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The existence, uniqueness and stability of steady state for a class of first-order difference equations with application to the housing market dynamic

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A B S T R A C T

We study the existence, uniqueness and stability of the steady state for the dynamic described by a class of first-order difference equations. We then apply the result to analyse a housing market where the supply is linear and demand is a bounded and monotone decreasing function of price, derived from households' optimization behaivour. Under two linear price adjustment mechanisms, we prove the existence and uniqueness of an equilibrium, which is independent of the mechanisms. That is, the house price converges to a same steady state where it clears the market under both mechanisms. The result is general in the sense that we do not need to specify a particular form of demand function. Besides, the same approach can be utilized to analyse the dynamics of other markets.

The first-order difference equation plays a role in describing many dynamic phenomena, and has been studied extensively, for example its solution method [1,2,3] and oscillatory behaviour [4]. It is well known that the dynamic described by a first-order difference equation, $y_t = \alpha y_{t-1} + \eta$ where y is a scalar-valued variable of interest and α and η are parameters, admits a unique and stable steady state (the limit point of sequence $\{y_t\}_{t\geq 0}$) if $|\alpha| < 1$. In this note, we extend this result to a class of first-order difference equations, as follows:

$$y_t + F(y_t) = \alpha y_{t-1} + G(y_{t-1}) \tag{1}$$

where F and G are functions of y_t and y_{t-1} respectively, and the initial value y_0 is finite.

We then apply the result to analyse a housing market dynamic with a linear supply function and bounded and monotone decreasing demand function. The demand function is general in that no specific functional form is required, and is derived from households' optimization behaivour. We show the existence and uniqueness of a same equilibrium under two different linear price adjustment mechanisms.

1. Existence, uniqueness and stability of steady state

Lemma 1. *If (1) F and αG are monotone increasing; (2) $|F(y_t)| \leq c < \infty$, $|G(y_{t-1})| \leq d < \infty$; and (3) $|\alpha| < 1$, then Eq. (1) admits a unique*

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and stable steady state.

Proof: (1) The sequence $\{y_t\}_{t\geq 0}$ is bounded:

Since $|F(y_t)| \leq c < \infty$, $|G(y_{t-1})| \leq d < \infty$, we have $\alpha y_{t-1} - (c + d) \leq y_t \leq \alpha y_{t-1} + (c + d)$. By recursion, $\alpha^t y_0 - (c + d) \sum_{\tau=0}^{t-1} \alpha^\tau \leq y_t \leq \alpha^t y_0 + (c + d) \sum_{\tau=0}^{t-1} \alpha^\tau$. Taking the limit, $-\frac{c+d}{1-\alpha} \leq \liminf_{t\to\infty} y_t \leq \limsup_{t\to\infty} y_t \leq \frac{c+d}{1-\alpha}$ as $|\alpha| < 1$. Therefore, for any $\varepsilon > 0$, there exists $T_0 < \infty$ such that for all $t \geq T_0$, $y_t \in \left[-\frac{c+d}{1-\alpha} - \varepsilon, \frac{c+d}{1-\alpha} + \varepsilon\right]$. Subsequently, $\{y_t\}_{t\geq 0}$ is bounded.

(2) The sequence $\{y_t\}_{t\geq 0}$ is monotone:

The left-hand side of Eq. (1) is monotone increasing with respect to y_t, and hence we can invert it to obtain $y_t = \widetilde{F}^{-1} \circ \widetilde{G}(y_{t-1})$ where $\widetilde{F}(y_t) = y_t + F(y_t)$ and $\widetilde{G}(y_{t-1}) = \alpha y_{t-1} + G(y_{t-1})$.

- (i) $0 \leq \alpha < 1$: $\widetilde{G}$ is monotone increasing with respect to y_{t-1}. $\widetilde{F}^{-1} \circ \widetilde{G}$ is monotone increasing with respect to y_{t-1}. If not, there exists $y_{t-1} \leq y'_{t-1}$ with $\widetilde{F}^{-1} \circ \widetilde{G}(y_{t-1}) > \widetilde{F}^{-1} \circ \widetilde{G}(y'_{t-1})$. This implies $\widetilde{G}(y_{t-1}) > \widetilde{G}(y'_{t-1})$ as $\widetilde{F}$ is monotone increasing, contradicting the fact that $\widetilde{G}$ is monotone increasing.
- (ii) $-1 < \alpha < 0$: $\widetilde{G}$ is a monotone decreasing function. Similar reasoning establishes that $\widetilde{F}^{-1} \circ \widetilde{G}$ is monotone decreasing with respect to y_{t-1}.

Therefore, in either case, the sequence $\{y_t\}_{t\geq 0}$ is monotone. Since it is also bounded, it has a limit point. The limit point is the unique steady state of the dynamic described by Eq. (1). The steady state is globally stable as for any initial value y_0, the sequence $\{y_t\}_{t\geq 0}$ is monotone and eventually falls into a bounded interval.

Lemma 2. *If (1) F and αG are monotone increasing; (2) $|F(y_t)| \leq c < \infty$, $|G(y_{t-1})| \leq \beta y_{t-1} + d$ where $d < \infty$; and (3) $|\alpha - \beta| < 1$ and $|\alpha + \beta| < 1$, then Eq. (1) admits a unique and stable steady state.*

Lemma 3. *If (1) F and αG are monotone increasing; (2) $|F(y_t)| \leq \gamma y_t + c$ where $c < \infty$, $|G(y_{t-1})| \leq d < \infty$; and (3) $|\gamma| < 1$, $|\frac{\alpha}{1+\gamma}| < 1$ and $|\frac{\alpha}{1-\gamma}| < 1$, then Eq. (1) admits a unique and stable steady state.*

Lemma 4. *If (1) F and αG are monotone increasing; (2) $|F(y_t)| \leq \gamma y_t + c$ where $c < \infty$, $|G(y_{t-1})| \leq \beta y_{t-1} + d$ where $d < \infty$; and (3) $|\gamma| < 1$, $|\frac{\alpha-\beta}{1+\gamma}| < 1$ and $|\frac{\alpha+\beta}{1-\gamma}| < 1$, then Eq. (1) admits a unique and stable steady state.*

Remark: Proofs of Lemmas 2–4 are similar to that of Lemma 1, and hence not reported, which are available upon request. The boundedness of F and G ensures that $\liminf y_t$ and $\limsup y_t$ do not explode to infinity. The monotonicity of F and G guarantees that the sequence $\{y_t\}_{t\geq 0}$ eventually become monotone. Other than these two requirements, F and G can be any function, for example being discontinuous.

Below are some examples of first-order difference equations that admit a unique and stable steady state: $y_t = 0.5y_{t-1} + 10 + \left(\frac{1}{y_t} - 1\right) 1(y_t \geq 1)$, $y_t = 0.5y_{t-1} + \frac{1}{1+e^{-y_{t-1}}}$, $y_t + \frac{1}{1+e^{-y_t}} = 0.5y_{t-1}$ and $y_t + \frac{1}{1+e^{-y_t}} = 0.5y_{t-1} + \frac{1}{1+e^{-y_{t-1}}}$. One may wish to scrap the requirement of increasing monotonicity of F and αG, which however is not feasible. Counter examples include $y_t = 0.5y_{t-1} + (-1)^t$ and $y_t = 0.5y_{t-1} + 2\cos(y_{t-1})$.

2. Housing market dynamic

Consider a housing market where (1) there are H households that make optimal decisions on house purchase; (2) the supply function is linear, namely $S_t = \delta P_t$, where S the quantity of supply, P is the price, $\delta > 0$ and the subscript t denotes time; and (3) the price adjustment is linear.

In each period, households make optimal decision on whether to purchase houses, which leave the market once purchased. If they decide to purchase, their budget allows them to buy M_{it} houses at price P_t, and they rent the houses with a rental income of R per house per period. If instead they decide not to purchase houses, they can invest elsewhere and receive a rate of return of r_{it} (an outside option which is the opportunity cost of investing in the housing market). Since different households have different budgets and outside options, we assume M_{it} and r_{it} are i.i.d. random variables, which are also independent of each other, with $E[M_{it}] = m < \infty$ and a cumulative distribution function (CDF) $\Phi(r)$ for r_{it}.

Households' optimal purchase behaviour is characterized by an indicator function $\chi\left(r_{it} \leq \frac{R}{P_t}\right)$. That is, if the rate of return from investing in the housing market is higher than their outside option, they will purchase houses. Therefore, the total housing demand can be written as $D_t = \sum_{i=1}^H M_{it}\chi\left(r_{it} \leq \frac{R}{P_t}\right)$. Taking expectation with respect to M_{it} and r_{it}, we obtain $E[D_t] = \sum_{i=1}^H E\left[M_{it}\chi\left(r_{it} \leq \frac{R}{P_t}\right)\right] = \sum_{i=1}^H E[M_{it}]E\left[\chi\left(r_{it} \leq \frac{R}{P_t}\right)\right] = mH\Phi\left(\frac{R}{P_t}\right)$. Accordingly, the aggregate house demand can be written as follows :[1]

[1] Sun [5] investigates a similar demand function in estimating the demand for a COVID-19 vaccine.

$$D_t = mH\Phi\left(\frac{R}{P_t}\right) + \omega$$

where ω is an error term with $E[\omega] = 0$ and we fix $\omega = 0$ in our following analysis,[2] and the rental income (R) positively affects the aggregate house demand.

The equilibrium of the housing market is the price dynamic that (1) admits a unique and stable steady state and (2) clears the market at the steady state.

Proposition: Existence of Equilibrium

(1) With the price adjustment being $P_t = P_{t-1} + \lambda(D_{t-1} - S_{t-1})$, where $\lambda > 0$, there exists a unique and stable equilibrium for the housing market described above if $0 < \lambda\delta \leq 1$;
(2) With the price adjustment being $P_t = P_{t-1} + \lambda(D_t - S_t)$, where $\lambda > 0$, there exists a unique and stable equilibrium for the housing market.

Furthermore, both price adjustment mechanisms lead to a same equilibrium.

Proof: (1) Plug the demand and supply into the price adjustment equation $P_t = P_{t-1} + \lambda(D_{t-1} - S_{t-1})$, we obtain the price dynamic as follows:

$$P_t = (1 - \lambda\delta)P_{t-1} + \lambda mH\Phi\left(\frac{R}{P_{t-1}}\right) \tag{2}$$

which satisfies the hypotheses of Lemma 1 if $0 < \lambda\delta \leq 1$.

(2) Plug the demand and supply functions into $P_t = P_{t-1} + \lambda(D_t - S_t)$, we obtain the following price dynamic:

$$P_t - \frac{1}{1 + \lambda\delta}\lambda mH\Phi\left(\frac{R}{P_t}\right) = \frac{1}{1 + \lambda\delta}P_{t-1} \tag{3}$$

It is clear that Eq. (3) satisfies the hypotheses of Lemma 1.

Therefore, Eqs. (2) and (3) admit a unique and stable steady state, denoted as P^*. Plug P^* into the price adjustment equations, we can obtain $D = S$, namely P^* clears the market. Hence, there exists a unique and stable equilibrium for the housing market. In addition, from Eqs. (2) and (3), we can obtain the equilibrium price, implicitly defined by $\delta P^* - mH\Phi\left(\frac{R}{P^*}\right) = 0$, which does not depend on the price adjustment mechanisms.

The proposition considers two different price adjustment mechanisms. The first one is a standard linear price adjustment mechanism that has been used in studying the housing market dynamic (see for example Dieci and Westerhoff [6]). The second price adjustment mechanism allows for the current-period price to respond to current-period excess demand $(D_t - S_t)$, which can be the case in some markets. Nevertheless, since both adjustment mechanisms lead to the same equilibrium, such a distinction seems immaterial in the long run, despite the first mechanism requires a stronger restriction of parameter values.

Irrespective of the functional form of $\Phi(\cdot)$, the housing market described above always admits a unique equilibrium. Hence, the result is general in the sense that we are studying a class of demand functions. Note despite this application focuses on the housing market, one can use the same approach to model other markets.

If $\Phi(\cdot)$ is the CDF of a uniform random variable, namely $\Phi(r) = \frac{r}{\bar{r}}$ where $\bar{r}$ is the upper bound of the support (lower bound being 0), then the aggregate demand function is $D_t = mH\frac{R}{\bar{r}P_t}$ and the steady state price $P^* = \sqrt{\frac{mHR}{\delta\bar{r}}}$. If $\Phi(r) = 1 - e^{-br}$, namely exponentially distributed with mean $1/b$, the aggregate demand function is $D_t = mH - mHe^{-b\frac{R}{P_t}}$ and the steady state price is implicitly defined by $\delta P^* + mHe^{-b\frac{R}{P^*}} - mH = 0$, which has a unique solution. If $\Phi(r) = 1 - \underline{r}/r$, $r \geq \underline{r} > 0$, namely Pareto distributed with scale and shape parameters of $\underline{r}$ and 1 respectively, then the aggregate demand is linear, $D_t = mH - \frac{mH\underline{r}}{R}P_t$, $0 \leq P_t \leq \frac{R}{\underline{r}}$, and the steady state price is $P^* = \frac{mHR}{R\delta + mH\underline{r}}$.

3. Concluding remarks

This note studies the existence, uniqueness and global stability of steady state for a class of first-order difference equation, where we derive sufficient conditions that ensure a unique and globally stable steady state is achieved. We then applied the result to investigate the housing market dynamic, where the demand function is general and derived from households' optimisation behaviour and the supply function and price adjustment are linear. We show the existence of an equilibrium in the housing market under two different price adjustment mechanisms. The same approach can be utilised to analyse markets of other types

[2] Alternatively, one can rewrite the aggregate demand as $D_t = H\frac{1}{H}\sum_{i=1}^{H} M_{it}\chi\left(r_{it} \leq \frac{R}{P_t}\right) = mH\Phi\left(\frac{R}{P_t}\right) + \omega$ where the second equality is in the sense of almost sure convergence by the Strong Law of Large Numbers.

Declaration of competing interest

The authors declare that they have no competing interests.

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