as they work out the problem ... student developed procedural flow charts, they represent a student's maths knowledge in a visual way. (Ms. Simon).

Identifying students' competencies early was seen as central to successful problem solving as it provided opportunities for early intervention. Results showed that teachers viewed procedural flowcharts as a resource that could be used to identify gaps in skills, level of understanding and misconceptions that could affect successful and meaningful execution of a problem-solving task. Going through a student-developed flowchart during problem solving provided the teachers with insight into the student's level of understanding of the problem and how the effectiveness of the procedures proposed to address the problem. This is critical for tasks that require students to develop a report detailing the solution at the end of developing the solution. Teachers can get the opportunity to gain an insight of the proposed solution before the student commit to write the report. The procedural flowchart provides the bigger picture of the solution plan which might expose gaps in knowledge.

I found it quite useful because I can identify what kids or which kids are competent in what, which sort of problem-solving skills. And I can identify misconceptions that students have or gaps in students understanding. (Participant 1, phase three study)
It also to me highlights gaps in students' knowledge in unique ways that students intend to reach a solution because the use of the procedural flow chart encourages students to explain the steps or procedures behind any mathematical manipulation that you know they're intending to use. And it's something that was much more difficult to determine prior to using procedural flow charts... I've also used you know, student developed procedural

flow charts to ascertain how narrow or wide the students' knowledge is and that's also something that wasn't obvious to make a judgement about prior to using procedural flow charts. (Ms. Simon)

Problem solving was seen as student-centred. If procedural flowcharts could be used to support problem solving, then they could facilitate an environment where students were the ones to do most of the work. The students could develop procedural flowcharts showing how they will solve a PSMT task using concepts and procedures they have learnt. The open-ended nature of the problem in a PSMT provides opportunities for diverse solutions that are validated through mathematical justifications. The visual nature of procedural flowcharts makes them more efficient to navigate compared to text.

Mathematics goes from being very dry and dusty to being something which is actually creative and interesting and evolving, starting to get kids actually engaging and having to back themselves. (Participant 7, phase three study)
As a teacher, I find that procedural flowcharts are a really efficient way to ascertain the ways that students have considered and how they are going to solve a problem ... It engages the students from start to finish, you know in different ways this method demands students to compare, interpret, analyse, reason, evaluate, and to an extent justify as they develop this solution. (Ms. Simon)

Similarly, results showed that procedural flowcharts could be used as a resource to promote collaborative learning and scaffolding. Students could be asked to collaboratively develop a procedural flowchart or could be provided with one to follow as they worked towards solving the problem. Collaborative development of procedural flowcharts can support problem solving as students can bring their different mathematical understanding to develop a solution from different perspectives.

Sometimes, you know, I get students to work on it in groups as they share ideas and get that mathematisation happening. So, it's really helpful there ... I looked at the PSMT and its Marking Guide, and develop a more detailed procedural flowchart for students to use as a scaffold to guide them through the process. So, procedural flowcharts provide a structure in a more visual way for students to know what to do next. (Ms. Simon)

Ms. Simon shared her detailed procedural flowchart in Fig. 3 that she used to guide students in PSMTs.

The participants also observed that procedural flowcharts could be used to promote opportunities for solution evaluation which played an important role in problem solving. Loops can be introduced in procedural flowcharts to provide opportunities for reflection and reasoning as alternative paths provide flexibility while the solution is being developed. Following Fig. 4 are participants' comments referring to the figure which was among procedural flowcharts shared with participants as examples of how they can be used to teach syllabus identified Mathematical Methods concepts. The Mathematical Methods syllabus expects students to "recognise the distinction between functions and relations and use the vertical line test to determine whether a relation is a function" (QCAA, 2018 p. 20).

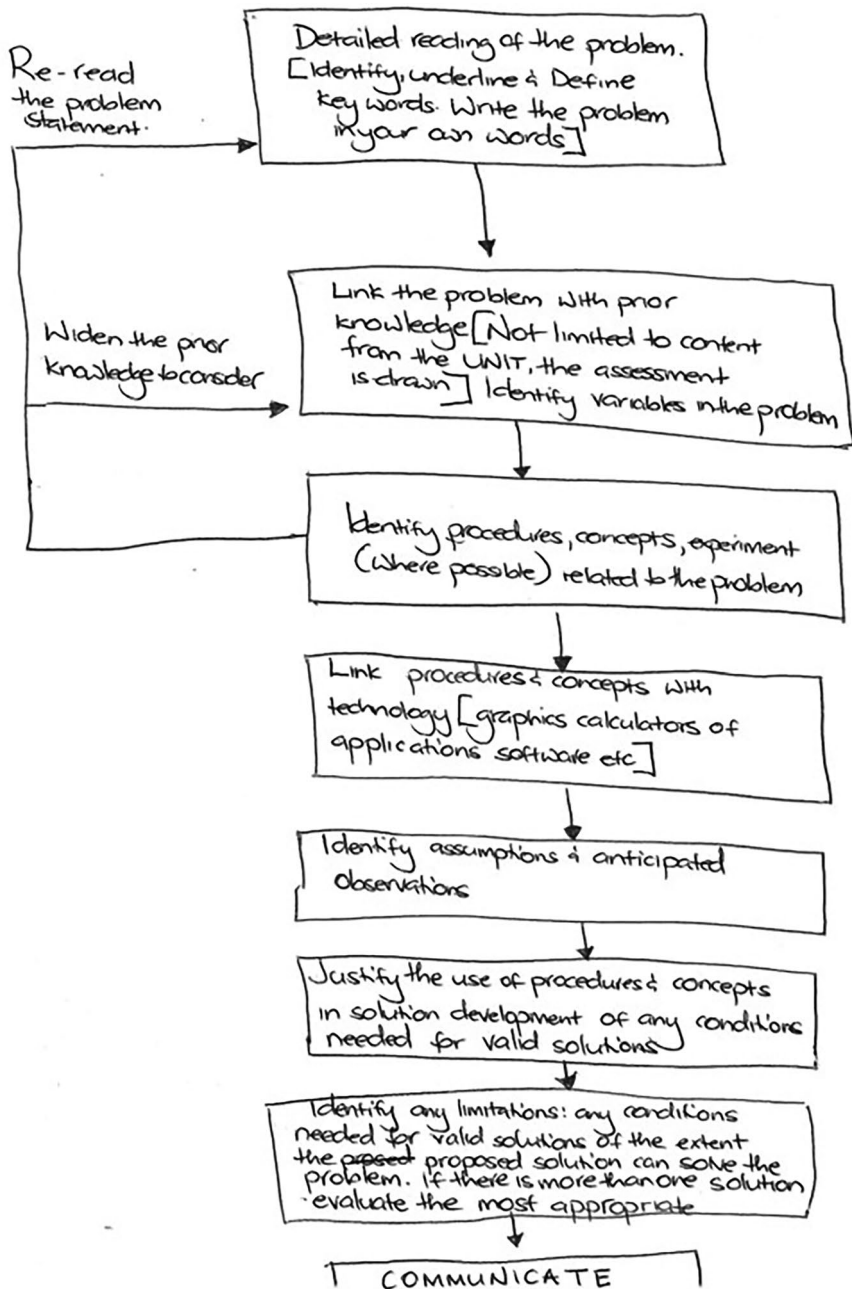


Fig. 3 Ms. Simon's procedural flowchart on problem solving

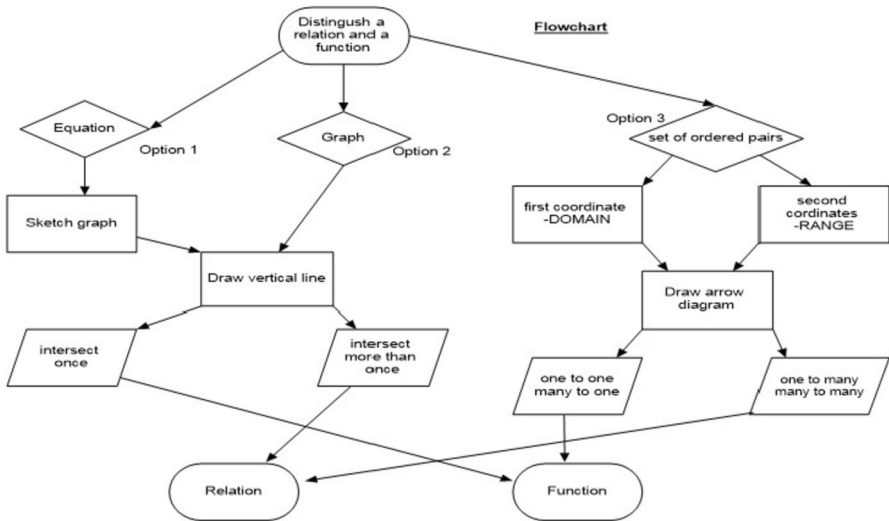


Fig. 4 Procedural flowchart on distinguishing functions and relations

The cycle approach, the feeding back in the feeding back out that type of stuff, you know, that is when we starting to teach students how to think. (Participant 7, phase three study)

Complex procedural flowcharts like the one you provided guide students in making key decisions as they work through solutions which is key to critical thinking and judgement and these two are very important in maths. (Participant 8, phase three study)

I also sincerely believe that procedural flowcharts are a way to get students to develop and demonstrate the critical thinking skills, which PSMTs are designed to assess. Students inadvertently have to use their critical thinking skills to analyse and reason as they search for different ways to obtain a solution to the problem presented in the PSMT ... the use of procedural flowcharts naturally permits students to develop their critical thinking skills as it gets their brain into a problem-solving mode as they go through higher order thinking skills such as analysis, reasoning and synthesis and the like ... this visual way of presenting solution provides students with opportunities to think differently, which they're not used to do, and it leads them to reflect and compare. (Ms. Simon)

Problem solving of non-routine problems uses a structure that should be followed. Resources that are intended to support problem solving in students can be used to support the integration of the stages involved in problem solving.

Theme 2 The utility of procedural flowcharts in supporting the integration of the four stages of mathematics problem solving.

Procedural flowcharts can support the flow of ideas and processes in the four stages during problem-solving and modelling task in Mathematical Methods subject. Literature synthesis in this study identified the four stages as:

- Identification of problem and mathematics strategies than can solve the problem.
- Implementation.
- Evaluation and justification.
- Communicating the solution.

Similarly, QCAA flowchart on PSMT identifies the four stages as formulate, solve, evaluate and verify, and communicate.

The logical sequencing of the stages of mathematics problem solving is crucial to solving and communicating the solution to the problem. Development of procedural flowcharts can play an important role in problem solving through fostering the logical sequencing of processes to reach a solution. Participants noted that the development of procedural flowcharts provides opportunities for showing the flow of ideas and processes which lay out an overview of how different stages connect into a bigger framework of the solution. Furthermore, it can help show how different pieces of a puzzle interconnect, in this case how all the components of the solution interconnect and develop to address the problem. In fact, procedural flowcharts can be used to show how the different mathematics concepts students have learnt can be brought together in a logical way to respond to a problem.

Procedural flowcharts help students sum up and connect the pieces together... connect the bits of knowledge together. (Participant 4, phase three study)

Really good how it organises the steps and explains where you need to go if you're at a certain part in a procedure. (Participant 2, phase three study)

Potentially, it's also an excellent visual presentation, which shows a student's draft of their logical sequence of processes that they're intending to develop to solve the problem ... So, the steps students need to follow actually flows logically. So really given a real-life scenario they need to solve in a PSMT students need to mathematise it and turn it into a math plan, where they execute their process, evaluate and verify it and then conclude ... so we use procedural flowcharts to reinforce the structure of how to approach problem-solving ... kids, you know, they really struggling, you know, presenting things in a logical way, because they presume that we know what they're thinking. (Ms. Simon)

Developing procedural flowcharts provided students with opportunities to plan the solution informed by the stages of problem solving. Teachers could reinforce the structure of problem solving by telling students what they could expect to be included on the procedural flowchart. Procedural flowcharts can be used as a visual tool to highlight all the critical stages that are included during the planning of the solution.

I tell the students, "I need to see how you have interpreted the problem that you need to solve. I need to see how you formulated your model that involves the process of mathematisation, where you move from the real world into the maths world, and I need to see all the different skills you're intending to use to arrive at your solution." (Ms. Simon)

Similarly, procedural flowcharts could visually represent more than one strategy in the "identify and execute mathematics procedures that can solve the problem" stage, thereby providing a critical resource to demonstrate flexibility. When there are multiple ways of addressing a problem, developing a procedural flowchart can provide an opportunity of showing all possible paths or relationships between different paths to the solution, thus promoting flexibility. Procedural flowcharts provide an opportunity to show how different procedures can be used or integrated to solve a problem.

Students are expected to show evidence that they have the knowledge of solving the problem using several ways to get to the same solution. So, it goes beyond the students' preferred way of answering a question and actually highlights the importance of flexibility when it comes to processes and strategies of solving a problem ... By using procedural flowcharts, I'm saying to the students, "Apart from your preferred way of solving the problem, give me a map of other routes, you can also use to get to your destination." (Ms. Simon)

The results also indicated that procedural flowcharts could be used to identify strengths and limitations of procedures in the "evaluate solution" stage and thus demonstrate the reasonableness of the answer. Having more than one way of solving a problem on a procedural flowchart helps in comparing and evaluating the most ideal way to address the problem.

And I'm finding that, you know, as students go through, and they compare the different processes, you know, the strengths and limitations, literally stare them in the face. So, they don't have to. They're not ... they don't struggle as much as they used to in coming up with those sorts of answers ... it's also a really easy way that once the students reach the next phase, which is the evaluating verified stage, they can go back to their procedural flow chart and identify and explain strengths and limitations of their model ... It's a convenient way for students to show their reasonableness of their solution by comparing strengths and weaknesses of all the strategies presented on the procedural flowchart, something that they've struggled with in the past. (Ms. Simon)

The results from the interview show that the procedural flowcharts supported efficient communication of the steps to be followed in developing the solution to the problem. Student-developed procedural flowcharts allowed the teacher to have an insight and overview of the solution to the problem earlier in the assessment task. In addition, they provided an alternative way of presenting their solution to the teacher.

I expect students to use the procedural flowchart as a way to communicate to me how they're planning to solve the scenario in the PSMT...It's also one of the parts that students are expected to hand in to me on one of the check points, and I find it a really efficient way for me to look at, you know, a proposed individual students processes, and provide relevant feedback to the student to consider in a really efficient way...I just found that it helps students communicate their solution to a problem in lots of different ways that challenges students to logically present a solution. (Ms. Simon)

She went on to say,

Students also found it challenging to communicate their ideas in one or two paragraphs, when more than one process or step was required to solve the problem. So, I found that, you know, procedural flowcharts, have filled this gap really nicely, as that provides students with a simple tool that they can use to present a visual overview of the processes they've chosen to use to solve the problem. And so, for me, as a teacher, procedural flowcharts are an efficient way for me to scan the intended processes that an individual student is proposing to use to solve the problem in their authentic way and provide them with valuable feedback.

In summary, the teacher's experiences, views and perceptions showed that procedural flowcharts can be a valuable resource in supporting students in all four stages of problem solving.

Students' artefacts

The student-generated flowcharts in this part of the research gave an insight into students' understanding as they planned how to solve the problem presented to them. Students were expected to use the problem-solving stages to successfully develop solutions to problems. Their de-identified procedural flowcharts are shown in Figs. 5, 6 and 7.

Students 1 and 2 also collaboratively developed a procedural flowchart, shown as Fig. 7.

Discussion

This discussion is presented as two sections: (1) how developing procedural flowcharts can support mathematics problem solving and (2) how developing procedural flowcharts support the integration of the different stages of mathematics problem solving. This study although limited by sample size highlighted how developing procedural flowcharts can support mathematics problem solving, can reinforce the structure of the solution to a problem and can help develop metacognitive skills among students. The different stages involved in problem solving inform the process of developing the solution to the problem. The focus on problem-based learning has signified the need to introduce resources that can support students and teachers in

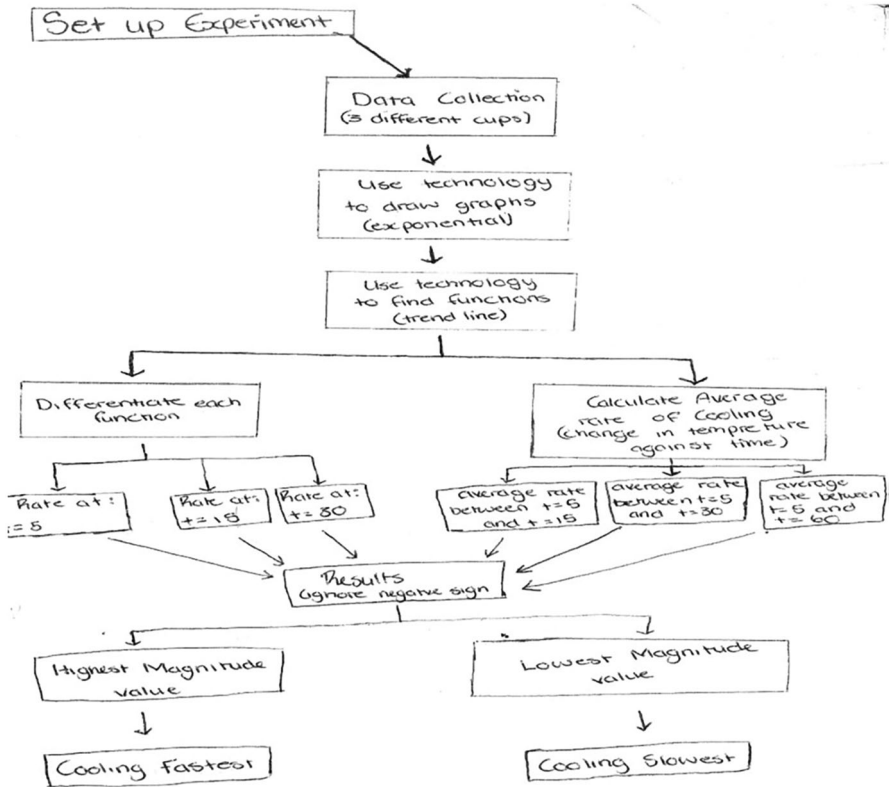


Fig. 5 Procedural flowchart developed by student 1

developing and structuring solutions to problems. Results from this study have also provided discussion points on how procedural flowcharts can have a positive impact in mathematics problem solving.

Procedural flowcharts can support mathematics problem solving

Procedural flowcharts help in visualising the process of problem solving. The results described in this study show that student-generated flowcharts can provide an overview of the proposed solution to the problem. The study noted that students preferred developing procedural flowcharts rather than writing how they planned to find a solution to the problem. The teachers also preferred visual aids because they were easier and quicker to process and facilitated understanding of the steps taken to reach the solution. These results are consistent with the findings of other researchers (McGowan & Boscia, 2016; Raiyn, 2016). The results are also consistent with Grosskinsky and colleagues' (2019) findings that flowcharts break complex information into different tasks and show how they are connected, thereby enhancing understanding of the process. Consequently, they allow teachers to provide timely

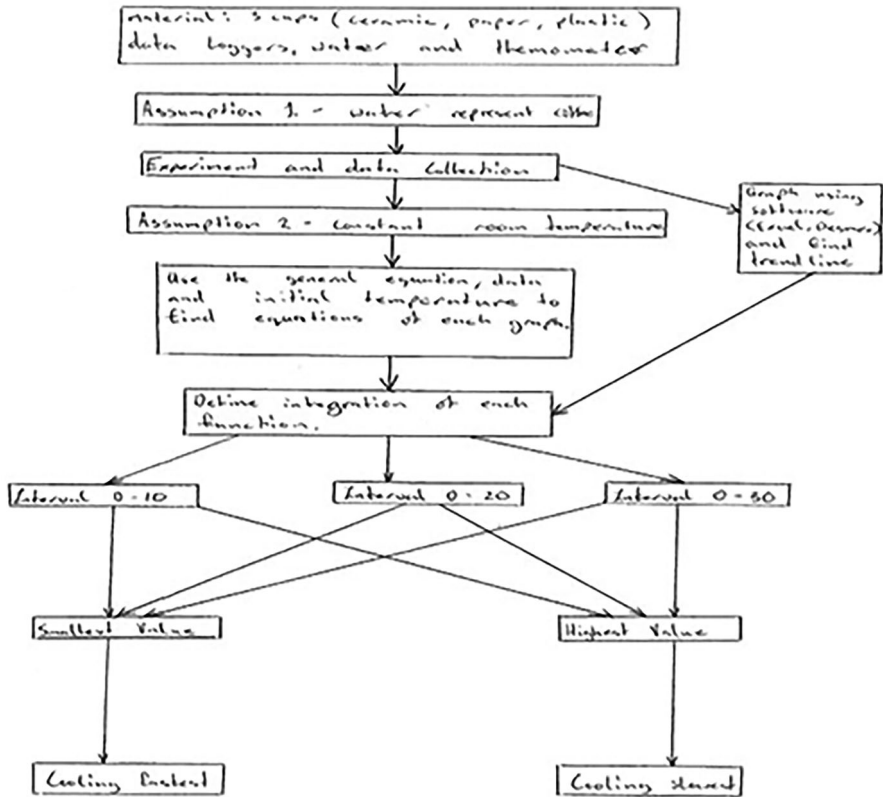


Fig. 6 Procedural flowchart developed by student 2

feedback at a checkpoint compared to the time a teacher would take to go through a written draft. Procedural flowcharts connect procedures and processes in a solution to the problem (Chinofunga et al., 2022). Thus, the feedback provided by the teacher can be more targeted to a particular stage identified on the procedural flowchart, making the feedback more effective and worthwhile. The development of a procedural flowchart during problem solving can be viewed as a visual representation of students' plan and understanding of how they plan to solve the problem as demonstrated in Figs. 5, 6 and 7.

In this study, Ms. Simon noted that procedural flowcharts can represent students' knowledge or thinking in a visual form, which is consistent with Owens and Clements' (1998) findings that visual representations are cognitive constructs. Consequently, they can facilitate evaluation of such knowledge. This study noted that developing procedural flowcharts can provide opportunities to identify gaps in students' understanding and problem-solving skills. It also noted that providing students with opportunities to develop procedural flowcharts may expose students' misconceptions, the depth and breadth of their understanding of the problem and

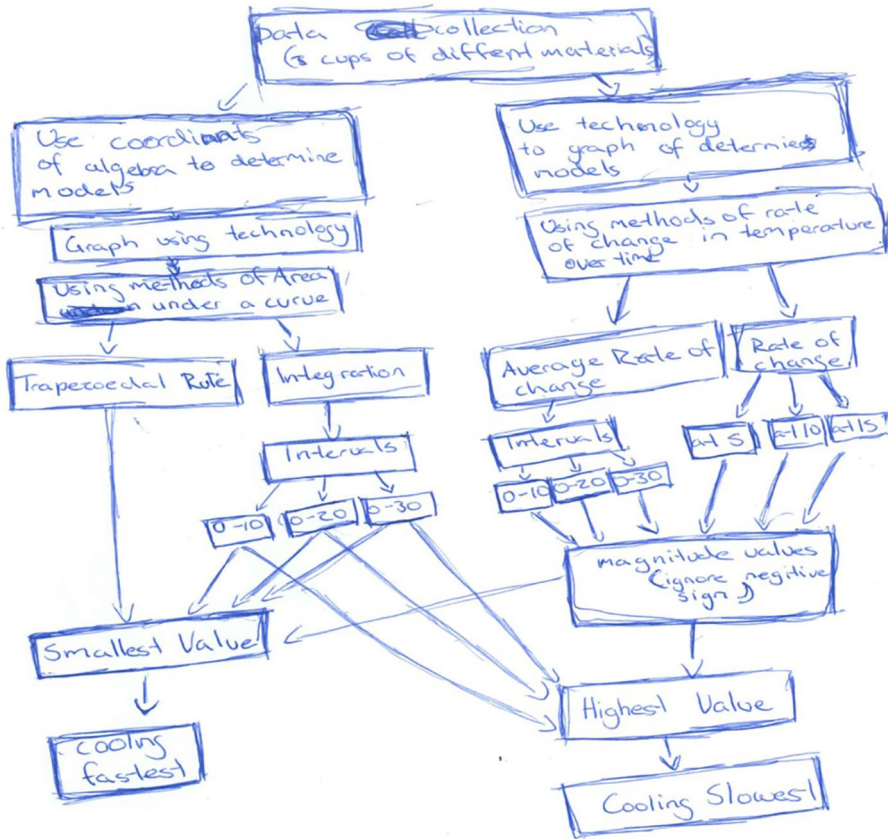


Fig. 7 Collaboratively developed procedural flowchart

how they plan to solve the problem. This is supported by significant research (Grosskinsky et al., 2019; Norton et al., 2007; Vale & Barbosa, 2018), which identified flowcharts as a resource in helping visualise and recognise students' understanding of a problem and communication of the solution. Thus, providing teachers with opportunities to have an insight into students' thinking can facilitate intervention early in the process. The results in this study showed that when students develop their own plan on how to respond to a problem, they are at the centre of their learning. However, scaffolding and collaborative learning can also support problem solving.

Vygotsky (1978) posited that in the Zone of Proximal Development, collaborative learning and scaffolding can facilitate understanding. In this study, the results indicated that a teacher-developed procedural flowchart can be used to guide students in developing a solution to a problem. These results are consistent with Davidowitz and Rollnick's study that concluded that flowcharts provide a bigger picture of how to solve the problem. In Queensland, the QCAA has developed a flowchart (see Appendix 1) to guide

schools on problem-solving and modelling tasks. It highlights the significant stages to be considered during the process and how they relate to each other. Teachers are encouraged to contextualise official documents to suit their school and classes. In such cases, a procedural flowchart acts as a scaffolding resource in directing students on how to develop the solution to the problem. The findings are consistent with previous literature that flowcharts can give an overall direction of the process, help explain what is involved, may help reduce cognitive load and allow students to focus on complex tasks (Davidowitz & Rollnick, 2001; Norton et al., 2007; Sweller et al., 2019).

In addition to being a scaffolding resource, results showed that procedural flowcharts can be developed collaboratively providing students with an opportunity to share their solution to the problem. Being a scaffolding resource or a resource to use in a community of learning highlights the importance of procedural flowcharts in promoting learning within a zone of proximal development, as posited by Davidowitz and Rollnick (2001). Scaffolding students to problem solve and develop procedural flowcharts collaboratively provides students with the opportunity to be at the centre of problem solving.

Research has identified problem solving as student-centred learning (Ahmad et al., 2010; Karp & Wasserman, 2015; Reinholz, 2020; Vale & Barbosa, 2018). The process of developing the procedural flowcharts as students plan for the solution provides students with opportunities to engage more with the problem. Results showed that when students developed procedural flowcharts themselves, mathematics learning transformed from students just being told what to do or follow procedures into something creative and interesting. As students develop procedural flowcharts, they use concepts they have learnt to develop a solution to an unfamiliar problem (Matty, 2016), thus engaging with learning from the beginning of the process until they finalise the solution. The results indicated that developing procedural flowcharts promoted students' ability to not only integrate different procedures to solve the problem but also determine how and when the conditions were ideal to address the problem, providing opportunities to justify and evaluate the procedures that were used.

Deeper understanding of mathematics and relationships between concepts plays an important role in problem solving, and the results from this study showed that different procedures can be integrated to develop a solution to a problem. The participants observed that developing procedural flowcharts could support the brainstorming ideas as they developed the flowchart, as ideas may interlink in a non-linear way. Moreover, students are expected at different stages to make key decisions about the direction they will need to take to reach the solution to the problem, as more than one strategy may be available. For example, student 1 planned on using only technology to develop the models while student 2 considered both technology and algebra. This showed that student 2 applied flexibility in using alternative methods, thus demonstrating a deeper understanding of the problem. Equally important, Ms. Simon observed that as students developed their procedural flowcharts while planning the steps to reach a solution, they were required to analyse, conceptualise, reason, analyse, synthesise and evaluate, which are important attributes of deeper understanding. Fostering deeper understanding of mathematics is the key goal of using problem solving (Kim et al., 2012; King, 1995; Moon, 2008; QCAA, 2018).

The results are additionally consistent with findings from Owens and Clements (1998) and Roam (2009), who posited that visual aids foster reasoning and show cognitive constructs. Similarly, logical sequencing of procedures and ways to execute a strategy expected when developing procedural flowchart can support deeper understanding, as posited by Parvaneh and Duncan (2021). When developing procedural flowchart, students are required to link ideas that are related or feed into another, creating a web of knowledge. Students are also required to identify the ways in which a concept is applied as they develop a solution, and this requires deeper understanding of mathematics. Working collaboratively can also support deeper and broader understanding of mathematics.

The procedural flowchart that was developed collaboratively by the two students demonstrated some of the skills that they did not demonstrate in their individual procedural flowcharts. Like student 2, the collaboratively developed flowchart included use of technology and algebra to determine the models for the three different cups. The students considered both rate of change and area under a curve in the task analysis. Apart from planning to use rate at a point, average rate and definite integration, they added the trapezoidal rule. Both average rate and definite integration were to be applied within the same intervals, building the scope for comparison. The trapezoidal rule would also compare with integration. The complexity of the collaboratively developed procedural flowchart concurred with Rogoff and others (1984) and Stone (1998), who suggested that a community of learning can expand current skills to higher levels than individuals could achieve on their own. It seems the students used the feedback provided by the teacher on their individually developed procedural flowcharts as scaffolding to develop a much more complex procedural flowchart with competing procedures to address the problem. Their individually developed flowcharts might have acted as reference points, as their initial plans were still included in the collaboratively developed plan but with better clarity. This observation is consistent with Guk and Kellogg (2007), Kirova and Jamison (2018) and Ouyang and colleagues (2022), who noted that scaffolding involving peers, teacher and other resources enhances complex problem-solving tasks and transfer of skills.

Supporting the integration of the different stages of mathematics problem solving

When students develop procedural flowcharts, it supports the logical sequencing of ideas from different stages into a process that ends with a solution. Problem solving follows a proposed order and procedural flowcharts visually display decision and/or action sequences in a logical order (Krohn, 1983). They are used when a sequenced order of ideas is emphasised, such as in problem solving (Cantatore & Stevens, 2016). This study concurs with Krohn, Cantatore and Stevens, as the results showed that procedural flowcharts could be used to organise steps and ideas logically as students worked towards developing a solution. Students' procedural flowcharts are expected to be developed through the following stages: problem identification, problem mathematisation, planning and execution and finally evaluation. Such a structure can be reinforced by teachers by sharing a generic problem-solving flowchart outlining the stages so that students can then develop a problem-specific version.

Importantly, students' artefacts in Figs. 5, 6 and 7 provided evidence of how procedural flowcharts support the different stages of problem-solving stages to create a logical and sequential flow of the solution (see Appendix 1). Similarly, Ms. Simon noted that while her students had previously had problems in presenting the steps to their solution in a logical way, she witnessed a significant improvement after she asked them to develop procedural flowcharts first. Further, the results are consistent with Chinofunga et al.'s (2022) work that procedural flowcharts can support procedural flexibility, as they can accommodate more than one procedure in the "identify and execute mathematics procedures that can solve the problem" stage. Thus, stages that require one procedure or more than one procedure can all be accommodated in a single procedural flowchart. Evaluating the different procedures is also a key stage in problem solving.

As students develop the solution to the problem and identify ways to address the problem, they also have to evaluate the procedures, reflecting on the limitations and strengths of the solutions they offer. Ms. Simon observed that her students had previously struggled with identifying strengths and weaknesses of different procedures. However, she noted that procedural flowcharts gave students the opportunity to reflect and compare as they planned the solution. For example, students could have the opportunity to reflect and compare rate at a point, average rate and integration so they can evaluate which strategy can best address the problem. The artefacts identified the different procedures the students used in planning the solution, enabling them to evaluate the effectiveness of each strategy. Thus, enhancing students' capacity to make decisions and identify the optimal strategy to solve a problem aligns with the work of McGowan and Boscia (2016). Similarly, Chinofunga and colleagues' findings noted that developing procedural flowcharts can be effective in evaluating different procedures as they can accommodate several procedures. The different stages that need to be followed during problem solving and the way the solution to the problem is logically presented are central to how the final product is communicated.

In this study, procedural flowcharts were used to communicate the plan to reach the solution to a problem. The length of time given to students to work on their problem-solving tasks in Queensland is fairly long (4 weeks) and students may struggle to remember some key processes along the way. Developing procedural flowcharts to gain an overview of the solution to the problem and share it with the teacher at an early checkpoint is of significant importance. In this study, Ms. Simon expected her students to share their procedural flowcharts early in the process for her to give feedback, thus making the flowcharts a communication tool. The procedural flowcharts developed by the students in Figs. 5, 6 and 7 show how students proposed solving the problem. This result lends further support to the NCTM (2000) findings that visual representations can help students communicate their thinking before applying those thoughts to solving a problem. Ms. Simon also noted that before introducing students to procedural flowcharts, they did not have an overall coherent structure to follow, which presented challenges when they wanted to communicate a plan that involved more than one strategy. However, the students' artefacts were meaningful, clearly articulating how the solution to the problem was being developed, thus demonstrating that procedural flowcharts can

provide the structure that supports the coherent and logical communication of the solution to the problem by both teachers and students (Norton et al., 2007). The visual nature of the students' responses in the form of procedural flowcharts is key to communicating the proposed solution to the problem.

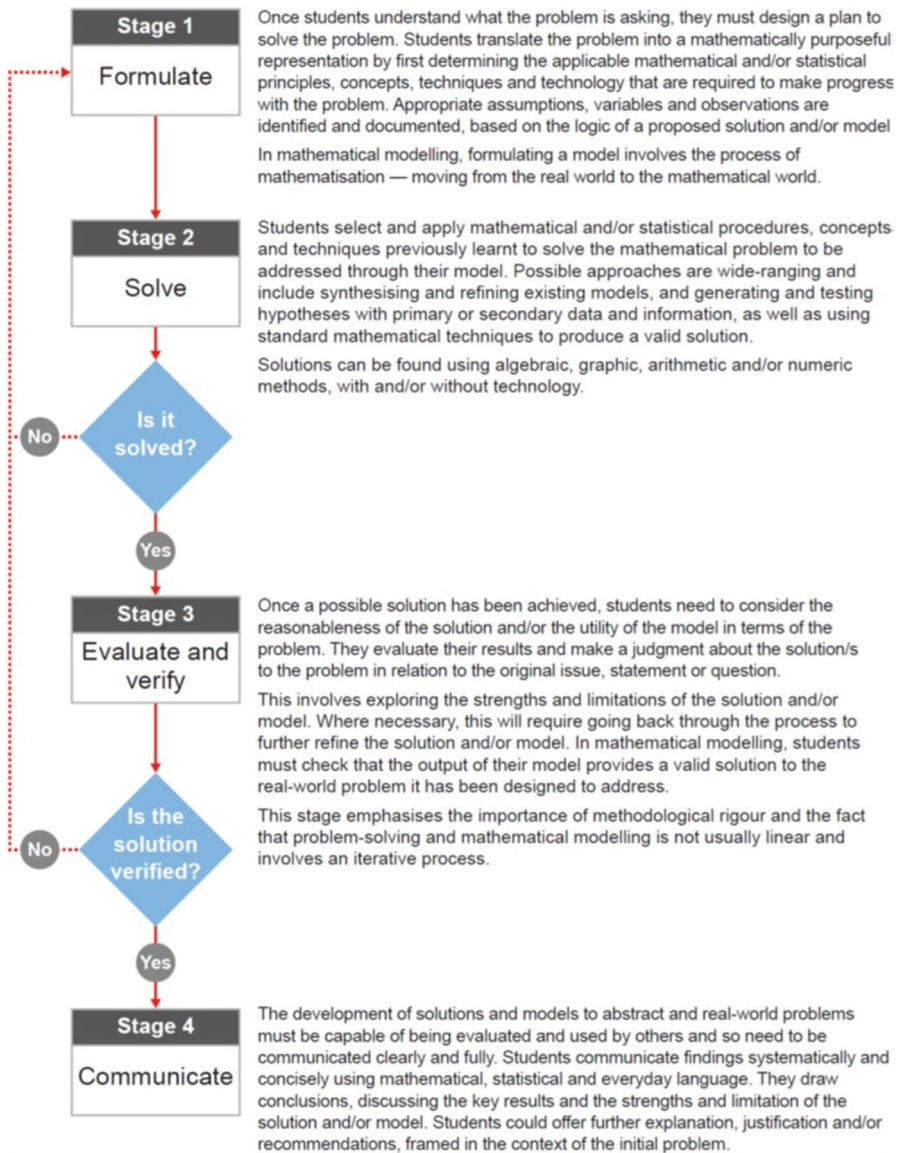
Visual representations are a favourable alternative to narrative communication. Procedural flowcharts can help teachers to check students' work faster and provide critical feedback in a timely manner. Ms. Simon noted that the use of procedural flowcharts provided her with the opportunity to provide feedback faster and more effectively earlier in the task because the charts provided her with an overview of the whole proposed solution. Considering that students are expected to write a report of 2000 words or 10 pages on the task, the procedural flowchart provides the opportunity to present large amounts of information in just one visual representation. Raiyn (2016) noted that visual representations can be a quicker way to evaluate a solution and represent large amounts of information.

Conclusion

The procedural flowcharts that were created by students in this study demonstrate that they can be effective in supporting the development of problem-solving skills. This study suggests that including procedural flowcharts in problem solving may support teachers and students in communicating efficiently about how to solve the problem. For students, it is a resource that provides the solution overview, while teachers can consider it as a mental representation of students' thinking as they plan the steps to reach a solution. Student-developed procedural flowcharts may represent how a student visualises a solution to a problem after brainstorming different pathways and different decision-making stages.

Moreover, as highlighted in this study, the visual nature of procedural flowcharts may offer a diverse range of support for problem solving. Procedural flowcharts make it easy to process and provide timely feedback that in turn might help students engage with the problem meaningfully. Furthermore, they may also provide a structure of the problem-solving process and guide students through the problem-solving process. Navigating through stages of problem solving might be supported by having students design procedural flowcharts first and then execute the plan. Indeed, this study showed that the ability of procedural flowcharts to represent multiple procedures, evaluation stages or loops and alternative paths helps students reflect and think about how to present a logically cohesive solution. Importantly, procedural flowcharts have also been identified as a resource that can help students communicate the solution to the problem. Procedural flowcharts have been noted to support deeper understanding as it may facilitate analysis, logical sequencing, reflection, reasoning, evaluation and communication. Although the in-depth study involved one teacher and three artefacts from her students, which is a very small sample to be conclusive, it identified the numerous advantages that procedural flowcharts bring to mathematics learning and teaching, particularly in terms of supporting the development of problem-solving skills. The study calls for further investigation on how procedural flowcharts can support students' problem solving.

Appendix 1 An approach to problem solving and mathematical modelling



Appendix 2 Phases three and four thematic analysis themes

	A	B	C	D	E	F	G	H
1	Framework on Content Sequencing	Candidate theme - Procedural flowcharts can support the integration of the four stages of mathematics problem-solving.	Candidate theme - Support interconnection of mathematical concepts.	Candidate theme - A tool for consolidation and assessment of teaching and learning of mathematics.	Candidate theme - Support procedural fluency when learning mathematics.	Candidate theme - Support student participation in teaching and learning of mathematics procedures.	Candidate theme - Procedural flowcharts can support mathematics problem-solving.	Candidate theme - Procedural flowcharts can support the integration of the four stages of mathematics problem-solving.
2	backward mapping/mapping out content	breaking down concepts to determine fundamental concepts students need to understand to access new concepts	teacher provided concept map for students to visualise how concepts link to have a bigger picture of the topic, unit or level.	class developed concept maps linking other knowledge of foundational concepts with new knowledge	it organises the steps and explains where you need to go if you're at a certain part in a procedure	individual student created procedural flowcharts after teaching and learning	really good how it organises the steps and explains where you need to go if you're at a certain part in a procedure	can be a way to communicate to teachers how they're planning to solve the scenarios in the PSMT
3	hierarchical, spiralling and logical development of concepts	linking prior knowledge to new concepts in the unit	developing a concept map with the class over a unit or topic for students to understand conceptual connections	students map drawing simple families, permutations, combining and complex problem	can help identify misconceptions that students have and holes or gaps in students understanding	teacher created procedural flowcharts for students to use or targeting students who have not achieved fluency or for students with identified learning	mathematics goes from being very dry and dusty to being something which is actually creative and interesting and explores as students experiment procedures, experiment procedures	it is steps in diagrammatic form which can be used to display how all problem solving stages are related and addressed as students are

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Declarations

Ethics approval This research was approved by the Human Research Ethics Committee, James Cook University (Approval Number H8201).

Informed consent Informed consent was sought from all participants in accordance with the above ethics approval conditions. Before seeking consent from participants, permission was first obtained from the Queensland Department of Education and school principals of participating schools. The information sheets and consent forms were sent to participating mathematics teachers. An information sheet and consent form for students to share their artifacts was sent through their teacher and consent was obtained directly from them as young adults (17–18 years). Consent forms and the data collected has been deidentified to protect participants and stored in the university repository.

Conflict of interest The authors declare no competing interests.

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