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# Random fractional partial differential equations and solutions for water movement in soils: Theory and applications

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## Abstract

This paper analyses a set of random fractional partial differential equations (rfPDEs) for water movement in soils. The rfPDEs for both rigid and swelling soils are solved for both a random flux boundary condition (BC), and random concentration BC. Solutions from a random flux BC are presented for the large-time and small-time situations with the large-time solution as a very simple method for determining the flux through the surface of the soil. The equation of cumulative infiltration is presented with random parameters of the rfPDE subject to a random concentration BC. The simulations using the results of the rfPDE for the two types of BCs yielded encouraging and stable results based on two sets of field data: the first set of the data was measurements at a single site while the second set was from 26 measurements in a small catchment. The results suggest that the presented procedures are very useful methods for the interpolation, extrapolation, and prediction of hydrological variables and parameters such as water content, hydraulic conductivity or the flux through the surface of the soil. The methodologies presented in this paper are able to reveal and reproduce the realistic hydrological processes in nature which are often stochastic and random.

KEYWORDS

random flux boundary, random fractional partial differential equations (rfPDEs), random soil parameters, soils, water movement

## 1 | INTRODUCTION

The natural processes in the environment and the environment itself are variable and their uncertainties are common in time and space. The movement of water in soils is a typical example which is affected by many factors such as the non-uniformity of soil properties and uncertain exchanges of water, other materials and the energy inside and outside of the soil. Many existing deterministic assumptions specified in models for water movement in soils may not be the optimal choice although the use of deterministic models is an unquestioned norm. To account for these uncertainties, stochastic or random models are expected to better represent realistic water movement in soils, which is the key focus of this paper.

Partial differential equations (PDEs) play a central role in describing water flow, solute transport and related processes overland, in soils and aquifers as well as in water bodies (Bear, 1971). Based on the fractional PDEs (fPDEs) for water flow in soils (Su, 2014, 2021), random fractional partial differential equations (rfPDEs) for water movement in soils are investigated and examples are analysed in this paper. The rfPDEs discussed in this paper are extensions of the fPDEs by considering the stochastic parameters, initial condition (IC), boundary conditions (BCs) or source terms which constitute the mathematical problem of water movement in soils.

In this paper, the following conventions are made:

1. in this Introduction section, Yevjevich's (1974) convention is followed that regards the words *stochastic*, *random* and *probabilistic* as synonyms in generic descriptions. In the rest of the paper, the two words, stochastic and random, refer to different forms of PDEs, ODEs or fPDEs.
2. a *stochastic* fPDE (sfPDE) refers to those special forms of PDEs for the process of water flow or solute movement in porous media that are driven by a *stochastic source term* in the form of fractional Brownian motion (fBm) which is Gausian with unbounded trajectories (Burgos et al., 2022) or the principal variable is stochastic (Makahane & Atangana, 2022).
3. *random* fPDE (rfPDE) refers to an fPDE with a *random parameter or parameters* which appear, *at least*, in one of the following forms: (1) the coefficient(s) or parameter(s) of an rfPDE; (2) order(s) of fractional derivatives; (3) initial condition (IC); (4) boundary conditions (BCs), and (5) source terms. The randomness could be any probability distribution functions in the terms from (1) to (5) without a specific requirement for its distribution such as fBm for an sfPDE (Burgos et al., 2022). The random PDE was considered an emergent mathematical subject since the surveys by Bharucha-Reid (1973) appeared (Casabán et al., 2018). In this paper, we further consider random fPDEs or rfPDEs for water movement in soils, particularly infiltration subject to two types of random inputs on the surface of the soil known as the boundary condition in addition to the random hydraulic conductivity and stochastic moisture content.

The debate whether an event is *deterministic* or *stochastic* has been an opinion towards philosophy and science for about 3000 years since the Greek civilisation (Yevjevich, 1974). The two approaches in the quantitative analysis of water flow in geological strata and on the earth surface appeared more than a century ago.

The classic PDEs with either constant or *deterministic* functions as coefficients are based on integer calculus and emerged from the situations when ordinary differential equations (ODEs) failed to model some physical phenomena such as vibration of strings, waves in liquids and in the gravitational field, and propagation of sound (Evans et al., 1999). The earliest PDEs were proposed for the analysis of imaginary fluids in terms of hydrodynamics, a word coined by Euler in 1734 as a pure mathematic topic (Evans et al., 1999). Principles in hydrodynamics were applied to real fluids to form hydraulics (Daugherty et al., 1989) which was further extended to create hydrology at large scales.

In hydrodynamics and hydraulics, a set of integer-based PDEs known as the Navier–Stokes equations (NSEs) evolved over time and appear in the literature as the bridge stones of fluid mechanics. The simplified NSEs appear as different models including Darcy's law for the velocity of water movement in porous media, and the Saint-Venant equations for flow on the surface and/or in channels which have been simplified further to form a number of other models such as the kinematic wave equation for overland flow, and so forth (Chow et al., 1988).

Models based on the integer calculus have dominated hydrology, soil science and related fields since the middle 1800s when Darcy (1856) presented a differential equation for the analysis of water flow in porous media. Darcy's experiment and analysis marked the modern era of the formal mathematical analysis of water flow in soils and aquifers. Boussinesq (1904) presented a set of PDEs for hydrodynamics, particularly one PDE for water flow in aquifers which is widely used today. Shortly after those developments, Gardener and Widstoe (1921) proposed PDEs for water movement in soils, and one of which was applied further to develop an equation of infiltration which essentially predates the well-known Horton equation (Horton, 1939). However, Gardner and Widstoe's formulation with the density as the principal variable for water flow is not used today, instead Richards' (1931) potential-based PDE, widely known as the Richards equation, has been the most commonly used mathematical model in the context of unsaturated flow in porous media.

The earlier developments and applications of PDEs were mainly concerned with deterministic systems until 1827 when Robert Brown observed random motion of pollens suspended in water (Gardiner, 1985, p. 2–3). Lord Rayleigh (J.W. Strutt, 1902) first considered the statistical description of the random motion following Brown's observations with no substantial findings. It was Einstein (1905) whose work about the nature of Brown motion *must* be regarded as the beginning of the stochastic modelling of natural phenomena (Gardiner, 1985, p. 2–3).

The emergence of fPDEs for water movement in soils is among the latest developments in hydrology and soil science, which is a result of the gradual improvement in our understanding of material movement in the environment. The development of fPDEs for generic "environmental processes" by Compte (1997) is one particularly relevant example. Comprehensive developments on the related topics have recently been presented by the author (Su, 2021).

The author (Su, 2014, 2021) has shown that the fPDE can be derived for water movement in soils from the theory of the continuous-time random walks (CTRW), where the variables for space and time are random variables and the probability density function (pdf) of the random walks is the principal variable in the fPDE. This means that fPDEs for water flow and solute transport in porous media derived from the CTRW theory are stochastic models. However, the nomenclature for these kinds of equations in the similar literature does not conform to the conventional definition of a stochastic fPDE (or sfPDE) because neither is a stochastic source term essential nor are their coefficients random.

The fPDEs for water movement in soils appeared around three decades ago, and the literature on their detailed analysis and applications have not been comparable to those for integer PDEs such as the Richards equation. Three issues are clearly worthwhile investigating for water movement in soils, which are (1) the variability in the parameters in the fPDEs; (2) variability of the initial condition and/or (3) boundary conditions to reflect more natural conditions and uncertain environments. Examples of random conditions and uncertain environments include a variable rainfall on the surface of the soil, and the random or probabilistic changes in the soil properties along soil profiles and across the field.

The types of distributions of soil particles across a soil profile partially determine the soil properties which are reflected in model parameters such as the diffusivity and hydraulic conductivity of the soil. According to Hartemink (2016), reports in the English literature alone indicated that Donaldson (1852) seemed to be the first to report physical investigations of soils, particularly soil texture and particle sizes so that quantitative investigations of soil properties along soil profiles would have taken place in the late 1800s.

When a mathematical model from continuum fluid mechanics is introduced to model water flow in porous media with discrete flow paths resulting from complex pore geometries, some kinds of averaging methods are required to assess a quantity of the flow across the porous media. At the pore scale, the spatial distribution of pores could be highly variable, and one kind of analysis is analogous to electric signals superimposed with random noises. By defining a representative elementary volume (REV) (Marle, 1967), an REV functions as a noise filter similar to the signal filter in electrical engineering that filters out the random noises and produces a smooth signal (Yeh & Stephens, 1988, p. 12). Mathematically, the process is formulated as

$$H(x) = \int h(x+\eta) f(\eta) d\eta, \tag{1}$$

where $H(x)$ is the output as a smoothed, continuous, function; $h(x)$ is the original, noisy and irregular signal, and $f(\eta)$ is the filter equivalent to a weighting factor.

In geological investigations, stochastic differential equations (SDEs) were also used for deriving descriptions of particle size distributions (Gripenberg, 1934; Krumbein, 1934) and earlier quantitative investigations clearly appeared which can be seen from these bibliographies in different languages. SDEs were further applied to analyse flow in heterogeneous porous media by Warren and Price (1961).

In addition to the SDEs, porosity and permeability as key parameters of flow and porous media were found to obey log-normal distributions by Law (1944) who was regarded as the first researcher to do so (Freeze, 1975). While the previous works were important pioneering studies on flow in porous media, mainly in the petroleum industry, the stochastic approach by Freeze (1975) on flow in aquifers was regarded by Dagan (2002) as the emergence of stochastic modelling in a hydrological context even though Matheron (1965, 1969, 1971, 1973) explored earlier stochastic distributions of hydrological variables and parameters in terms of geostatistics and stochastics (see also de Marsily, 1986).

Mueller (2009) provided a concise overview of the origin and development of sPDEs. It is seen from Mueller that SDEs were studied intensively throughout the twentieth century, but sPDEs were investigated much later. The sPDE has diverse origins with early work stemming from Zakai (1969) in filtering theory, and theoretical development by Pardoux (1972, 2007) and Krylov and Rosovski (1981) (see also Dalang et al., 2009; Walsh, 1986). rPDEs emerged as a mathematical subject in the early 1970s marked by Bharucha-Reid's (1973) survey (Casabán et al., 2018).

In more applied fields such as geosciences and engineering, a brief historical review of stochastic modelling of flow in porous media was provided by Warren and Price (1961). Stochastic modelling of water flow in soils and transport in porous media in general was reported by Freeze (1975), Matheron and de Marsily (1980), Gelhar (1986), Cushman (1987), Serrano (1990), Serrano and Unny (1990), Simmons et al. (1995) and Dagan (2002). Reports on geostatistical, spatial stochastic analyses of flow can be found in Matheron (1965, 1969, 1971, 1973), and geospatial random processes and parameter estimation are detailed in a comprehensive monograph by Christakos (1992).

Stochastic modelling and issues in surface hydrology were provided by Yevjevich (1974), Klemeš (1978), Yevjevich (1987), Wright et al. (2020) and Beven (2021). A futuristic opinion on hydrological modelling, including stochastic approaches, was discussed by Yevjevich (1991).

The above survey indicates that the investigations reported to date include the applications of PDEs, sPDEs and fPDEs for water flow in porous media, statistical/stochastic descriptions of porous media for flow parameters (diffusivity and conductivity etc.) and porous media itself with the REV concept, initial condition or boundary conditions, and source terms. Earlier investigations on porous media can be traced back to the middle and late 1800s, stochastic modelling of natural environmental processes started in 1905 and the diverse origins of the stochastic approaches started around the 1960s.

In this paper, the rfPDE is solved subject to two types of random BCs: one BC is for a random flux at the surface of the soil and the second BC is a random water content on the surface. Solutions derived from these two types of BCs are then used for assessing important hydrological processes:

1. the solution derived from the random flux BC is used to compute the variable rate of infiltration into the soil which is very important but very difficult to determine in practice, and.
2. the solution from the random water content on the boundary is used to derive an equation of cumulative infiltration. With the equations of cumulative infiltration, the random parameters are used to generate random interpolation and extrapolation of cumulative infiltration based on field data measured both at a site (Talsma & van der Lelij, 1976) and from 26 sites at a catchment scale (Sharma et al., 1980). Detailed analyses and discussion of the results are presented in relevant sections, which demonstrate that the methods and procedures presented in this paper are very robust, stable, consistent, and reliable for hydrological applications.

## 2 | MATHEMATICAL PRELIMINARIES AND BACKGROUND FOR WATER FLOW IN SOILS

### 2.1 | Random models and concepts

In terms of water flow in soils, the earlier presentation (Su, 2014) established the connection between the CTRW concept and anomalous water flow in both rigid and swelling soils. The CTRW theory models the motion of particles or water parcels with two probabilities for the two stages of random particle movements: one probability relates to the motion (or jump) length and the second probability to the waiting time of the particles before the next movement in a sequence of two states. Each jump length and the waiting time are independent random variables (irv), and each probability is independently identically distributed (iid) (Gorenflo et al., 2007; Tejedor & Metzler, 2010). The iid positive waiting times are denoted by $T_1, T_2, T_3, \ldots$, each having the same probability density function (pdf), $\varphi(t), t > 0$, and the iid random jumps are denoted by $X_1, X_2, X_3, \ldots$ in a real domain, $\mathbf{R}$, each having the same pdf $w(x), x \in \mathbf{R}$. With these definitions, the probability density of the particle (or water parcel) movement in the soil can be written in the Laplace-Fourier domain as (Gorenflo et al., 2007; Gorenflo & Mainardi, 2005)

$$\hat{\tilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} + |\kappa|^{\gamma} i^{\omega \,\mathrm{sign}\kappa}}, \tag{2}$$

where $\kappa$ and $s$ are the Fourier and Laplace transform variables, respectively; $\beta$ is the exponent for the probability of the waiting time intervals between two consecutive steps; $\gamma$ is the exponent for the probability of the length of steps for the random walks; $\omega$ is the skewness acting on the space variable, $|\omega| \le \min\{\gamma, 2-\gamma\}$, and $i^{\omega \,\mathrm{sign}\kappa} = \exp[i(\mathrm{sign}\kappa)\omega\pi/2]$.

In the symmetrical case of $\omega = 0$, Equation (2) can be inverted to the following fractional diffusion-wave equation (fDWE) (Gorenflo & Mainardi, 2009).

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}}, \; u(x,0) = \delta(x). \tag{3}$$

The fDWE in Equation (3) with the Dirac delta function, δ(x), as an initial condition results from the asymptotic or long-time approximation of the CTRW model with the two transitional pdfs for the length of jumps, $P(X > x)$, and waiting time intervals, $P(J > t)$, obeying power laws, that is, $P(X > x) \approx x^{-\gamma}$, and $P(J > t) \approx t^{-\beta}$ (Meerschaert, 2011).

The left-hand side of Equation (3) is the Caputo fractional derivative with respect to time, $t$, while the right-hand side of it is the Riesz-Feller fractional derivative (RFFD) with respect to space, $x$. The connections between RFFD and other fractional derivatives such as Riemann-Liouville fractional derivatives (RLFD) and Caputo fractional derivatives (CFD) can be found in Gorenflo et al. (2002). In the symmetrical case of $\omega = 0$, the RFFD is simply the Riesz potential, and the difference between RFFD and RLFD is the factor (see details in Appendix A),

$$c_{+}(\beta, 0) = c_{-}(\beta, 0) = \frac{1}{\sin(\pi\beta/2)}, \tag{4}$$

which can be incorporated in the diffusion coefficient so that RFFD and RLFD can be conveniently used interchangeably.

With convection due to a shift jump size distribution in the CTRW theory (Zhang et al., 2009), a set of fPDEs were presented earlier (Su, 2014) and one of these fPDEs to be analysed in this paper is of the form

$$\frac{\partial^{\beta}\theta}{\partial t^{\beta}} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial^{\eta}\theta}{\partial z^{\eta}}\right) - \frac{\partial K(\theta)}{\partial z} \tag{5}$$

with

$$0 < \beta \le 1; \quad 0 < \eta \le 1 \tag{6}$$

and $\theta$ is the water content; $D(\theta)$ and $K(\theta)$ are the diffusivity and hydraulic conductivity functions, respectively; $z$ is the depth of the soil and $t$ is time. The diffusivity and hydraulic conductivity is related through the relationship (Philip, 1969)

$$D(\theta) = K(\theta)\frac{d\psi}{d\theta}, \tag{7}$$

where $\psi$ is the water potential in the unsaturated soil.

An extension of the CTRW theory and fPDE to swelling soils has also been provided which can be written as (Su, 2014),

$$\frac{\partial^{\beta}\theta}{\partial t^{\beta}} = \frac{\partial}{\partial m}\left[D_m(\theta)\frac{\partial^{\eta}\theta}{\partial m^{\eta}}\right] - (\gamma_n\alpha - 1)\frac{dK_m(\theta)}{d\theta}\frac{\partial\theta}{\partial m}, \tag{8}$$

where $m$ is the material coordinate; $\gamma_n$ is the particle specific gravity; $\alpha$ is the gradient (or slope) of the shrinkage curve, which is a ratio on the graph of the specific volume, $v$, versus water content or moisture ratio, $\theta$; $D_m(\theta)$ is the material diffusivity given by

$$D_m(\theta) = \frac{K_m(\theta)}{1+\theta}\frac{d\Phi}{d\theta} \tag{9}$$

with $\Phi$ being the unloaded matrix potential, and $K_m(\theta)$ being the unsaturated material hydraulic conductivity defined as (Smiles & Raats, 2005, Equation (29), for a negative sign in their Equation (28))

$$K_m(\theta) = \frac{k_m}{\theta_s}(\gamma_n\alpha - 1) \tag{10}$$

with $k_m$ being the saturated material hydraulic conductivity (Smiles & Raats, 2005) given a

$$k_m = K(\theta)\theta_s, \tag{11}$$

where $K(\theta)$ is the conventional unsaturated hydraulic conductivity.

Based on the previous development (Su, 2014, 2021), we further investigate random boundary conditions and parameters in fPDEs for water flow in soils.

## 2.2 | Methods for solutions of random fPDEs

There are different methods for the solutions of random fPDEs (or rfPDEs), and methods used in this paper are based on the random Laplace transform method detailed by Casabán et al. (2015) and mean square Laplace transform by Burgos et al. (2022). Interested readers are referred to related literature for other methods such as random differential operational calculus (Villafuerte et al., 2010), random Laplace transform (Casabán et al., 2015), random Fourier transform (Casabán et al., 2018), and mean square Laplace transform (Burgos et al., 2022) etc.

# 3 | RANDOM FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS FOR WATER MOVEMENT IN SOILS

Here we are mainly concerned with rfPDEs for water movement in soils. For simplicity, let us define that $\theta$ is a random variable, and Equation (5) with $\eta = 1$ applies to vertical water movement in soils,

$$\frac{\partial^\beta \theta}{\partial t^\beta} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\partial K(\theta)}{\partial z} \tag{12}$$

with

$$0 < \beta \le 1. \tag{13}$$

## 3.1 | Parameters and their forms

### 3.1.1 | Diffusivity and hydraulic conductivity

While the relationship between the diffusivity and hydraulic conductivity is defined in Equation (7), a large number of functions have been found to be suitable for the diffusivity (Philip, 1960a, 1960b). In practice, power functions are important and convenient for the diffusivity (Philip, 1992),

$$D(\theta) = D_0\theta^b \tag{14}$$

and the hydraulic conductivity

$$K(\theta) = K_0\theta^k, \tag{15}$$

where $D_0$, $b$, $K_0$, and $k$ are constants, which need to be determined experimentally.

### 3.1.2 | Dimensions of the parameters

When a fractional order is introduced into the PDE such as in Equation (12), there are different options for defining the dimensions of the fPDE. One option is to accept and use the new dimensions in an fPDE, and the second option is to introduce a new parameter to the original PDE (Kilbas et al., 2006, p. 464) so that the usual dimensions of the parameters are retained while ensuring correct dimensions in the fPDE. With the approach by Kilbas et al. (2006), the new fractional diffusion coefficient and hydraulic conductivity in Equation (12) are, respectively, updated as

$$D_f = D(\theta)\tau^{1-\beta} \tag{16}$$

and

$$K_f = K(\theta)\tau^{1-\beta}, \tag{17}$$

where $\beta$ is the order of the fPDE in Equation (12); $D(\theta)$ and $K(\theta)$ are the diffusion coefficient and hydraulic conductivity, respectively, in the integer-based PDE, and $\tau$ is the new time parameter which accommodates the new dimensions. With this approach, Equation (12) is updated to be

$$\frac{\partial^\beta \theta}{\partial t^\beta} = \tau^{1-\beta}\left[\frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\partial K(\theta)}{\partial z}\right]. \tag{18}$$

## 3.2 | The possible random terms in an fPDE for water flow in soils

The randomness of the flow problem can appear in any of the following terms and/or their combinations:

1. Equation (12) and its IC and BCs such as the flux into or out of the soil;
2. $D(\theta)$ and/or $K(\theta)$;
3. $\beta$ (Li et al., 2009; Sun et al., 2009, 2011).
   The order of fractional derivatives can take different forms. Reports in the literature suggest that the order of fractional derivatives can be one of seven types, which ranges from a constant to a random function (for a brief summary, see Su, 2021, p. 109). In this paper, we use a constant $\beta$ that only varies with different soils. We are particularly interested in the random flux of water through the surface of the soil, random water content on the surface of the soil and relevant solutions while the randomness of $D(\theta)$ and $K(\theta)$ are allowed.

# 4 | SOLUTIONS SUBJECT TO A RANDOM FLUX BOUNDARY CONDITION ON THE SURFACE OF THE SOIL

## 4.1 | The random flux boundary condition for rigid soils

For rigid soils, solutions of the rfPDE in Equation (18) is solved for soil water movement subject to a random flux BC or the BC of the third kind with a semi-infinite profile:

$$\theta=\theta_i,\quad t=0,\, z>0, \tag{19}$$

$$\tau^{1-\beta}\left[K(\theta)-D(\theta)\frac{\partial\theta}{\partial z}\right]=\tau^{1-\beta}r,\quad t>0,\, z=0, \tag{20}$$

$$\frac{\partial\theta}{\partial z}=0,\quad t>0,\, z\to\infty, \tag{21}$$

where $\theta_i$ is the initial water content; $r$ is the random flux on the surface of the soil with the dimension $[L/T]$, and it is positive for inflow into the soil and negative for evaporation from the soil. Note that the random flux accompanied by a dimensional correction factor $\tau^{1-\beta}$; $K(\theta)$ and $D(\theta)$ are the hydraulic conductivity and diffusivity, respectively. All these quantities could take forms of random variables (Casabán et al., 2015), deterministic functions or constants as reported in the literature.

For swelling soils, the corresponding IC and BCs are given below,

$$\theta=\theta_i,\quad t=0, m>0, \tag{22}$$

$$\tau^{1-\beta}\left[(\gamma_n\alpha-1)K_m(\theta)-D_m(\theta)\frac{\partial\theta}{\partial m}\right]=\tau^{1-\beta}r,\, t>0,\, m=0, \tag{23}$$

$$\frac{\partial\theta}{\partial m}=0,\quad t>0,\, m\to\infty. \tag{24}$$

In the derivation of the solutions, the reduced water content is used,

$$\vartheta=\frac{\theta-\theta_i}{\theta_s-\theta_i}, \tag{25}$$

where $\theta_s$ is the saturated value of $\theta$.

We look for solutions of Equation (18) with $b=0$, and $k=1$ for the diffusivity and hydraulic conductivity as discussed in Su (2010, 2012, 2014) with conditions in Equations (19), (20) and (21). Applying the random Laplace transform (Casabán et al., 2015) to Equation (18) and the conditions in Equations (19), (20) and (21) yields the following result,

$$D_0\frac{d^2\tilde{\vartheta}}{dz^2}-K_0\frac{d\tilde{\vartheta}}{\partial z}-\tau^{\beta-1}s^{\beta}\tilde{\vartheta}=0, \tag{26}$$

$$\tilde{\vartheta}=0,\quad t=0,\, z>0, \tag{27}$$

$$\tau^{1-\beta}\left[K_0\tilde{\vartheta}-D_0\frac{\partial\tilde{\vartheta}}{\partial z}\right]=\tau^{1-\beta}r,\quad t>0,\, z=0, \tag{28}$$

$$\frac{\partial\tilde{\vartheta}}{\partial z}=0,\quad t>0,\, z\to\infty, \tag{29}$$

where $s$ is the random Laplace transform variable and $\tilde{\vartheta}$ is the random Laplace transform of $\vartheta$.

The solution of a similar problem defined by Equation (26) subject to Equations (27), (28) and (29) () has been given in the Laplace domain (Gershon & Nir, 1969; van Genuchten & Alves, 1982)

$$\tilde{\vartheta}(z,s)=\frac{r}{K_0}\left(\frac{2\tau^{1-\beta}K_0}{u+\tau^{1-\beta}K_0}\right)\exp\left[\frac{(\tau^{1-\beta}K_0-u)\tilde{z}}{2\tau^{1-\beta}D_0}\right] \tag{30}$$

with

$$u=\tau^{1-\beta}K_0\left(1+\frac{4D_0s^{\beta}}{\tau^{3(1-\beta)}K_0^2}\right)^{1/2}. \tag{31}$$

Equation (30) is very complex in the Laplace domain and cannot be easily inverted analytically, and numerical inversion is required. In this paper, we present two forms of asymptotic solutions below which correspond to large-time and small-time asymptotic results.

## 4.2 | Large-time solutions for rigid and swelling soils

### 4.2.1 | Large-time solution for rigid soils

In this method, an exact inversion of Equation (30) is not visibly possible, and numerical approximations can be carried out for certain situations. Here we are only interested in the large time situation when $t\to\infty$ or $s\to 0$, $u\approx\tau^{1-\beta}K_0$ in Equation (31) so that Equation (30) can be inverted to yield a solution of the form

$$\vartheta(z,t)=\frac{r}{K_0}, \tag{32}$$

which, with the original variables restored, is

$$\theta=\frac{(\theta_s-\theta_i)r}{K_0}+\theta_i \tag{33}$$

with the dimensions of $K_0$ and $r$ as $[L/T]$.

Equation (33) is very easy to use for determining the water content given the influx on the surface and the conductivity of the soil. In fact, in most situations it is much more convenient to measure the water content and the conductivity of the soil because the accurate influx is unknown for it is the net inflow rate rather than the gross rainfall intensity or irrigation rate. For this reason, Equations (32) and (33) are rearranged to yield

$$r=\vartheta(z,t)K_0, \tag{34}$$

which, with the original variables restored, is

$$r=K_0\left(\frac{\theta-\theta_i}{\theta_s-\theta_i}\right). \tag{35}$$

In these solutions from Equations (32) to (35), the influx $r > 0$ is for infiltration and $r < 0$ for evaporation.

### 4.2.2 | Large-time solution for swelling soils

Referring to the previous section, the related solution and results for swelling soils are

$$\theta = \frac{(\theta_s - \theta_i) r}{(\gamma_n \alpha - 1) K_0} + \theta_i \tag{36}$$

and

$$r = (\gamma_n \alpha - 1) K_0 \left( \frac{\theta - \theta_i}{\theta_s - \theta_i} \right), \tag{37}$$

where $r > 0$ is for infiltration and $r < 0$ for evaporation.

## 4.3 | Small-time approximation

### 4.3.1 | Small-time approximation rigid soils

The derivation of the small-time solution is detailed in Appendix B with the aid of random Laplace transform (Casabán et al., 2015), and the solution for small time is presented as Equation (B10),

$$\vartheta(z,t) = \frac{K_0 r}{D_0^{1/2}} \exp\left(\frac{K_0 z}{2D_0}\right) t^{(\beta/2)-1} \phi\left(-\beta/2,\, \beta/2; -\frac{z}{D_0^{1/2} t^{\beta/2}}\right), \tag{38}$$

where

$$\phi\left(-\beta/2, \beta/2; -\frac{z}{D_0^{1/2} t^{\beta/2}}\right) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma[(1-k)\beta/2]} \left(-\frac{z}{D_0^{1/2} t^{\beta/2}}\right)^k \tag{39}$$

is the Wright function (Gorenflo et al., 1999; Kilbas et al., 2006). The temporal component of Equation (38) can be written as

$$\Phi\left(-\beta/2, \beta/2; -\frac{z}{D_0^{1/2} t^{\beta/2}}\right) = t^{(\beta/2)-1} \phi\left(-\beta/2, \beta/2; -\frac{z}{D_0^{1/2} t^{\beta/2}}\right) \tag{40}$$

which is the generalized Wright function (GWF).

Stanković (1970), Equation (31) showed that the GWF in Equation (40) has an asymptotic expression for $\frac{z}{D_0^{1/2} t^{\beta/2}} \to \infty$, that is,

$$\Phi\left(-\beta/2, \beta/2; -\frac{z}{D_0^{1/2} t^{\beta/2}}\right) \sim \frac{\sin(\pi\beta/2)\Gamma(1-\beta/2)}{\pi} \left(\frac{z}{D_0^{1/2} t^{\beta/2}}\right)^{(\beta/2)-1}, \tag{41}$$

which enables Equation (38) to be written as

$$\vartheta(z,t) \sim \frac{K_0 r}{D_0^{1/2}} \exp\left(\frac{K_0 z}{2D_0}\right) \frac{\sin(\pi\beta/2)\Gamma(1-\beta/2)}{\pi} \left(\frac{z}{D_0^{1/2} t^{\beta/2}}\right)^{(\beta/2)-1} \tag{42}$$

for $\frac{z}{D_0^{1/2} t^{\beta/2}} \to \infty$.

### 4.3.2 | Small-time approximation for swelling soils

Referring to the Equation (38) in the previous section, the parallel solution for swelling soils is

$$\vartheta(m,t) = \frac{(\gamma_n \alpha - 1) K_0 r}{D_{m0}^{1/2}} \exp\left(\frac{(\gamma_n \alpha - 1) K_0 m}{2D_{m0}}\right) t^{(\beta/2)-1} \phi\left(-\beta/2, \beta/2; -\frac{m}{D_{m0}^{1/2} t^{\beta/2}}\right), \tag{43}$$

where

$$\phi\left(-\beta/2, \beta/2; -\frac{m}{D_{m0}^{1/2} t^{\beta/2}}\right) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma[(1-k)\beta/2]} \left(-\frac{m}{D_{m0}^{1/2} t^{\beta/2}}\right)^k \tag{44}$$

with $D_{m0}$ for swelling soils which is equivalent to $D_0$ for rigid soils.

The corresponding asymptotic result of Equation (43) for swelling soils is

$$\vartheta(m,t) \sim \frac{(\gamma_n \alpha - 1) K_0 \sin(\pi\beta/2)\Gamma(1-\beta/2) r}{\pi D_{m0}^{1/2}} \exp\left(\frac{(\gamma_n \alpha - 1) K_0 m}{2D_{m0}}\right) \left(\frac{m}{D_{m0}^{1/2} t^{\beta/2}}\right)^{(\beta/2)-1} \tag{45}$$

for $\frac{m}{D_{m0}^{1/2} t^{\beta/2}} \to \infty$.

## 4.4 | Examples: determination of the random flux into and out of the soil using the large-time solution

In this section, two examples are demonstrated to compute the random flux into and out of the soil surface based on Equation (35) with the given random soil water content $\theta$ and random hydraulic conductivity $K_0$. The first example is generated from a small variability in the soil water content and the second example from a large variability in the water content of the soil.

Both examples are based on the same hydraulic conductivity. Note that due to the random nature of the variables, each simulation is different from the previous result even though the same values are assigned to these parameters.

### 4.4.1 | Random influx with a small variability in the water content of the soil

With the measured water content, $\theta$, and hydraulic conductivity of the soil, $K_0$, as random values, the random infiltration flux, $r$, is computed using Equation (35). As the random variable could take any forms of probability distributions (Casabán et al., 2018), it is assumed here that both the water content and hydraulic conductivity of the soil have the normal distributions. The random variables representing both $\theta$ and $K_0$ were generated by the random number generator, *randn*, in MATLAB (Mathworks, Inc., 2002).

The input values for these random parameters are hypothesised as: the mean $\theta = 0.3$ with a standard deviation $\sigma = 0.02$, and the mean $K_0 = 10.0\,\text{mm/h}$ with a standard deviation $\sigma = 1.0\,\text{mm/h}$. Then the computed random flux, $r$, is shown together with the inputs in Figure 1.

While it is feasible to determine the random infiltration flux, $r$, in Figure 1 using Equation (35) with both the measured water content, $\theta$, and hydraulic conductivity of the soil, $K_0$, as random values, a constant $K_0$ can also be used for simplicity.

#### 4.4.2 | Random influx with a large variability in the water content of the soil

The input values for this example are hypothesised as: the mean $\theta = 0.3$ with a standard deviation $\sigma = 0.06$, and the mean $K_0 = 10.0$ mm/h with a standard deviation $\sigma = 1.0\,mm/hour$. The computed random flux, $r$, is shown together with the inputs in Figure 2.

The computation of the flux using Equation (35) or (37) is very easy, and this procedure is probably one of the simplest inverse problems. The simulated results presented in Figures 1 and 2 are straightforward, which simply show that the larger the influx, the higher the water content in the soil.

## 5 | SOLUTIONS SUBJECT TO A CONCENTRATION BOUNDARY CONDITION

### 5.1 | Solutions for swelling soils

With the following initial and boundary conditions,

$$\theta = \theta_i, \quad t = 0, \; m > 0, \tag{46}$$

$$\theta = \theta_0, \quad t > 0, \; m = 0, \tag{47}$$

$$\theta \to 0, \quad t > 0, \; m \to \infty \tag{48}$$

the solution of Equation (8) with $\eta = 1$ was presented earlier (Su, 2010) with deterministic parameters. In the current analysis, the parameters of the fPDE in Equation (8) are treated as random parameters and related results are discussed here.

With a constant diffusivity, and linear hydraulic conductivity of the form

$$K = K_1 + K_0\theta \tag{49}$$

and the initial and boundary conditions in Equations (46), (47) and (48), the cumulative infiltration is similar to the one given earlier (Su, 2010)

$$I(t) = At + St^{\beta/2} \tag{50}$$

and the infiltration rate is given by differentiating Equation (50) with respect to time,

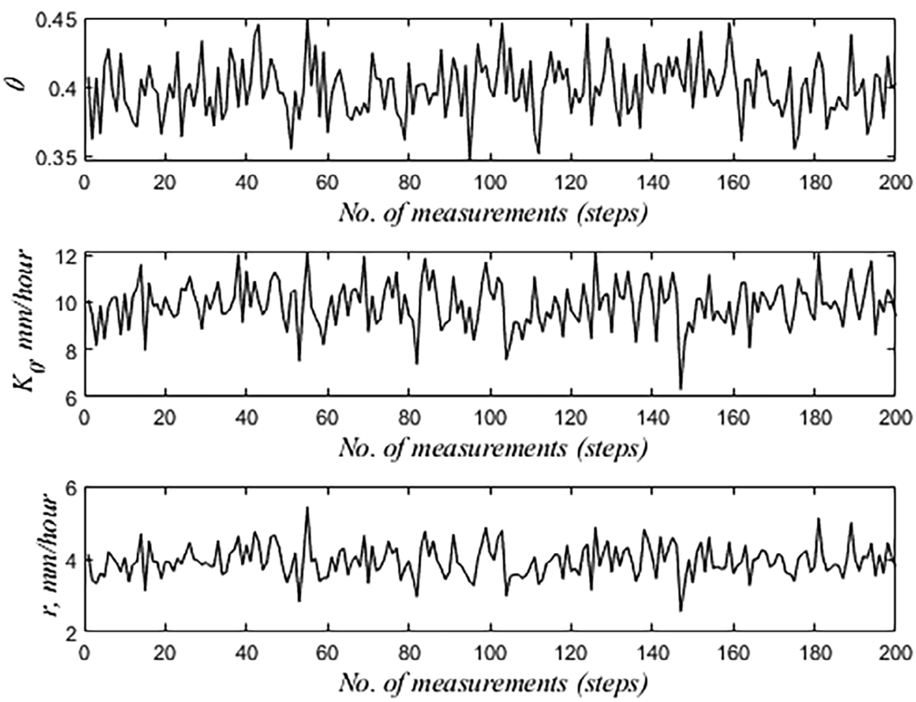


**FIGURE 1** Computed random fluxes into and out of the soil, $r$ from the measured water content, $\theta$, and hydraulic conductivity, $K_0$.

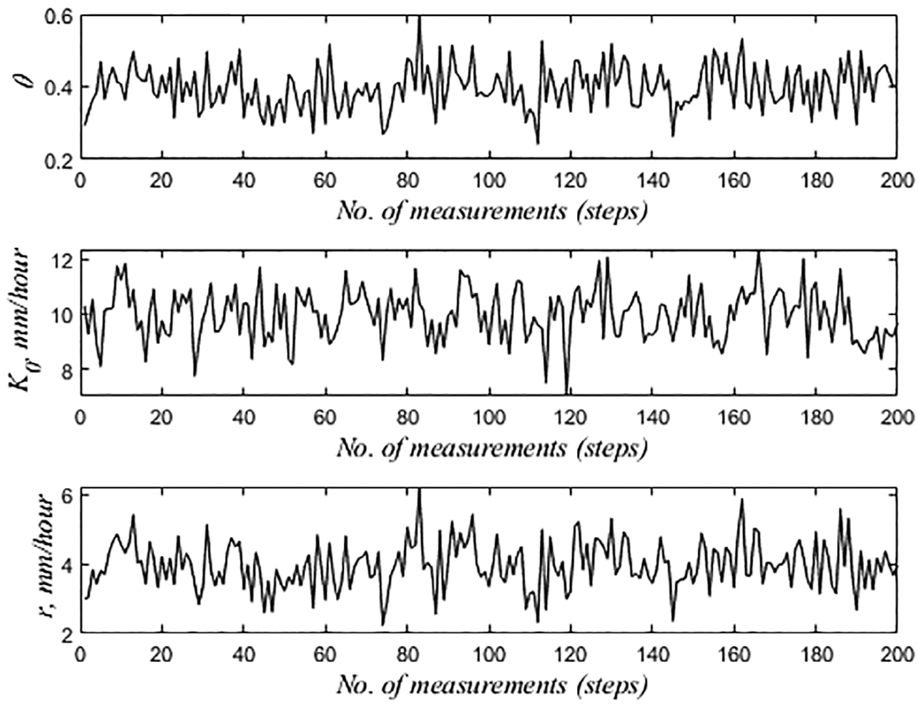


**FIGURE 2** Computed random fluctuations of the soil water flux from a higher water content, $\theta$, and hydraulic conductivity, $K_0$.

$$i(t)=A+\frac{\beta}{2}St^{(\beta/2)-1}, \tag{51}$$

where $A$ is the final infiltration rate, and $S$ is the anomalous sorptivity given by

$$S=\frac{(\theta_0-\theta_i)\Gamma[1-\beta/2]D_m^{1-\beta/2}}{2[K_0(\gamma_n\alpha-1)]^{1-\beta}}. \tag{52}$$

When the anomalous sorptivity is known, rearranging Equation (52) yields the anomalous diffusivity for the swelling media,

$$D_m=\left\{\frac{2S[K_0(\gamma_n\alpha-1)]^{1-\beta}}{\Gamma[1-\beta/2](\theta_0-\theta_i)}\right\}^{2/(2-\beta)}. \tag{53}$$

## 5.2 | Solutions for rigid soils

For rigid or non-swelling soils, with the initial and boundary conditions are

$$\theta=\theta_i,\quad t=0,\quad z>0, \tag{54}$$

$$\theta=\theta_0,\quad t>0,\ z=0, \tag{55}$$

$$\theta\to 0,\quad t>0,\ z\to\infty \tag{56}$$

the equation for cumulative infiltration is

$$I(t)=At+St^{\beta/2} \tag{57}$$

and the infiltration rate is given by differentiating Equation (57) with respect to time,

$$i(t)=A+\frac{\beta}{2}St^{(\beta/2)-1}, \tag{58}$$

where $A$ is the final infiltration rate, and $S$ is the anomalous sorptivity given by

$$S=\frac{(\theta_0-\theta_i)\Gamma[1-\beta/2]D^{1-\beta/2}}{2K_0^{1-\beta}}. \tag{59}$$

When the anomalous sorptivity is known, rearranging Equation (59) results in the anomalous diffusivity for the swelling media,

$$D=\left\{\frac{2SK_0^{1-\beta}}{\Gamma[1-\beta/2](\theta_0-\theta_i)}\right\}^{2/(2-\beta)}. \tag{60}$$

## 5.3 | Methods for determining values of the parameters

The final infiltration rate, $A$, and the anomalous sorptivity, $S$, derived in the previous analysis (Su, 2010) from the data of Talsma and van der Lelij (1976), are treated as random ones here, and the values of $A$ and $S$ were the results of optimal curve fitting which are regarded as the mean value in this example. Other methods based on the spectral analysis (Yu et al., 2010) can also be used when the number of measurements is large.

## 5.4 | Examples

In this section, examples are presented for the application of the random infiltration equation in Equation (50) and its parameters. These examples are based on data measured in the field for both a single location (examples [1] and [2]) and a number of measurements at a catchment scale (example [3]).

### 5.4.1 | Measurements and random simulation at a single site

*Randomly simulated parameters A and S*

The data used in this study was collected by Talsma and van der Lelij (1976) from a rice field near Coleambally, New South Wales, Australia. The dominant soil type is Wunnamurra clay and Tuppal clay. An earlier study (Su, 2010), which presented an fPDE and verified using the data of cumulative infiltration by Talsma and van der Lelij (1976), showed that $A = 1.29$ mm/day, $S = 48.58$ mm/day$^{1/2}$, and $\beta = 0.2385$ for this clay soil complex.

Based on Equation (50) which is identical in structure to the equation presented earlier (Su, 2010) but with different definitions, the parameter values derived (Su, 2010) are $A = 1.29$ mm/day, $S = 48.58$ mm/day$^{1/2}$ and $\beta = 0.2385$, and it is also assumed that a standard deviation of $A$ is $\sigma = 0.1$ mm/day and that of $S$ is $\sigma = 3.0$ mm/day$^{\beta/2}$. With these data, Figure 3 is generated using the *randn* algorithm in MATLAB which generates normal distributions for both $A$ and $S$. Note that with the introduction of the dimensional correction factor $\tau^{1-\beta}$ (Kilbas et al., 2006), the dimensions of $A$ and $S$ have the usual units and are different from the previous report in Su (2010).

While the final infiltration rate, $A$, in Figure 3 is determined as a random variable, it can also be treated as a constant for simplicity which is different from the sorptivity as shown in Equation (52) as a function of different parameters.

*Interpolation of the cumulative infiltration subject to random A and S*

Based on the earlier findings (Su, 2010) using data from a single site (Talsma & van der Lelij, 1976), here we treat $A = 1.29$ mm/day, $S = 48.58$ mm/day$^{\beta/2}$ as the random variables with $\beta = 0.2385$, and standard deviation of $\sigma = 0.1$ mm/day for $A$ and that of $\sigma = 3.0$ mm/day$^{\beta/2}$ for $S$ as demonstrated in Figure 3. Then the random variability in cumulative infiltration in Equation (50) subject to these random parameters is computed and graphed in Figure 4.

The seemingly stable and consistent trend of the cumulative infiltration generated by the normal distributions for both random variables $A$ and $S$ suggest that the methodologies based on the rfPDE is an excellent tool for interpolation, extrapolation, and prediction.

### 5.4.2 | Measurements and random simulation at a catchment scale

*Spatial variability of infiltration parameters A and S*

In this example, the random concept and the related equation of cumulative infiltration and its parameters demonstrated in the

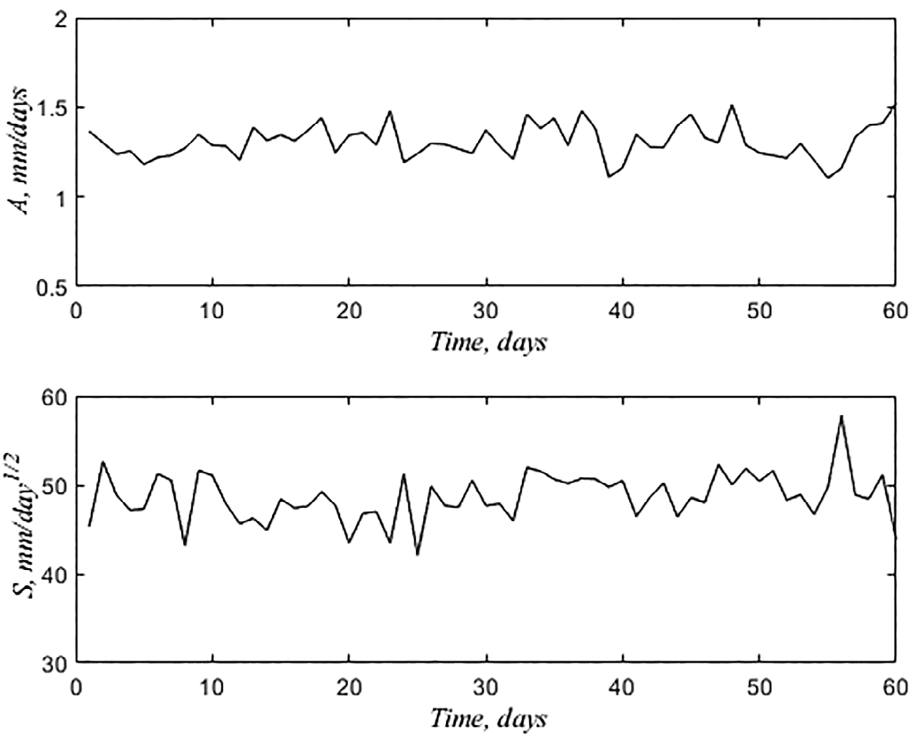


**FIGURE 3** The variability of the two random variables $A$ and $S$ for a single location.

previous examples are applied to a catchment scale. The catchment, where the data on infiltration were collected from 26 sites by Sharma et al. (1980), was a 9.6-ha small catchment in the Southern Great Plains near Chickasha, Oklahoma, USA. Three soil types were identified as Renfrow silt loam, Grant silt loam and Kingfisher silt loam.

Equation (50) was fitted to the measured data of Sharma et al. (1980) for infiltration at a catchment scale, which resulted in $A=0.0348$, $S=0.5869$, and $\beta=1.0$. In comparison with the results in the previous example with data of Talsma and van der Lelij (1976), the

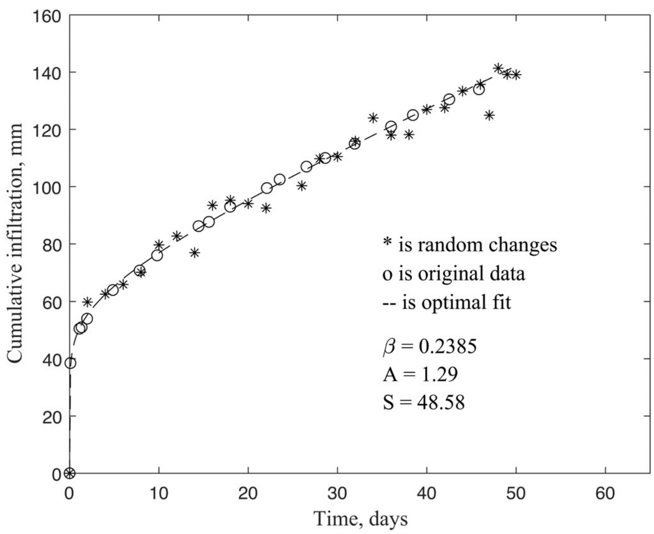


**FIGURE 4** Comparison of an optimal fit and the random simulation of cumulative infiltration data measured in the field (Talsma & van der Lelij, 1976).

value of $\beta$ is much larger because the soils in this catchment are silt loams rather than clay complex. Based on these derived values, a standard deviation of $\sigma=0.002$ cm/min for $A$ and $\sigma=0.025$ cm/min$^{1/2}$ for $S$ were used to generate Figure 5.

*Interpolation of the cumulative infiltration subject to random A and S*

The original data of Sharma et al. (1980) with the optimal fitting and the computed interpolation and extrapolation of cumulative infiltration in Equation (50) with random $A$ and $S$ generated in Figure 5 was used ito generate Figure 6.

It can be seen from Figures 4 and 6 that the interpolation at a single location and a catchment scale as well as the extrapolation in Figure 6 are very stable and consistent, which provide confidence for interpolation, extrapolation, and prediction if they are needed.

## 6 | CONCLUSIONS AND DISCUSSION

It is shown from the survey of the literature that it has been more than 100 years since 1905 when stochastic modelling of natural phenomena was regarded as a new scientific approach. Deviating from the early stage of stochastic PDE models reported since the 1960s till early 2000s for soil water movement, this paper demonstrates the application of random fractional PDEs (or rfPDEs). The key points are as follows based on the analysis with two types of boundary conditions for the rfPDE:

1. It is demonstrated here that the rfPDE approach with random parameters yields a realistic and excellent tool for the analysis of water movement in soils with two types of boundary conditions.

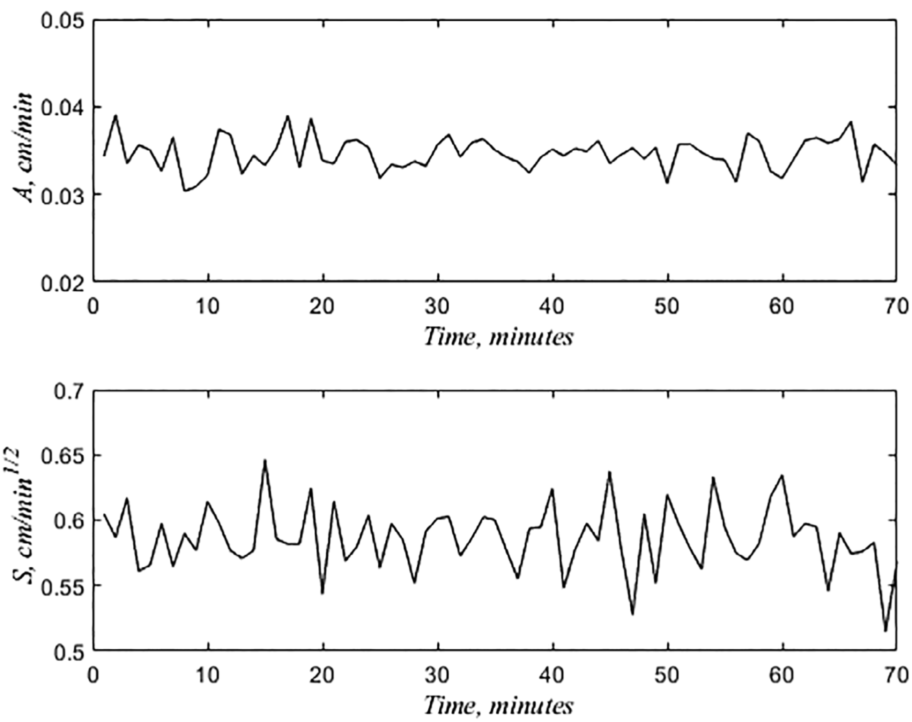


**FIGURE 5** The variability of the two random variables $A$ and $S$ for a catchment.

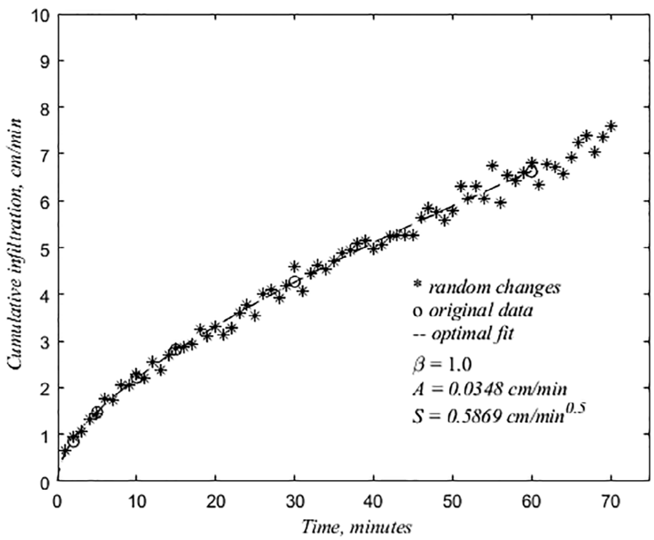


**FIGURE 6** Comparison of an optimal fit and the random simulation of cumulative infiltration data measured at 26 sites in a catchment (Sharma et al., 1980).

2. In the examples with the flux boundary condition, it is shown that the method presented here is stable and realistic for computing the fluxes through the soil, which are difficult to measure in practice even though their definitions are clear. The presented solutions as methods for determining the flux or either of the other two quantities are recommended for large time situation which is close, but not identical, to the steady-state scenarios. It is yet to determine the threshold when the large time situation is exact in the soil water content. For small time solutions, numerical methods must be used to approximately invert Equation (30) from the Laplace domain.
3. For infiltration subject to a concentration boundary condition demonstrated in Section 5, the final infiltration rate $A$, and the sorptivity, $S$, have been demonstrated as random quantities in computing cumulative infiltration. The effect of the order of fractional derivatives, $\beta$, is yet to be analysed once it is treated as a random quantity. With the same random quantities of $A$ and $S$, the infiltration rate is expected to follow a similar trend. In this paper, equations of infiltration with orders of temporal fractional derivatives, $\beta$, are analysed only. With $\eta \neq 1$ in Equations (5) and (8), the analyses and equations of infiltration presented earlier (Su, 2014) can also be used if the parameters in the fPDEs (Su, 2014) are defined as random ones.
4. Compared to the measured cumulative infiltration, the computed cumulative infiltration from the random variables $A$ and $S$ are stable and consistent. This fact is consistent for data from both single location and at a catchment scale and suggests that the equation of cumulative infiltration derived from the solution of the rfPDE is an excellent tool for interpolation, extrapolation and prediction.
5. The examples presented here are for vertical random flow only. For two-dimensional and three-dimensional random flows, the spatial variability of the random parameters in the fPDEs or similar models can be assessed using geostatistical methods such as those demonstrated by Matheron (1965, 1969, 1971) and Christakos (1992).
6. The simulated results for infiltration subject to a random flux in Figures 1 and 2 and for cumulative infiltration in Figures 4 and 6 are very stable and consistent. These findings are very encouraging and demonstrate the capacities of the methodologies presented in this paper for potential practical applications. Practitioners in hydrology are often faced with the reality that data from the field not only vary significantly in their measured values in a random way, they are also limited by the number of measurements in addition to limited time intervals or periods. With the methodologies presented in this paper, the stochastic variability of the values of a parameter in a hydrological model for water movement in soils can be simulated, and the interpolation, extrapolation and prediction of water movement in soils such as infiltration can be computed.

Overall, this approach based on the rfPDE has been demonstrated to be a very simple and stable method for generating realistic quantities for hydrological parameters when their measurement is difficult such as the flux through the surface of the soil. The methodologies presented here are innovative and were not reported in the literature in hydrological and soil sciences to date, and the approach is a step forward in the understanding of realistic hydrological processes in nature.

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## DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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## APPENDIX A

Riesz-Feller fractional derivatives (RFFD) and Riemann-Liouville fractional derivatives (RLFD): The RFFD becomes the Liouville fractional derivative for $\omega = \pm\gamma$ with the positive sign for the forward fractional derivative and the negative sign for the backward fractional derivative (Ortigueira and Trujillo, 2012, p. 5155–5156). Gorenflo et al. (2002) detailed the connections between the RFFD, ${}_xD_\theta^\beta$, and Riemann-Liouville fractional derivative (RLFD), ${}_xD_\pm^\beta$,

$$ {}_xD_\theta^\beta = -\left[c_+(\beta,\theta)\,{}_xD_+^\beta + c_-(\beta,\theta)\,{}_xD_-^\beta\right], \tag{A1} $$

where

$$ c_+(\beta,\theta) = \frac{1}{\sin(\beta\pi)}\sin\left[(\beta-\theta)\frac{\pi}{2}\right], \tag{A2} $$

$$ c_-(\beta,\theta) = \frac{1}{\sin(\beta\pi)}\sin\left[(\beta+\theta)\frac{\pi}{2}\right]. \tag{A3} $$

In the symmetrical case of $\omega = 0$, Gorenflo et al. (2002, Equation (A.8)) showed that the difference between the RFFD and the RLFD is a constant only,

$$ c_+(\beta,0) = c_-(\beta,0) = \frac{1}{\sin(\pi\beta/2)} \tag{A4} $$

then

$$ {}_xD_0^\beta = \frac{1}{\sin(\pi\beta/2)}\,{}_xD_\pm^\beta \tag{A5} $$

and

$$ {}_xI_0^\beta = \frac{1}{\sin(\pi\beta/2)}\,{}_xI_\pm^\beta. \tag{A6} $$

## APPENDIX B

Small-time solutions subject to a flux BC or the BC of the third kind as a random variable

$$ \widetilde{\vartheta} = 0, \quad t = 0, \quad z > 0, \tag{B1} $$

$$ K_0\widetilde{\vartheta} - D_0\frac{\partial\widetilde{\vartheta}}{\partial z} = \pm r(s), \quad t > 0, \quad z = 0, \tag{B2} $$

$$ \frac{\partial\widetilde{\vartheta}}{\partial z} = 0, \quad t > 0, \quad z \to \infty, \tag{B3} $$

where $s$ is the random Laplace transform variable and $\widetilde{\vartheta}$ is the random Laplace transform of $\vartheta$. The solution of the above problem in the random Laplace domain is similar to that for steady-state solute transport (van Genuchten & Alves, 1982, p. 57)

$$ \widetilde{\vartheta}(z,s) = \frac{r(s)}{K_0}\left(\frac{2K_0}{u+K_0}\right)\exp\left[\frac{(K_0-u)z}{2D_0}\right] \tag{B4} $$

with

$$ u = K_0\left(1 + \frac{4D_0 s^\beta}{K_0^2}\right)^{1/2} \tag{B5} $$

and $\widetilde{\vartheta} = \widetilde{\vartheta}_0$ for $z = 0$. For small time $t$ or large value of $s^\beta$, the term $\frac{4D_0 s^\beta}{K_0^2} \gg 1$ applies so that Equation (B5) can be approximated by

$$ u = 2D_0^{1/2}s^{\beta/2} \tag{B6} $$

then Equation (B4) can be approximated for small time by

$$ \widetilde{\vartheta}(z,s) = \frac{r(s)}{K_0}\exp\left(\frac{K_0 z}{2D_0}\right)\left(\frac{2K_0}{K_0 + 2D_0^{1/2}s^{\beta/2}}\right)\exp\left(-\frac{z}{D_0^{1/2}}s^{\beta/2}\right) \tag{B7} $$

or

$$ \widetilde{\vartheta}(z,s) = \frac{r(s)}{K_0}\exp\left(\frac{K_0 z}{2D_0}\right)\left(\frac{2K_0}{\left(2D_0^{1/2} + K_0 s^{-\beta/2}\right)}\right)s^{-\beta/2}\exp\left(-\frac{z}{D_0^{1/2}}s^{\beta/2}\right) \tag{B8} $$

For small time or large $s^{\beta/2}$, the term $K_0 s^{-\beta/2}$ is neglected compared to $2D_0^{1/2}$, Equation (B8) is cast into

$$ \widetilde{\vartheta}(z,s) = \frac{K_0 r(s)}{D_0^{1/2}}\exp\left(\frac{K_0 z}{2D_0}\right)s^{-\beta/2}\exp\left(-\frac{z}{D_0^{1/2}}s^{\beta/2}\right). \tag{B9} $$

The inverse Laplace transform of Equation (B9) is given as (Stanković, 1970; Gorenflo et al., 1999)

$$ \vartheta(z,t) = \frac{K_0 r(t)}{D_0^{1/2}}\exp\left(\frac{K_0 z}{2D_0}\right)t^{(\beta/2)-1}\phi\left(-\beta/2,\beta/2;-\frac{z}{D_0^{1/2}t^{\beta/2}}\right), \tag{B10} $$

where

$$ \phi\left(-\beta/2,\beta/2;-\frac{z}{D_0^{1/2}t^{\beta/2}}\right) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma[(1-k)\beta/2]}\left(-\frac{z}{D_0^{1/2}t^{\beta/2}}\right)^k \tag{B11} $$

is the Wright function (Gorenflo et al., 1999; Kilbas et al., 2006) with $\sum_{k=0}^{\infty}$ being the summation of terms $k = 0, 1, 2 + \ldots$.