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Some aspects of one-relator groups

by

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Contents

Introduction

For an introduction to the theory of one-relator groups see Baumslag 1964, and Magnus, Karrass, and Solitar 1966. There are three main themes in the work which follows.

The first theme is the determination of the Abelian subgroups of a one-relator group. This investigation was prompted by a conjecture of Baumslag 1964 that the additive group of rationals is not a subgroup of a one-relator group. The Abelian subgroups of one-relator groups have now been completely determined; they are free Abelian of rank ≤ 2 or the additive group of n-adic rationals, n a positive integer or finite cyclic groups.

Theorem (See Theorem 1.2.3) Let $G = gp(a, b, c, \ldots | R)$ be a torsion-free one-relator group. Then no non-trivial element has more than finitely many prime divisors. Moreover a non-trivial element is not divisible by more than finitely many powers of a prime p, if p is greater than the length of the relator.

Thus the additive group of rationals is not a subgroup of a torsion-free one-relator group, in fact;

Corollary (See Corollary 1.2.4) The additive group of rationals is not a subgroup of a one-relator group.

In the case of one-relator groups with torsion, one can say much more.

Theorem (See Theorem 2.3.2) The Abelian subgroups of a one-relator group with torsion are cyclic.

Corollary (See Corollary 2.3.3) The soluble subgroups of a one-relator group with torsion are cyclic.

Corollary (See Corollary 2.3.4) The centralizer of every non-trivial element of a one-relator group with torsion is cyclic.

The second theme is the problem of extending the Freiheitssatz. This theorem, proved by Magnus 1930, is the basic result in the theory of one-relator groups. Let

$$G = gp(a, b, c, \ldots \mid R(a, b, c, \ldots))$$

where R is cyclically reduced, and suppose the generator a occurs non-trivially in R. Then the Freiheitssatz states that b, c, ... freely generate a subgroup of G. But something more than this is true, and we seek to extend the Freiheitssatz by proving that, for some integer m

$$a^m, b, c, \ldots$$

freely generate a subgroup of G in certain rather general cases. The first result in this direction was obtained for certain two generator groups by Mendelsohn and Ree 1967. Here we prove the following.

Theorem (See Theorem 1.3.11) Let $G = gp(a, b, c, \ldots \mid R)$

where a, b, c occur non-trivially in R with $\sigma_a(R) = 0$. Then for one of the generators a, b, or c, (say b) there exists an integer m such that, for all integers $\alpha > m$

$$a,\ b^{\alpha},\ c,\ \ldots$$

freely generate a subgroup of G.

Again one can say more for groups with torsion. The basic result for such groups is the

Spelling Theorem. (See Theorem 2.1.1) Let $G = gp(a, b, \ldots \mid R^n)$ $n > 1$, where R is cyclically reduced. Suppose that two words $W(a, b, \ldots)$, $V(b, \ldots)$, where W is a freely reduced word containing a non-trivially and V does not contain a, define the same element of G. Then W contains a subword which is identical with a subword of $R^{\pm n}$ of length greater than $(n - 1)/n$ times the length of R^n.

From this theorem one can prove the following extension of the Freiheitssatz.

Corollary (See Corollary 2.1.6) Let $G = gp(a, b, c, \ldots \mid R^n)$ $n > 1$, where R is cyclically reduced involving a, b non-trivially, and suppose β is any integer which does not divide the a-exponents in R^n. Then a^{β}, b, c, ... freely generate a subgroup of G.

The third theme is concerned with algorithmic problems in one-relator groups; more specifically we are primarily concerned with the word problem and the conjugacy problem in one-relator groups with torsion. These algorithmic problems, proposed by Dehn 1911 are fundamental problems in the presentation theory of groups. In general they are unsolvable. For free groups both the word problem and the conjugacy problem are solvable. In 1932 Magnus used an ingenious application of the Freiheitssatz to prove that one-relator groups have a solvable word problem. However Magnus, Karrass, and Solitar 1966 pointed out that there are some unsatisfactory aspects of the solution, for the algorithm appears to be unnecessarily complicated. In the case of less than one-sixth groups investigated by Greendlinger 1960 there is a simpler algorithm. Using the Spelling Theorem a trivial proof of the solvability of the Word problem for one-relator groups with torsion can be given, (See Corollary 2.14) and the algorithm which emerges is of the required degree of simplicity, and provides a bridge between the work of Magnus and that of Tartakovskii 1949 and Greendlinger 1960. Another similar problem related to a problem of Lyndon 1962 is the following:

<u>Corollary</u> (See Corollary 2.1.5) Let $G = gp(a, b, c, \ldots \mid R^n)$ $n > 1$ and let $\underset{\approx}{W}$, $\underset{\approx}{Z}$ be subsets of the generators. Then there is an algorithm to determine for an

arbitrary element $g \in G$ if $g = w(\underset{\approx}{W})z(\underset{\approx}{Z})$ for some words w, z.

As for the conjugacy problem very little has been done. Greendlinger 1960b, 1964 has proved the solvability of the conjugacy problem for less than one-sixth groups, and Soldatova 1967 has extended the result to certain less than one-fourth groups. The conjugacy problem for the free product of two free groups with a cyclic subgroup amalgamated has been solved by Lipschutz (unpublished). In this work we show that all one-relator groups with torsion have a solvable conjugacy problem.

<u>Theorem</u> (See Theorem 3.2.3) Let G be a one-relator group with torsion. Then the conjugacy problem and the extended conjugacy problem relative to the subgroup generated by any subset of the generators are solvable in G.

The problem of finding an algorithm to determine whether or not an arbitrary element of a group is a power has been investigated by Reinhart 1962 and by Lipschutz 1965 and 1968. We prove the following result.

<u>Theorem</u> (See Theorem 3.3.1) Let $G = gp(a, b, c, \ldots \mid R^n)$ $n > 1$. Given $g \in G$ there is an algorithm to determine the roots of g.

There are a few miscellaneous results which emerge in the work. The first concerns the Frattini subgroup.

Theorem (See Theorem 1.3.12) A one-relator group has trivial Frattini subgroup if

(a) it is torsion-free with more than two generators

or (b) it has torsion with more than one generator.

The next result concerns a residual property.

Corollary (See Corollary 2.1.7) Let $G = gp(a, b, c, \ldots \mid R^n)$ $n > 1$. Then G is residually a two-generator one-relator group with torsion.

The following theorem of Baumslag and Steinberg 1964 may be proved quite easily.

Theorem (See Theorem 1.3.7) Let $w(x_1, x_2, \ldots, x_n)$ be an element of a free group F freely generated by $x_1, x_2, \ldots, x_n, x$ which is neither a proper power nor a primitive. If $g_1, g_2, \ldots, g_n, g$, generate a free group G and are connected by the relation

$$w(g_1, g_2, \ldots, g_n) = g^m \qquad m > 1$$

then the rank of G is at most $n - 1$.

There is a common strata to these three main themes, namely the technique for proving them. The technique is as follows. In a one-relator group with more than one generator occuring non-trivially in the cyclically reduced relator one can, without too much disruption of the group, arrange for the exponent sum on one of the generators to be zero. Let G be such a one-relator group, and let N be the normal subgroup of G generated by the remaining generators. This is usually a complicated group, infinitely generated and infinitely related. But it has one nice property; it is the direct limit of a chain of subgroups of N,

$$N_0,\ gp(N_0,N_1),\ gp(N_{-1},N_0,N_1),\ gp(N_{-1},N_0,N_1,N_2),\ \ldots$$

where the N_i are isomorphic one-relator groups, with the length of the relator less than that of the original group G. We thus have a basis for an induction argument to prove the one-relator group G has some specified group theoritic property $\underline{\underline{P}}$. Thus the induction hypothesis would be that all one-relator groups with relator length less than the length of the relator of G have the property $\underline{\underline{P}}$. Now the normal subgroup N is well situated in G, for it has infinite cyclic factor group. For the properties of interest here, we can show that in order to prove G has the property $\underline{\underline{P}}$ it will suffice to prove that N has the property $\underline{\underline{P}}$. We now use the nice structure of N. By the induction hypothesis N_0, and in fact each N_i, has the property $\underline{\underline{P}}$. Thus the first term in the chain above has property $\underline{\underline{P}}$.

Does the second term, $gp(N_0, N_1)$ have property $\underline{\underline{P}}$? Well the $gp(N_0, N_1)$ is a generalized free product of N_0 and N_1 amalgamating a subgroup generated by a common subset of the generators of N_0 and N_1, so the basic problem is this; when does a generalized free product of two groups each having the property **P**, have property **P**. For example it is known (Neumann 1954) that the generalized free product of torsion-free groups is torsion-free. In order for some property **P** of the factors to be inherited by a free product with amalgamation it is usually necessary to put conditions on the amalgamated subgroup. For example the generalized free product of residually finite groups is residually finite if the amalgamated subgroup is finite. Thus we will have to find for each property **P** (and there will be a different one in each chapter) an appropriate type of subgroup, call it a **q**-subgroup, such that the following proposition holds.

Proposition 1 If $C = \{A * B \; ; \; J\}$ is the generalized free product of the factors A and B amalgamating the subgroup J, and A and B have the property **P**, and J is a **q**-subgroup of A and B, then C has the property **P**.

For the groups with which we are concerned we know from the Freiheitssatz that the amalgamated subgroup is free. But freeness is not usually sufficient. In our case we will need the following proposition:

Proposition 2 Any subset of the generators of a one-relator group G generates a **q**-subgroup of G.

The Propositions 1 and 2 will then allow us to take one step up the chain and prove $gp(N_0, N_1)$ has the property $\underline{\underline{P}}$.

Now the third term of the chain, $gp(N_{-1}, N_0, N_1)$ is again a generalized free product, of the two factors $gp(N_0, N_1)$ and N_{-1} amalgamating a subgroup J_{-1} generated by a common subset of the generators of N_0 and N_{-1}. From the two preceeding paragraphs we know that both these factors have the property $\underline{\underline{P}}$. All we need in view of Proposition 1 is for the amalgamated subgroup to be a $\underline{\underline{q}}$-subgroup of both factors. From Proposition 2, J_{-1} is a $\underline{\underline{q}}$-subgroup of N_{-1}. In order for the amalgamated subgroup to be a $\underline{\underline{q}}$-subgroup of the first factor it suffices to have the following result.

<u>Proposition 3</u> A $\underline{\underline{q}}$-subgroup of a $\underline{\underline{q}}$-subgroup is a $\underline{\underline{q}}$-subgroup, and if $C = \{A * B ; J\}$ where the amalgamated subgroup J is a $\underline{\underline{q}}$-subgroup of the factors A and B, then the factors A and B are $\underline{\underline{q}}$-subgroups of C.

With this result we can proceed to the third term in the chain. For N_0 is by Proposition 3 a $\underline{\underline{q}}$-subgroup of $gp(N_0, N_1)$ and J_{-1} is by Proposition 2 a $\underline{\underline{q}}$-subgroup of N_0, hence J_{-1} is a $\underline{\underline{q}}$-subgroup of $gp(N_0, N_1)$. One continues stepping up the chain by repeating these arguments, for each successive term is formed by a similar generalized free product construction. Hopefully the direct limit of the chain of groups with property $\underline{\underline{P}}$ will also have property $\underline{\underline{P}}$. This then would prove that N has the property $\underline{\underline{P}}$.

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Notation

For the readers convenience we list some of the notations used.

$gp(a,b,c,\ldots \mid \ldots,R_i,\ldots)$	the group generated by $a,b,c,\ldots$ with defining relators $\ldots,R_i,\ldots$.
$gp(X)$	the subgroup generated by the set X.
$gp(A,B,\ldots)$	the subgroup generated by the subgroups $A,B,\ldots$.
$\{A * B\}$	the free product of A and B.
$\{A * B \,;\, J\}$	the generalized free product of A and B amalgamating a subgroup J.
$\{A * B \,;\, x = y\}$	the generalized free product of A and B with $x \in A$, $y \in B$ identified.
$\lambda(g)$	the length of g as a word in a free group.
$\lvert g\rvert$	the length of g as a word in normal form in a generalized free product.
$\sigma_a(R)$	the exponent sum of the generator a in the word R.

$\Phi(G)$	the Frattini subgroup of G.
$w(a,b,c,\ldots)$	a word in the letters a,b,c, ... Often this is abbreviated to w, and thus is identified with an element w in a group. We have tried to be as relaxed as possible with regard to this notation, and will use words and elements interchangeably. Thus the equation $$w_1(a,b,c) = w_2(x,y,z)$$ will denote that $w_1(a,b,c)$ and $w_2(x,y,z)$ interpreted as elements of the group in question, are equal.
£	the lifting transformation.
$[G, G]$	the commutator subgroup of G.
$\delta^i(G)$	the i-th term of the soluble series of G. $\delta^0(G) = G$, $\delta^1(G) = [G, G]$.

Chapter 1

Torsion-free one-relator groups

The problem we are concerned with in this chapter is to determine what Abelian groups occur as subgroups of torsion-free one-relator groups. This problem has been solved: the Abelian subgroups are free-Abelian of rank ≤ 2, or those subgroups of the additive group of rationals which are divisible by only finitely many primes.

However, given a particular one-relator group one cannot say precisely what Abelian subgroups will occur! What one would like is an algorithm to determine what Abelian subgroups occur in a particular group presented as a one-relator group. It is conjectured that for torsion-free groups an Abelian subgroup which is not free Abelian will occur as a subgroup if and only if, to within cyclic permutations, the relator is a word of the form $S^{-1}T^{\alpha}ST$ for some words S, T which do not commute as elements of a free group, and some integer α, $|\alpha| > 1$. It has been shown that there is an algorithm to determine if a word R is of this form.

As has been remarked in the introduction, the basic tool for proving our results is the generalized free product, and the appropriate condition to place on the amalgamated subgroup will now be studied.

Section 1.1 The theory of p-pure subgroups.

A positive integer $n > 1$ is a divisor of an element g in a group G if there exists a root $x \in G$ such that $x^n = g$. If the element g has a divisor n, then g is said to be divisible by n. Let H be a subgroup of G. Then H is p-pure in G if for all $g \in G$ and integers r such that $g^{p^r} \in H$ there exists an element $h \in H$ such that $h^{p^r} = g^{p^r}$. Let π be a set of prime numbers. Then H is π-pure in G if H is p-pure in G for all $p \in \pi$.

Lemma 1.1.1 A p-pure subgroup of a p-pure subgroup of G is a p-pure subgroup of G.

Proof Let K be a p-pure subgroup of H and H a p-pure subgroup of G. Suppose $g^{p^r} \in K$. Then $g^{p^r} \in H$ and since H is p-pure in G there exists an element $h \in H$ such that $g^{p^r} = h^{p^r}$. Hence $h^{p^r} \in K$ and since K is p-pure in H there exists an element $k \in K$ such that $k^{p^r} = h^{p^r}$. Hence K is a p-pure subgroup of G.

Lemma 1.1.2 Let C be a generalized free product $C = \{A * B ; K\}$ where the amalgamated subgroup K is a p-pure subgroup of A and B. Then A and B are p-pure subgroups of C.

Proof From the symmetry between A and B in C it will suffice to prove that A is a p-pure subgroup of C. Let $a \in A$ and suppose for some $g \in C$,

$$g^{p^r} = a.$$

It must be shown that there exists an element of A whose p^r-th power is a. Let $|g|$ denote the length of g when g is written in normal form.

If $|g|$ is even then g is cyclically reduced so $|g^{p^r}| = p^r|g|$. But $|a| \leq 1$ hence $|g| = 0$. This implies $g \in A$ whence g is the required p^r-th root.

If $|g| = 1$ then $g \in A$ or $g \in B$. If $g \in A$ there is nothing to prove, so suppose $g \in B$. Then $g^{p^r} \in A$ and $g^{p^r} \in B$ whence $g^{p^r} \in K$. Since K is a p-pure subgroup of B there exists an element $k \in K$ such that $k^{p^r} = g^{p^r} = a$, whence a has the required p^r-th root in A.

If $|g|$ is odd and $|g| \geq 3$ one proceeds by induction. Suppose inductively that the result has been established for all elements with length $< n$ and let $|g| = n$ where n is odd.

It is possible to choose an element s, either in A or B such that

$$g = s^{-1}\underset{\sim}{g}s \qquad\qquad |\underset{\sim}{g}| < |g|.$$

If $s \in A$ then

$$s^{-1}\underset{\sim}{g}^{p^r}s = a$$

or

$$\underset{\sim}{g}^{p^r} = sas^{-1} \in A.$$

By the induction hypothesis there exists an element $a_1 \in A$ such that

$$a_1{}^{p^r} = sas^{-1}$$

hence

$$(s^{-1}a_1 s)^{p^r} = s^{-1}a_1{}^{p^r}s = a.$$

Thus $s^{-1}a_1 s$ is the required p^r-th root of a in A.

If $s \in B$, without loss of generality assume $\underset{\sim}{g}$ when written in normal form begins with an element of A. Then $\underset{\sim}{g}^{p^r}$ when written in normal form begins with an element of A. But

$$s^{-1}\underset{\sim}{g}^{p^r}s = a$$

implies

$$\underset{\sim}{g}^{p^r} = sas^{-1} \in A \cap B.$$

By the induction hypothesis there exists an element $b \in B$ such that $b^{p^r} = \underset{\sim}{g}^{p^r}$. Now

$$(s^{-1}bs)^{p^r} = s^{-1}b^{p^r}s = s^{-1}\underset{\sim}{g}^{p^r}s = a$$

where $|s^{-1}bs| \leq 1$. Hence from the above there exists in A the required p^r-th root of a.

This completes the proof of Lemma 1.1.2.

If A_1 is a p-pure subgroup of A then A_1 is a p-pure subgroup of C, using the notation of the previous Lemma. In particular K is a p-pure subgroup of C. Note also that free factors of a group are p-pure subgroups where p is any prime.

<u>Lemma 1.1.3</u> Let $C = \{A * B \ ; \ J\}$ where J is p-pure in A and B. If no non-trivial element of the factors A and B is divisible by all powers of the prime p then no non-trivial element of C is divisible by all powers of p.

<u>Proof</u> Let v be any non-trivial element of C. If v is divisible by only finitely many powers of p then any conjugate of v will be divisible by only finitely many powers of p. It suffices therefore to assume v is cyclically reduced.

If $|v| > 1$ where $|v|$ denotes the length of v in normal form, then any root of v must be cyclically reduced with length > 1. Hence v has only a finite number of divisors since each divisor must divide $|v|$.

If $|v| \leq 1$ assume without loss of generality that $v \in A$. From Lemma 2.1.2, A is a p-pure subgroup of C, and as there are only finitely many powers of p dividing v in A, there can be only finitely many powers of p dividing v in C.

<u>Lemma 1.1.4</u> Let $C = \{A * B \ ; \ J\}$ where J is π-pure in A and B and π is the set of all but a finite number of primes. If A and B are groups in which every non-trivial element is divisible by at most a finite number of primes, then every non-trivial element in C is divisible by at most a finite number of primes.

Proof The proof is similar to the proof of Lemma 1.1.3 and so is omitted.

Section 1.2 The Abelian subgroups of torsion-free one-relator groups.

We are now ready to apply the theory developed in the previous section to one-relator groups. It will be shown that any subset of the generators of a one-relator group generates a p-pure subgroup where p is any prime greater than the length of the relator. First we need a lemma to simplify the problem.

Lemma 1.2.1 In order to prove that any subset of the generators of a one-relator group generates a p-pure subgroup where p is any prime greater than the length of the relator it suffices to prove that in all groups

$$G = gp(a, b, \ldots, t \mid R)$$

where R is a cyclically reduced word involving $a, b, \ldots, t$ non-trivially, the $gp(b, \ldots, t)$ is p-pure in G where p is any prime greater than $\lambda(R)$.

Proof Let $H = gp(x_1, x_2, \ldots \mid R)$ be any one-relator group. Without loss of generality one may assume R is cyclically reduced. Let $\{y_1, y_2, \ldots\}$ be any subset of the generators of H, and put

$$Y = gp(y_1, y_2, \ldots)$$

If $\{y_1, y_2, \ldots\}$ is not a proper subset of $\{x_1, x_2, \ldots\}$

then $Y = H$ and so Y is certainly a p-pure subgroup of H. If the set is empty there is nothing to prove. Let $\{y_1, y_2, \ldots\}$ be a proper non-empty subset of $\{x_1, x_2, \ldots\}$.

Firstly suppose that every generator in R is in the set $\{y_1, y_2, \ldots\}$. Then the generators of H may be split into two disjoint subsets

$$\{y_1, y_2, \ldots\}, \quad \{z_1, z_2, \ldots\}$$

where no z-generator appears in R. These two subsets generate free factors Y, Z respectively such that

$$H = Y * Z.$$

Hence Y is a p-pure subgroup of H since it is a free factor of H.

Secondly suppose that there exists a generator say x_i which is not in $\{y_1, y_2, \ldots\}$ but appears in R. Let $X = gp(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots)$. It is clear that $Y \subset X$; in fact, by the Freiheitssatz, Y is a free factor of X. It will suffice therefore to prove that X is a p-pure subgroup of H.
Let

$$G = gp(a, b, \ldots, t \mid R)$$

be obtained from H by deleting those generators which do not occur in R, and rewriting x_i as a and the remaining generators appearing in R as $b, \ldots, t$ in some order. Then

$$H = \{G * X \,;\, gp(b, c, \ldots, t)\}.$$

In order to prove that X is a p-pure subgroup of H it will suffice to prove that the amalgamated subgroup is a p-pure subgroup of both G and X. But it is a p-pure subgroup of X since it is a free factor of X. Hence the problem is reduced to proving $gp(b, c, \ldots, t)$ is a p-pure subgroup of G where the relator R of G is cyclically reduced and involves a, b, ..., t non-trivially.

<u>Lemma 1.2.2</u> Let $G = gp(a, b, \ldots \mid R)$ be a one-relator group with R cyclically reduced. Then any subset of the generators of G generates a p-pure subgroup of G if $p > \lambda(R)$.

<u>Proof</u> The lemma will be proved by induction on $\lambda(R)$. If R involves no more than one generator the lemma is trivially true. Suppose the lemma is true for all groups with relator length $< \lambda(R)$. By the previous lemma it will suffice to prove that

$$H = gp(b, c, \ldots, t)$$

is a p-pure subgroup of

$$G = gp(a, b, c, \ldots, t \mid R)$$

where $p > \lambda(R)$, and all the generators a, b, c, ..., t appear non-trivially in R. For simplicity of notation, assume the generators of G are a, b, c, t.

Suppose in G there exists an element $g(a, b, c, t)$

and an element $w(b, c, t) \in gp(b, c, t)$ such that for some integer r,

$$g^{p^r} = w. \qquad (1)$$

It will be shown that w has a p^r-th root in $gp(b, c, t)$. One now proceeds as in the proof of the Freiheitssatz. The proof is divided into various cases depending on $\sigma_a(R)$, the exponent sum of a in R.

<u>Case 1</u> Let $\sigma_a(R) = 0$. Put $N = gp_G(b, c, t)$. Then an element of G belongs to N if and only if its exponent sum on a is zero. Since $w \in N$ then $\sigma_a(g) = 0$. Thus to prove the lemma in this case it will suffice to prove that H is a p-pure subgroup of N. To obtain a presentation of N one uses a Reidemeister-Schreier rewriting process. Let

$$b_k = a^{-k}ba^k, \quad c_k = a^{-k}ca^k, \quad t_k = a^{-k}ta^k$$

where k ranges over all integers. Then rewriting $a^{-k}Ra^k$ one has a new word R_k where $\lambda(R_k) < \lambda(R)$. Also rewriting w and g one obtains $w(b_0, c_0, t_0)$ and $\underset{\sim}{g}(b_i, c_i, t_i)$ where i ranges over the integers. The $\underset{\sim}{g}$-symbol indicates that $g(b, c, t)$ when rewritten changes to a different word entirely while w when rewritten has precisely the same word form except that the letters are subscripted by zero.

It must now be shown that $w(b_0, c_0, t_0)$ has a p^r-th root in $H = gp(b_0, c_0, t_0)$. A presentation for N is

$$N = gp(b_j,\ c_j,\ t_j \mid R_j \text{ (all integers } j)).$$

As usual N is constructed from "smaller" one-relator groups using a generalized free product construction. For each integer i define

$$N_i = gp(b_i,\ \ldots,\ b_{\mu+i},\ c_j,\ t_j \text{ (all integers } j) \mid R_i)$$

where, without loss of generality zero and μ are taken to be the smallest and largest b-subscripts respectively in R_0. Then using the Freiheitssatz N may be constructed as

$$N = \bigcup_{k=0}^{\infty} K_k, \quad \text{where}$$

$K_0 = N_0$

$K_1 = \{K_0 * N_1 \ ; \ J_1\}$

$K_2 = \{K_1 * N_{-1} \ ; \ J_{-1}\}$

and in general

$$K_{2n} = \{K_{2n-1} * N_{-n} \ ; \ J_{-n}\} \qquad n \neq 0$$

$$K_{2n+1} = \{K_{2n} * N_{n+1} \ ; \ J_{n+1}\}$$

where if $i > 0$

$$J_i = gp(b_i,\ \ldots,\ b_{\mu+i-1},\ c_j,\ t_j \text{ (all integers } j))$$

and if $i < 0$

$$J_i = gp(b_{i+1},\ \ldots,\ b_{\mu+i},\ c_j,\ t_j \text{ (all integers } j)).$$

One is able to exploit the induction hypothesis because the building blocks N_i which go into the construction of N are one-relator groups with a relator of shorter length than R. Hence p, being a prime $> \lambda(R)$, will be $> \lambda(R_i)$.

<u>Remark</u> We will frequently be using the process of constructing N from one-relator group as outlined above, but with minor variations. For this reason it is convenient to abbreviate the description in the following way. We will state what N and N_0 are; thus

$$N = gp_G(b, c, t)$$

and

$$N_0 = gp(b_0, \ldots, b_\mu, c_i, t_i \text{ (all integers i) } | R_0).$$

We will always use $0, \mu$ as the smallest and largest subscripts on the generator in R_0 that is singled out as above. It is understood that K_i, N_i, J_i are obtained in a similar manner to those above.

To show H is a p-pure subgroup of N it will suffice to show H is a p-pure subgroup of K_i for all positive integers i. Here it is convenient to introduce another induction argument. Inductively suppose that

$H, N_{-m/2}, N_{m/2}$ are p-pure subgroups of K_m (m even)

and

$H, N_{-(m+1)/2}, N_{(m+1)/2}$ are p-pure subgroups of K_m (m odd).

Suppose m is even.

Then $K_{m+1} = \{K_m * N_{(m+2)/2} ; J_{(m+2)/2}\}$ and $J_{(m+2)/2}$ is a p-pure subgroup of $N_{m/2}$ and $N_{(m+2)/2}$. Since $N_{m/2}$ is a p-pure subgroup of K_n by the present induction hypothesis then $J_{(m+2)/2}$ is a p-pure subgroup of K_m and $N_{(m+2)/2}$. Hence K_m and $N_{(m+2)/2}$ are p-pure subgroups of K_{m+1}. Since by the present induction hypothesis H is a p-pure subgroup of K_m, it follows that H is a p-pure subgroup of K_{m+1}, and $N_{(m+2)/2}$ is a p-pure subgroup of K_{m+1}. Since $N_{-m/2}$ is a p-pure subgroup of K_m then $N_{-m/2}$ is a p-pure subgroup of K_{m+1}.

Suppose m is odd.

Then $K_{m+1} = \{K_m * N_{-(m+1)/2} ; J_{-(m+1)/2}\}$ and $J_{-(m+1)/2}$ is a p-pure subgroup of $N_{-(m+1)/2}$ and $N_{-(m-1)/2}$. Since $N_{-(m+1)/2}$ is a p-pure subgroup of K_n by the present induction hypothesis, then $J_{-(m+1)/2}$ is a p-pure subgroup of K_m and $N_{-(m+1)/2}$. Hence K_m and $N_{-(m+1)/2}$ are p-pure subgroups of K_{m+1}. Since by the present induction hypothesis H is a p-pure subgroup of K_m, it follows that H is a p-pure subgroup of K_{m+1}, and $N_{-(m+1)/2}$ is a p-pure subgroup of K_{m+1}. Since $N_{(m+1)/2}$ is a p-pure subgroup of K_m then $N_{(m+1)/2}$ is a p-pure subgroup of K_{m+1}.

Putting these together one has

H, $N_{-(m+1)/2}$, $N_{(m+1)/2}$ are p-pure subgroups of $K_{(m+1)}$ if m+1 is even, and

H, $N_{-(\overline{m+1}-1)/2}$, $N_{(\overline{m+1}+1)/2}$ are p-pure subgroups of K_{m+1} if m+1 is odd. It is easy to verify the hypothesis for m = 0, 1, 2. This completes the proof of the statement that H is a p-pure subgroup of K_k for all positive integers k. Hence H is a p-pure subgroup of N. Thus there exists in H an element $h(b_0, c_0, t_0)$ with

$$h^{p^r} = w.$$

Thus h(b, c, t) is the required p^r-th root of w in H.

Case 2 Suppose G has two generators a, b and $\sigma_a(R) \neq 0$. Then equation (1) is

$$g^{p^r}(a, b) = b^s. \qquad (2)$$

This relation in G implies a free equality

$$g^{p^r} b^{-s} = \prod_i S_i R^{\epsilon_i} S_i^{-1} \qquad \epsilon_i = \pm 1$$

where S_i are elements of the free group generated by a, b. Hence considering exponent sums on both sides one has

$$p^r \sigma_a(g) = (\Sigma \epsilon_i) \sigma_a(R),$$

and

$$p^r \sigma_b(g) - s = (\Sigma \epsilon_i) \sigma_b(R).$$

If $\Sigma \epsilon_i = 0$ then p^r divides s and so b^{s/p^r} is the required p^r-th root.

If $\Sigma \epsilon_i \neq 0$ then $p^r \sigma_a(g) \sigma_b(R) = \sigma_a(R)(p^r \sigma_b(g) - s)$. Since $p > \sigma_a(R)$ one has p^r divides $p^r \sigma_b(g) - s$ whence p^r divides s, so b^{s/p^r} is the required p^r-th root in gp(b).

Case 3 Suppose $\sigma_a(R) \neq 0$ and $\sigma_b(R) = \sigma_b(g) = 0$. Here one takes

$$N = gp_G(a,\ c,\ t)$$

and as before, constructs N from

$$N_0 = gp(a_0,\ \ldots,\ a_\mu,\ c_i,\ t_i \text{ (all integers i)} \mid R_0).$$

Using the same argument as in Case 1 one may prove $gp(c_i,\ t_i$ (all integers i)) is a p-pure subgroup of N. The equation (1) when rewritten in N will be

$$\underset{\sim}{g}^{p^r}(a_i,\ c_i,\ t_i) = \underset{\sim}{w}(c_i,\ t_i)$$

where i ranges over the integers, hence $\underset{\sim}{w}$ has a p^r-th root in $gp(c_i,\ t_i)$. This implies w has a p^r-th root in gp(b, c, t) as was required.

<u>Case 4</u> Suppose G has more than two generators, $\sigma_a(R) \neq 0$ and no generator has exponent sum zero on both g and R. Let

$$\sigma_a(R) = \alpha_1, \qquad \sigma_b(R) = \beta_1, \qquad \sigma_c(R) = \gamma_1,$$
$$\sigma_a(g) = \alpha_2, \qquad \sigma_b(g) = \beta_2, \qquad \sigma_c(g) = \gamma_2.$$

<u>Subcase 4.1</u> Suppose $\alpha_2\gamma_1 - \alpha_1\gamma_2 \neq 0$. Here one proceeds by embedding G in a larger group $\underset{\sim}{G}$ defined by

$$\underset{\sim}{G} = gp(a,\ \underset{\sim}{b},\ c,\ t \mid \underset{\sim}{R})$$

where

$$\underset{\sim}{R} = R(a\ \underset{\sim}{b}^{\beta_1\gamma_2-\beta_2\gamma_1},\ \underset{\sim}{b}^{\alpha_2\gamma_1-\alpha_1\gamma_2},\ c\ \underset{\sim}{b}^{\alpha_1\beta_2-\alpha_2\beta_1},\ t)$$

where G maps into $\underset{\sim}{G}$ by the natural extension of

$$a \to a\ \underset{\sim}{b}^{\beta_1\gamma_2-\beta_2\gamma_1}$$
$$b \to \underset{\sim}{b}^{\alpha_2\gamma_1-\alpha_1\gamma_2}$$
$$c \to c\ \underset{\sim}{b}^{\alpha_1\beta_2-\alpha_2\beta_1}$$
$$t \to t.$$

Note that $\sigma_{\underset{\sim}{b}}(\underset{\sim}{R}) = 0$. Under this mapping the relation (1) becomes

$$\underset{\sim}{g}^{p^r}(a, \underset{\sim}{b}, c, t) = \underset{\sim}{w}(b, c, t)$$

with $\sigma_{\underset{\sim}{b}}(\underset{\sim}{g}) = 0$. One now proves, as in Case 3 that $\underset{\sim}{w}$ has a p^r-th root $\underset{\sim}{h}$ in $gp(\underset{\sim}{b}, c, t)$.

Now

$$\underset{\sim}{G} = \{G * gp(\underset{\sim}{b}) \; ; \; b = \underset{\sim}{b}^{\alpha_2\gamma_1 - \alpha_1\gamma_2}\}$$

and $\underset{\sim}{w}$, $\underset{\sim}{h}$ are not conjugates of a power of $\underset{\sim}{b}$ for $\underset{\sim}{w}$, $\underset{\sim}{h}$ have $\sigma_{\underset{\sim}{b}}(\underset{\sim}{w}) = \sigma_{\underset{\sim}{b}}(\underset{\sim}{h}) = 0$. But $\underset{\sim}{w}$, $\underset{\sim}{h}$ commute so they belong to the same factor, namely G. Hence

$$\underset{\sim}{h} \in G \cap gp(\underset{\sim}{b}, c, t),$$

that is

$$\underset{\sim}{h} \in gp(b, c, t).$$

Thus w has the required p^r-th root in $gp(b, c, t)$.

<u>Subcase 4.2</u> Suppose $\alpha_2\gamma_1 - \alpha_1\gamma_2 = 0$. Since $\alpha_1 \neq 0$, if $\gamma_1 = 0$ then $\gamma_2 = 0$, contradicting the fact that no generator has exponent sum zero on both g and R. Thus one may assume $\alpha_1 \neq 0$, $\gamma_1 \neq 0$. In this case one embeds G in a new group $\underset{\sim}{G}$ defined by

$$\underset{\sim}{G} = gp(a, b, \underset{\sim}{c}, t \mid \underset{\sim}{R})$$

where

$$\underset{\sim}{R} = R(a\,\underset{\sim}{c}^{-\gamma_1}, b, \underset{\sim}{c}^{\alpha_1}, t)$$

under the natural extension of the mapping

$$a \to a\,\underset{\sim}{c}^{-\gamma_1}$$

$$b \to b$$

$$c \to \underset{\sim}{c}^{\alpha_1}$$

$$t \to t.$$

Now $\sigma_{\underset{\sim}{c}}(\underset{\sim}{R}) = 0$. The relation (1) becomes in $\underset{\sim}{G}$,

$$\underset{\sim}{g}^{p^r}(a,\ b,\ \underset{\sim}{c},\ t) = \underset{\sim}{w}(b,\ \underset{\sim}{c},\ t)$$

and $\sigma_{\underset{\sim}{c}}(\underset{\sim}{g}) = 0$. Again one may proceed as in Case 3 to show that $\underset{\sim}{w}$ has a p^r-th root $\underset{\sim}{h}$ in $gp(b,\ \underset{\sim}{c},\ t)$. But since neither $\underset{\sim}{w}$ nor $\underset{\sim}{h}$ are conjugate to a power of $\underset{\sim}{c}$, it follows that $\underset{\sim}{w}$, $\underset{\sim}{h}$ belong to the same factor G where

$$\underset{\sim}{G} = \{G * gp(\underset{\sim}{c})\ ;\ c = \underset{\sim}{c}^{\alpha_1}\}.$$

Thus $\underset{\sim}{h} \in G \cap gp(b,\ \underset{\sim}{c},\ t)$, that is

$$\underset{\sim}{h} \in gp(b,\ c,\ t).$$

Thus in every case w has the required p^r-th root in $gp(b,\ c,\ t)$. This completes the proof of the Lemma.

<u>Theorem 1.2.3</u> Let $G = gp(a,\ b,\ c,\ \ldots \mid R)$ be a torsion-free one-relator group. Then no non-trivial element has more than finitely many prime divisors. Moreover a non-trivial element is not divisible by more than finitely many powers of a prime p, if p is greater than the length of the relator.

<u>Proof</u> The theorem will be proved by induction on the length of the relator of G. If the length of the relator is 0 or 1 then the theorem is trivially true. Using the usual embedding process one may, without loss of generality, assume $\sigma_a(R) = 0$, and a, b occur non-trivially in R where R is cyclically reduced.

Let $N = gp_G(b, c, \ldots)$, and construct N in the usual way from

$$N_0 = gp(b_0, \ldots, b_\mu, c_i, \ldots \text{ (all integers i) } \mid R_0).$$

Now $\lambda(R_0) < \lambda(R)$ so by induction no non-trivial element of N_0 has more than finitely many prime divisors. Moreover a non-trivial element of N_0 is not divisible by more than finitely many powers of a prime p if p is greater than the length of the relator R_0. Since G/N is infinite cyclic it will suffice to prove the theorem for N.

Inductively suppose it has been shown that every non-trivial element of K_{k-1} has in K_{k-1} only finitely many prime divisors, and is not divisible by more than finitely many powers of a prime p where p is greater than $\lambda(R_0)$.

If k is even, say $k = 2n \neq 0$, then

$$K_k = \{K_{k-1} * N_{-n} \ ; \ J_{-n}\},$$

or if k is odd, say $k = 2n + 1$, then

$$K_k = \{K_{k-1} * N_{n+1} \ ; \ J_{n+1}\}.$$

In either case, by Lemma 1.2.2 the amalgamated subgroup is a p-pure subgroup of both factors; in fact it is π-pure where π is the set of all primes greater than $\lambda(R_0)$. One now uses Lemmas 1.1.3 and 1.1.4 to conclude that each non-trivial element of K_k has only finitely many prime divisors, and is not divisible by more than finitely many powers of a prime p, where p is greater than $\lambda(R_0)$. It has already been remarked that the result holds in $K_0(=N_0)$ so the result holds in K_k for any positive integer k.

Since K_k is a p-pure subgroup of N then no element of K_k can acquire new p^r-th divisors in N. Hence every element of K_k is divisible in N by only finitely many powers of the prime p. Since K_k is a π-pure subgroup of N where π is as above, then no element of K_k can acquire more than finitely many new prime divisors. Hence every element of K_k is divisible in N by only finitely many primes. This completes the proof of the theorem.

We can immediately confirm a conjecture of G. Baumslag 1964.

Corollary 1.2.4 The additive group of rationals is not a subgroup of a one-relator group.

Proof Let G be a one-relator group

$$G = gp(a, b, c, \ldots \mid R).$$

If G is torsion-free then no non-trivial element is divisible by all the primes, so the additive group of rationals is not a subgroup of a torsion-free one-relator group. Suppose G has torsion. This implies R is a proper power, say $R = V^m$, m an integer > 1, where V is not a proper power. Let F be the group freely generated by the elements a, b, c, ..., and let N_V, N_{V^m} be the normal subgroups of F generated by the words V, V^m respectively. From the Main Theorem of Cohen and Lyndon 1963 there is a transversal

$$X = \{\ldots, x, \ldots\}$$

for F mod N_V such that

$$N_V = gp(V^x(x \in X) \mid .),$$

is freely generated by the elements V^x. Now every element of F can be written xh where $x \in X$, $h \in N_V$. Hence

$$\begin{aligned} N_V/N_{V^m} &= gp(V^x \mid (V^m)^g, \text{ all } g \in F) \\ &= gp(V^x \mid (V^m)^{xh}, \text{ all } x \in X, h \in N_V) \\ &= gp(V^x \mid (V^m)^x, \text{ all } x \in X) \\ &= gp(V^x \mid (V^x)^m). \end{aligned}$$

Thus N_V/N_{V^m} is the free product of cyclic groups.

Now $G = F/N_{V^m}$ and put $H = F/N_V$. Then H has a presentation

$$H = gp(a, b, c, \ldots \mid V)$$

and so is a torsion-free one-relator group.
But

$$H = F/N_V = (F/N_{V^m})/(N_V/N_{V^m}) \simeq G/N$$

where N is the free product of cyclic groups. It has already been shown that no element of H is divisible by all the primes. Hence if an element g of G is divisible by all the primes, it must be contained in N.

Since all the roots of g lie in N, and no non-trivial element of N is divisible by all the primes, then no element of G is divisible by all the primes. This completes the proof of the corollary.

R.C. Lyndon has shown that the cohomological dimension of a torsion-free one-relator group is ≤ 2. Now the cohomological dimension of a free Abelian group of rank n is n, and of a direct product of an infinite cyclic group with a non-cyclic locally cyclic group is > 2. Since the cohomological dimension of a subgroup is less than or equal to the cohomological dimension of the group, it follows that the only Abelian subgroups of a torsion-free one-relator group are free Abelian of rank ≤ 2, or locally cyclic subgroups, in which every non-trivial element is divisible by at most finitely many primes.

Section 1.3 An extension of the Freiheitssatz.

In this section we prove the following result: if a one-relator group G involves more than two generators non-trivially and one of them has zero exponent sum in the relator of G, then one can choose one of the generators, say x, and an integer m such that for all integers $\alpha > m$, x^{α} and the generators other than x freely generate a subgroup of G. Because of the requirement that the exponent sum for one generator be zero, one is unable to use an induction argument in quite the same way as before.

It will be useful to have the following notation: let

$$G = gp(u, v, \ldots ; a, b, \ldots ; x, y, \ldots \mid R) \quad (1)$$

be a one-relator group where the generators are divided into three disjoint subsets. We call the generators u, v, ... the top generators, the a, b, ... the middle generators, and the x, y, ... the bottom generators. We assume always that a top and bottom generator, say u and x respectively, occur non-trivially in R where R is cyclically reduced. When a group is considered in this way it will be called a trisected group. A group is called bisected if the set of middle generators is empty.

Lemma 1.3.1 Let G be the trisected group (1) where u, x, y occur non-trivially in R. Then an equation

$$W(u, v, \ldots, a, b, \ldots) = Z(x, y, \ldots, a, b, \ldots) \quad (2)$$

where u occurs non-trivially in W implies that for some cyclic permutation of R, R is freely equal to a word in

$$u, v, \ldots, a, b, \ldots, w$$

where w is a word in x, y, ..., a, b, ...

Proof The lemma will be proved by induction on $\lambda(R)$. The lemma is easily shown to be true for the first few cases. For example if $R = uxuy^{-1}$ then $y = uxu$ and u, x freely generate G. Clearly no non-trivial word in uxu and x can have x removed thus establishing the lemma in this case. Suppose inductively the lemma is true for all relators with length $< \lambda(R)$, and that equation (2) takes place in G. It will be shown that R is of the required form.

<u>Case 1</u> Suppose $\sigma_x(R) = 0$. As usual let

$$N = gp_G(u, v, \ldots, a, b, \ldots, y, \ldots)$$

and construct N from

$$N_0 = gp(u_0, v_0, \ldots, a_j, b_j, \ldots, y_j, \ldots, \text{ (all } j) \mid R_0).$$

Note that if generators other than those displayed in N_0 were to occur in R_0 then equation (2) could not hold in G. For equation (2) takes place in N and expressed in terms of the generators of N will be

$$W(u_0, v_0, \ldots, a_0, b_0, \ldots) = Z(y_j, \ldots, a_j, b_j, \ldots).$$

This equation takes place in N_0. One now trisects N_0 as

$$N_0 = gp(u_0, v_0, \ldots ; a_0, b_0, \ldots ; y_i, \ldots, a_j, b_j, \ldots \ (j \neq 0) \mid R_0).$$

The relator R_0 has shorter length than that of R, so in order to apply the induction hypothesis it is only necessary to check that R involves non-trivially two bottom generators. But if for some integer r only y_r occurs in the bottom generators then x must have occurred in R only in the word form $x^{-r}yx^{r}$. Thus taking

$w = x^{-r}yx^{r}$ one has that R is a word in u, v, ..., a, b, ..., w.

Thus one may assume two bottom generators occur in R_0. Using the induction hypothesis R_0 is a word in

$$u_0, v_0, \ldots, a_0, b_0, \ldots, \bar{w}$$

where $\bar{w}$ is a word in $y_i, \ldots, a_i, b_i, \ldots$. Rewriting R_0 as the relator R of G one has R is a word in

$$u, v, \ldots, a, b, \ldots, w$$

where w is a word in x, y, ..., a, b, ...

<u>Case 2</u> Suppose $\sigma_x(R) = r \neq 0$, $\sigma_y(R) = -s \neq 0$. Here one may embed G in a group H

$$H = gp(u, v, \ldots ; a, b, \ldots ; \underset{\sim}{x}, \underset{\sim}{y}, \ldots \mid R_H),$$

where $R_H = R(u, v, \ldots, a, b, \ldots, \underset{\sim}{x}^s, \underset{\sim}{z}, \ldots)$ under the usual mapping

$$u \to u, \ v \to v, \ \ldots, \ a \to a, \ b \to b \text{ and}$$
$$x \to \underset{\sim}{x}^s$$
$$y \to \underset{\sim}{z} \quad \text{where } \underset{\sim}{z} = \underset{\sim}{x}^r\underset{\sim}{y} \text{ or } \underset{\sim}{y}\underset{\sim}{x}^r.$$

By a suitable choice of $\underset{\sim}{z}$ one can arrange for R_H to involve $\underset{\sim}{x}$, $\underset{\sim}{y}$ non-trivially. Clearly equation (2) when mapped into H will have the same form. Since $\sigma_{\underset{\sim}{x}}(R_H) = 0$ one may then proceed as in Case 1 to prove that R_H is a word in

$$u, v, \ldots, a, b, \ldots, \bar{w}$$

where $\bar{w}$ is a word in $\underset{\sim}{x}, \underset{\sim}{y}, \ldots, a, b, \ldots$. Now R_H is freely equal to $R(u, v, \ldots, a, b, \ldots, \underset{\sim}{x}^s, \underset{\sim}{z})$ so this implies that R is a word in

$$u, v, \ldots, a, b, \ldots, w$$

where w is a word in $x, y, \ldots, a, b, \ldots$. This completes the proof of the lemma.

Corollary 1.3.2 Let G be a trisected group (1) and suppose there is an equation

$$W(u, v, \ldots, a, b, \ldots) = Z(x, y, \ldots, a, b, \ldots)$$

where u occurs non-trivially in W. Then for a suitable cyclic permutation of R, R is a word in

$$w, a, b, \ldots, z$$

where w is a word in $u, v, \ldots, a, b, \ldots$, and z is a word in $x, y, \ldots, a, b, \ldots$.

Proof If from among the top and bottom generators only u, x occur non-trivially in R then the corollary is trivially true by taking $w = u$ and $z = x$. If x and y occur non-trivially then by the lemma above, there exists a word

$$z(x, y, \ldots, a, b, \ldots)$$

such that for a suitable cyclic permutation of R, R is a word in

$$u, v, \ldots, a, b, \ldots, z.$$

Similarly, or by symmetry, if u, v occur non-trivially then there exists $w(u, v, \ldots, a, b, \ldots)$ such that for a suitable permutation of R (and one which will not break up any word z), R is a word in

$w, a, b, \ldots, z.$

Lemma 1.3.3 Let G be a trisected group (1) and let R be a word $\bar{R}$ in $w, a, b, \ldots, x, y, \ldots$ where w is a word in $u, v, \ldots, a, b, \ldots$. Then if H is the subgroup of G generated by $w, a, b, \ldots, x, y, \ldots$ then H has the presentation

$$gp(p, a, b, \ldots, x, y, \ldots \mid \bar{R}(p, a, b, \ldots, x, y, \ldots)),$$

under the mapping $p \to w, a \to a, b \to b, \ldots, x \to x, y \to y, \ldots$.

Proof Now $G = gp(u, v, \ldots ; a, b, \ldots ; x, y, \ldots \mid \bar{R}(w(u, v, \ldots, a, b, \ldots), a, b, \ldots, x, y, \ldots))$ and by Tietze transformations

$$\begin{aligned} G &= gp(p, \bar{a}, \bar{b}, \ldots, u, v, \ldots, a, b, \ldots, x, y, \ldots \mid \\ &\qquad \bar{R}(p, \bar{a}, \bar{b}, \ldots, x, y, \ldots), \\ &\qquad p = w(u, v, \ldots, a, b, \ldots), \bar{a} = a, \bar{b} = b, \ldots), \\ &= \{A * B \,;\, J\} \end{aligned}$$

where $A = gp(p, \bar{a}, \bar{b}, \ldots, x, y, \ldots \mid \bar{R}(p, \bar{a}, \bar{b}, \ldots, x, y, \ldots))$

and

$$B = gp(u, v, \ldots, a, b, \ldots \mid .)$$

and J is the free subgroup freely generated by the identified elements

$$\bar{a} = a, \bar{b} = b, \ldots, p = w(u, v, \ldots, a, b, \ldots).$$

This decomposition of G is possible since from the Freiheitssatz $gp(w, \bar{a}, \bar{b}, \ldots)$ is free in the first factor. Thus the $gp(w, a, b, \ldots, x, y, \ldots)$ is the first factor which has the required presentation.

Corollary 1.3.4 Let G be a trisected group (1) and suppose

$$W(u, v, \ldots, a, b, \ldots) = Z(x, y, \ldots, a, b, \ldots) \tag{3}$$

Then if $w(u, v, \ldots, a, b, \ldots)$ is as defined above, W is a word in $w, a, b, \ldots$.

Proof From above one has a decomposition for G as a generalized free product. The right-hand side of equation (3) lies in one factor and the left-hand side of (3) lies in the other factor. Hence $W(u, v, \ldots, a, b, \ldots)$ lies in the amalgamated subgroup, which implies W is a word in $w, a, b, \ldots$.

In 1960 Greendlinger showed that if G is a less than one-sixth group, that is a group for which any two defining relators taken from a symmetrized set either cancel less than one-sixth of each other or are inverse to each other, then if a and b are disjoint generators of G with $a^\alpha = b^\beta \neq 1$ for integers α, β then $a^{\epsilon_1}b^{\epsilon_2}$, $\epsilon_i = \pm 1$ is a relator of G. One might have expected that in a one-relator group an equation $a^\alpha = b^\beta$ where a, b are distinct generators would hold only if the relator itself was a conjugate of $a^{\alpha_1}b^{\beta_1}$ for some integers α_1, β_1. This however is not the case for in

$$gp(a, b \mid abab^2)$$

one can easily show

$$a^2 = b^{-3}.$$

In fact it is an unsolved problem to determine all relators $R(a, b)$ such that an equation $a^\alpha = b^\beta$ holds in $gp(a, b \mid R(a, b))$. Magnus 1930 and Steinberg 1962 solved the problem for certain pairs of integers α, β.

However the following corollary shows that the next best thing is true.

Corollary 1.3.5 Let G be a bisected one-relator group

$$G = gp(u, v, \ldots ; x, y, \ldots \mid R)$$

with a non-trivial set of equations

$$w_i(u, v, \ldots) = z_i(x, y, \ldots)$$

where $i \in$ some index set. Then there exists an equation

$$\underline{\underline{W}}(u, v, \ldots) = \underline{\underline{Z}}(x, y, \ldots)$$

such that for all i

$$w_i = \underline{\underline{W}}^{n_i}, \ z_i = \underline{\underline{Z}}^{n_i}$$

for some integer n_i.

That is to say any equation between disjoint sets of generators in a one-relator group is merely a power of some one equation in the same disjoint sets of generators. We refer to the equation $\underline{\underline{W}} = \underline{\underline{Z}}$ as a basic disjoint equation. For different bisections of a group there may be different basic disjoint equations. Thus the corollary may be simply stated: every disjoint equation is a power of a basic disjoint equation. The proof follows easily from Corollary 1.3.4 by discarding the middle generators.

We now strengthen this result as follows.

Lemma 1.3.6 Let G be a trisected group (1) and let

$$\underline{\underline{W}}(u,v,\ldots) = \underline{\underline{Z}}(x,y,\ldots,a,b,\ldots) \tag{4}$$

be a basic disjoint equation. Then

$$W(u,v,\ldots,a,b,\ldots) = Z(x,y,\ldots,a,b,\ldots) \tag{5}$$

implies W is a word in $\underline{\underline{W}},a,b,\ldots$.

Proof The lemma is proved by induction on $\lambda(R)$. If there are no middle generators the result is true by the previous lemma. Assume the lemma has been proved for all groups with relator length $< \lambda(R)$, and assume some middle generators occur.

Case 1 Suppose $\sigma_x(R) = 0$. Let $N = gp_G(u,\ v,\ \ldots,\ a,\ b,\ \ldots,\ y,\ \ldots)$ and construct N from copies of

$N_0 = gp(u_0,v_0,\ldots;a_0,b_0,\ldots;y_i,\ldots,a_j,b_j,\ldots,j\neq 0 \mid R_0)$ regarded as a trisected group. Again note that R_0 will involve only those generators displayed otherwise equation (5) could not hold. When equation (5) is rewritten as an equation in N one has

$$\underline{\underline{W}}(u_0,\ v_0,\ \ldots) = \underline{\underline{Z}}(y_i,\ \ldots,\ a_i,\ b_i,\ \ldots)$$

which takes place in N_0. Clearly this is a basic disjoint equation in N_0. The relation (6) when rewritten as an equation in N is

$$W(u_0, v_0, \ldots, a_0, b_0, \ldots) = \underset{\approx}{Z}(y_i, \ldots, a_i, b_i, \ldots).$$

This equation also takes place in N_0. By the induction hypothesis W is a word in $\underline{\underline{W}}(u_0, v_0, \ldots)$ and $a_0, b_0, \ldots$. Interpreting this as a relation in G one sees that $W(u, v, \ldots, a, b, \ldots)$ is a word in $\underline{\underline{W}}(u, v, \ldots)$ and $a, b, \ldots$.

<u>Case 2</u> Suppose $\sigma_a(R) = 0$. Let $N = gp_G(u, v, \ldots, b, \ldots, x, y, \ldots)$ and let

$N_0 = gp(u_0, v_0, \ldots ; b_i, \ldots ; x_i, y_i, \ldots, (\text{all } i) \mid R_0)$.

Again only the generators displayed can occur in R_0. Rewriting equation (5) one obtains

$$\underline{\underline{W}}(u_0, v_0, \ldots) = \underset{\approx}{\underline{\underline{Z}}}(x_i, y_i, \ldots, b_i, \ldots)$$

which takes place in N_0, and is a basic disjoint equation in N_0. Equation (5) when rewritten will be

$$\prod_{i=1}^{n} W_i(u_{j_i}, v_{j_i}, \ldots, b_j, \ldots) = \underset{\approx}{Z}(x_i, y_i, \ldots, b_i, \ldots). \tag{6}$$

where each W_i involves u_{j_i} or v_{j_i}, ... non trivially. Now one observes that no W_i which is a word in the generators of N_{j_i} can be in any other N_j, $j \neq j_i$, because by the Freiheitssatz one cannot eliminate any u_{j_i}, v_{j_i}, ... One may now prove that each W_i is a word in $\underline{\underline{W}}(u_{j_i}, v_{j_i}, \ldots)$ and the b_i, ..., by an induction argument on the integer m. If $m = 1$ then

$$W_1(u_{j_1}, v_{j_1}, \ldots, b_j, \ldots) = \underset{\sim}{Z}(x_i, y_i, \ldots, b_i, \ldots)$$

and this takes place in N_{j_1}. By the original induction hypothesis this implies W_1 is a word in $\underline{\underline{W}}(u_{j_1}, v_{j_1}, \ldots)$ and $b_i, \ldots$. Suppose $m > 1$ and that the result has been established for all positive integers $< m$. Then taking the first factor W_1 on the left hand side it is clear that

$$W_1(u_{j_1}, v_{j_1}, \ldots, b_j, \ldots) = Z_1(x_i, y_i, \ldots, b_i, \ldots).$$

But such a relation implies W_1 is a word in $\underline{\underline{W}}(u_{j_1}, v_{j_1}, \ldots)$ and $b_i, \ldots$. Hence in equation (6) one may take W_1 to the right hand side and replace each $\underline{\underline{W}}(u_{j_1}, v_{j_1}, \ldots)$ by $\underline{\underline{Z}}(x_i, y_i, \ldots, b_i, \ldots)$. Thus one obtains

$$\prod_{i=2}^{m} W_i(u_{j_i}, v_{j_i}, \ldots, b_j, \ldots) = \overline{Z}(x_i, y, \ldots, b_i, \ldots).$$

But by the supplementary induction hypothesis this implies each W_i, $i = 2, \ldots, m$ is a word in $w(u_{j_i}, v_{j_i}, \ldots)$ and $b_i, \ldots$.

Using the same technique one can prove an interesting result which is somewhat analogous to the result of Greendlinger mentioned above. It is interesting because it is one of the rare occasions in one-relator groups when one can precisely determine the exact spelling of a relator from an equation in the group

Lemma 1.3.6* Let G be a bisected one-relator group

$$G = gp(u, v, \ldots ; x, y, \ldots \mid R)$$

and let $\underline{\underline{W}}(u, v, \ldots) = \underline{\underline{Z}}(x, y, \ldots)$ be a basic disjoint equation. If $\underline{\underline{Z}}$ is not a proper power then for a suitable cyclic permutation of $R^{\pm 1}$, the relator R is

$$\underline{\underline{W}}(u, v, \ldots)\, \underline{\underline{Z}}^{-1}(x, y, \ldots).$$

The proof follows that used above with a slight variation (in the case when G involves only two generators u, x), which the reader can easily supply.

This lemm is related to the following problem: If a free group G is generated by elements a, b, c satisfying the equation

$$a^k b^l = c^m$$

where $|k|, |l|, |m| \geqslant 2$, then the rank of G is at most 1. Lyndon 1959 proved this statement for $|k| = |l| = |m| = 2$, Schenkman 1959, Stallings 1959, and Baumslag 1960 for $|k| = |l| = |m|$, Schützenberger 1959, and Lyndon and Schützenberger 1962 for the general case, and Baumslag and Steinberg 1964 for a generalization.

<u>Theorem 1.3.7</u> (Baumslag and Steinberg 1964). Let $w(x_1, x_2, \ldots, x_n)$ be an element of a free group F freely generated by $x_1, x_2, \ldots, x_n$, x which is neither a proper power nor a primitive. If $g_1, g_2, \ldots, g_n$, g generate a free group G and are connected by the relation

$$w(g_1, g_2, \ldots, g_n) = g^m \qquad m > 1$$

then the rank of G is at most n - 1.

<u>Proof</u> If the rank of G is n + 1 then any n + 1 generat-

ors of G freely generate G, in particular $g_1, g_2, \ldots, g_n, g$. But this would imply that no non-trivial relation exists between $g_1, g_2, \ldots, g_n, g$, contradicting the hypothesis.

A necessary and sufficient condition for the rank G to be n is that $w(x_1, x_2, \ldots, x_n)x^{-m}$ has a primitive root R. For if it has a primitive root $R(x_1, x_2, \ldots, x_x, x)$ then let $N = gp_F(R)$ and F/N is free of rank n, and is generated by the n + 1 elements

$$x_1N, x_2N, \ldots, x_nN, xN,$$

satisfying the relation

$$w(x_1N, x_2N, \ldots, x_nN) = (xN)^m.$$

Consider the homomorphism ψ from F onto G defined by

$$x_i\psi = g_i \qquad (i = 1, 2, \ldots, n), \; x\psi = g.$$

Let $h_1, h_2, \ldots, h_n$ freely generate G. By a special case of Grushko's Theorem there exists a free generating set $y_1, y_2, \ldots, y_{n+1}$ of F such that

$$y_i\psi = h_i \qquad (i = 1, 2, \ldots, n), \; y_{n+1}\psi = 1.$$

The kernel of ψ is the normal closure of y_{n+1} in F. But

$$[w(x_1, x_2, \ldots, x_n)x^{-m}]\psi = 1.$$

Hence $w(x_1, x_2, \ldots, x_n)x^{-m}$ has a primitive root y_{n+1}.

Thus to prove the theorem it is sufficient to prove that $w(x_1, x_2, \ldots, x_n)x^{-m}$ does not have a primitive root. But by Lemma 1.3.6*any root must be a conjugate of $w(x_1, x_2, \ldots, x_n)x^{-m}$. Since a conjugate of a primitive is a primitive it is sufficient to show that $w(x_1, x_2, \ldots, x_n)x^{-m}$ is not primitive. But if it is primitive then

$$G = gp(g_1, g_2, \ldots, g_n, g \mid w(g_1, g_2, \ldots, g_n)g^{-m})$$

is a free group, and as mentioned before has rank < n + 1. Now

$$G/gp_G(w(g_1, g_2, \ldots, g_n))$$
$$= gp(g_1, g_2, \ldots, g_n \mid w(g_1, g_2, \ldots, g_n)) * gp(g \mid g^m).$$

Since w is not a primitive then the first factor cannot be generated by n - 1 elements (Magnus 1939). But by the Grushko-Neumann theorem on the number of generators of a free product, $G/gp_G(w)$ cannot be generated by fewer than n + 1 generators. The same is therefore true of G, so $w(x_1, x_2, \ldots, x_n)x^{-m}$ is not primitive. This proves the theorem.

We now introduce the concept of a word descending in a group. Let

$$N_0 = gp(a_0, \ldots, a_\mu, b_0, \ldots, b_\mu, \ldots \mid R_0)$$

where the subscripts in R_0 range from 0 to μ inclusive. Then a word $W(a_1, \ldots, a_\mu, b_1, \ldots, b_\mu, \ldots)$ is said to <u>descend one step</u> if

$$W(a_1,\ldots,a_\mu,b_1,\ldots,b_\mu,\ldots) = W_1(a_0,\ldots,a_{\mu-1},b_0,\ldots,b_{\mu-1},\ldots).$$

If further we <u>lift</u> the word on the right by increasing the subscripts by 1, symbolized by $\mathcal{L}$, thus

$W_1(a_0, \ldots, a_{\mu-1}, b_0, \ldots, b_{\mu-1}, \ldots) \pounds W_1(a_1, \ldots, a_\mu, b_1, \ldots, b_\mu, \ldots)$
and obtain a lifted word which descends another step, namely

$W_1(a_1, \ldots, a_\mu, b_1, \ldots, b_\mu, \ldots) = W_2(a_0, \ldots, a_{\mu-1}, b_0, \ldots, b_{\mu-1}, \ldots)$
then $W(a_1, \ldots, a_\mu, b_1, \ldots, b_\mu, \ldots)$ is said to descend two steps. Inductively one can define descension through k steps. Examples to illustrate the various possibilities of descension follow.

Example 1 Let $R_0 = (a_1a_0)^2$. Here it is possible to show (see the Spelling Theorem 2.1.1) that no non-trivial power of a_1 will descend. Hence descension is impossible with this relator.

Example 2 Let $R_0 = a_1{}^2a_0b_0$. Here $a_1{}^2$ descends one step, but no word in a_1, b_1 can descend more than one step.

Example 3 Let $R_0 = a_1{}^2a_0{}^{-1}$. Here $a_1{}^{2^k}$ will descend k steps. Thus

$$a_1{}^{16} = a_0{}^8 \pounds a_1{}^8 = a_0{}^4 \pounds a_1{}^4 = a_0{}^2 \pounds a_1{}^2 = a_0$$

descends four steps.

Example 4 Let $R_0 = a_1{}^2a_0{}^{-2}$. Here $a_1{}^2$ descends infinitely,

$$a_1{}^2 = a_0{}^2 \pounds a_1{}^2 = a_0{}^2 \pounds a_1{}^2 \ldots .$$

In a similar way one defines an ascension of a word through k steps. We aim to determine the spelling of a relator R_0 when unbounded descension or ascension occurs. To this end we first prove

<u>Lemma 1.3.8</u> Let $N_0 = gp(a_0, \ldots, a_\mu, b_0, \ldots, b_\mu, \ldots \mid R_0)$ and let $w_i = w(a_i, b_i, \ldots)$ for integers i, $0 \leq i \leq \mu$, where w is not a proper power as a word in a free group. If ${w_\mu}^n$ descends μ steps then

$${w_\mu}^n = W(w_0, w_1, \ldots, w_{\mu-1}).$$

<u>Proof</u> Since ${w_\mu}^n$ descends, by Corollary 1.3.5 there exists a basic disjoint equation which will clearly be, for some integer m

$${w_\mu}^m = z_1(a_0, \ldots, a_{\mu-1}, b_0, \ldots, b_{\mu-1}, \ldots). \quad (7)$$

There are two cases to consider.

<u>Case 1</u> Suppose ${w_\mu}^m$ descends μ steps. Then

$${w_\mu}^m = z_1 \mathbin{\pounds} z_1(a_1, \ldots, a_\mu, b_1, \ldots, b_\mu, \ldots).$$

But this is to descend again so using Lemma 1.3.6, a_μ, b_μ, must occur in z_1 lifted as ${w_\mu}^m$. This means that $a_{\mu-1}$, $b_{\mu-1}$, ... occur in z_1 in equation (7) as ${w_{\mu-1}}^m$. Thus equation (7) may be refined to read

$${w_\mu}^m = \bar{z}_1(a_0, \ldots, a_{\mu-2}, b_0, \ldots, b_{\mu-2}, \ldots, {w_{\mu-1}}^m). \quad (8)$$

Now repeating the descension

$$\begin{aligned} {w_\mu}^m &= \bar{z}_1(a_0, \ldots, a_{\mu-2}, b_0, \ldots, b_{\mu-2}, \ldots, {w_{\mu-1}}^m) \\ &\quad \pounds\ \bar{z}_1(a_1, \ldots, a_{\mu-1}, b_1, \ldots, b_{\mu-1}, \ldots, {w_\mu}^m) \\ &= \bar{z}_1(a_1, \ldots, a_{\mu-1}, b_1, \ldots, b_{\mu-1}, \ldots \\ &\quad \bar{z}_1(a_0, \ldots, a_{\mu-2}, b_0, \ldots, b_{\mu-2}, \ldots\ {w_{\mu-1}}^m)) \\ &\quad \pounds\ \bar{z}_1(a_2, \ldots, a_\mu, b_2, \ldots, b_\mu, \ldots \\ &\quad \bar{z}_1(a_1, \ldots, a_{\mu-1}, b_1, \ldots, b_{\mu-1}, \ldots\ {w_\mu}^m)). \end{aligned}$$

Now in order for this last word to descend the a_μ, b_μ, ... must occur as $w_\mu{}^m$. This can happen only if $a_{\mu-2}$, $b_{\mu-2}$, ... occur in (8) as $w_{\mu-2}{}^m$. Thus equation (8) may be refined to read

$$w_\mu{}^m = \bar{\bar{z}}_1(a_0, \ldots, a_{\mu-3}, b_0, \ldots, b_{\mu-3}, \ldots, w_{\mu-2}{}^m, w_{\mu-1}{}^m) \tag{9}$$

One repeats this argument until after μ descensions

$$w_\mu{}^m = W(w_0{}^m, w_1{}^m, \ldots, w_{\mu-1}{}^m).$$

Case 2 Suppose $w_\mu{}^m$ does not descend $\mu+1$ steps. Then one uses an argument similar to that above, until one strikes a step where no further descension is possible unless a power of $w_\mu{}^m$ is taken. But the effect of taking a power will introduce $w_\mu{}^m$ only if at that particular step, say the k-th step one has

$$V^{-1}w_\mu{}^rV$$

where V is a word in $a_0, \ldots, a_{\mu-k-1}, b_0, \ldots, b_{\mu-k-1}, \ldots, w_{\mu-k}{}^m, \ldots, w_\mu{}^m$. Taking the required power one then continues to show that V is a word in $w_0{}^m, w_1{}^m, \ldots, w_\mu{}^m$. Hence

$$w_\mu{}^m = V^{-1}w_{\mu-k}{}^rV$$ whence $w_\mu{}^n$ is a word in $w_0, w_1, \ldots, w_{\mu-1}$. This completes the proof of the lemma.

Actually we have proved more than we set out to do. For in Case 2 above, if V is non-trivial it is not too difficult to see that no word will descend unboundedly. Thus we have in the notation of Lemma 1.3.8

Lemma 1.3.9 If words $w_\mu^{\ n}$ descend unboundedly then either

$$w_\mu^{\ m} = w_0^{\ r}$$

or $$w_\mu^{\ m} = W(w_0^{\ m}, w_1^{\ m}, \ldots, w_{\mu-1}^{\ m}).$$

Lemma 1.3.10 Let $N_0 = gp(a_0, \ldots, a_\mu, b_0, \ldots, b_\mu, \ldots \mid R_0)$ and let $w_i = w(a_i, b_i, \ldots)$ for integers i, $0 \leqslant i \leqslant \mu$.

If

$$w_\mu^{\ n} = W(w_0, w_1, \ldots, w_{\mu-1})$$

then R_0 is a word in $w_0, w_1, \ldots, w_\mu$ for some cyclic permutation of R_0.

Proof Let N_0 be bisected with $a_\mu, b_\mu, \ldots$ as the top generators, and the remaining generators as bottom generators. By Corollaries 1.3.2 and 1.3.4, for some cyclic permutation of R_0, R_0 is a word in w_μ and some root of $W(w_0, \ldots, w_{\mu-1})$. But clearly a root of $W(w_0, \ldots, w_{\mu-1})$ will be a word in $w_0, w_1, \ldots, w_{\mu-1}$. Hence R_0 is a word in $w_0, w_1, \ldots, w_\mu$.

Theorem 1.3.11 Let $G = gp(a, b, c, \ldots \mid R)$ where a, b, c occur non-trivially in R with $\sigma_a(R) = 0$. Then for one of the generators a, b, or c, (say b) there exists an integer m such that for all integers $\alpha > m$

$$a, b^\alpha, c, \ldots$$

freely generate a subgroup of G.

Remark If $\sigma_a(R) \neq 0$ the result may be false, for example if R is $a^2b^2c^2$. Then there is for no integer m, a choice of a, b or c with the required freeness property.

Proof Let $N = gp_G(b, c, \ldots)$ and as usual let

$$N_0 = gp(b_0, \ldots, b_\mu, c_0, \ldots, c_\mu, \ldots \mid R_0)$$

where $0, \mu$ are the least and greatest subscripts appearing in R_0 for all b, c, ... The proof may be conveniently divided into two cases.

Case 1 Suppose there is a bound m_1 on the number of steps in the descension of words in $b_\mu, c_\mu, \ldots$, and a bound m_2 on the number of steps in an ascension of words in $b_0, c_0, \ldots$. Let m be any integer > $m_1 + m_2 + \mu$. It will be shown that if α is any integer > m,

$$a^\alpha, b, c, \ldots$$

freely generate a free group. For suppose

$$W(a^\alpha, b, c, \ldots) = 1$$

where W is a non-trivial word. Since $\sigma_a(W) = 0$ this may be rewritten as an equation in N,

$$\overline{W}(b_{\alpha n_i}, c_{\alpha n_i}, \ldots) = 1$$

for integers n_i. This relation may be split into a product of subwords, each subword being a word in the generators of $N_{\alpha n}$ for n integral. Thus

$$1 = \overline{W} = W_1(b_{\alpha n_1}, c_{\alpha n_1}, \ldots)W_2(b_{\alpha n_2}, c_{\alpha n_2}, \ldots)\ldots W_r(b_{\alpha n_r}, c_{\alpha n_r}, \ldots)$$

where $n_1 \neq n_2 \neq \ldots \neq n_r$.

This relation is impossible in N. For it if takes place in N_0 (without loss of generality one may assume W_1 comes from N_0 and N_0 is the least such $N_{\alpha n}$ occuring), then one has a non-trivial relation in N_0 which implies, by the Freiheitssatz that only the generators b_0, c_0, ... occur non-trivially. This implies that a occurs trivially in R contradicting the hypothesis of the theorem.

Suppose it takes place in $K_k = \{N_0 * gp(N_1, \ldots, N_k) ; J_1\}$. Then W_1 lies in N_0, and W_2 lies in the second factor. Suppose inductively that no word $\overline{W}$ of length r can lie in any N_i. This is clearly true if r = 2 for

$$W_1(b_0, c_0, \ldots)W_2(b_{n\alpha}, c_{n\alpha}, \ldots)$$

does not lie in N_0 since W_2 cannot descend sufficiently to lie in N_0. Similarly one shows that W_1 cannot ascend sufficiently to lie in any N_r to which W_2 can descend. Similarly one proves that $\overline{W}$ does not lie in N_i for any integer i, so $\overline{W}$ has length greater than 1 in K_k. This shows $\overline{W} \neq 1$ and hence there can be no relation between a^α, b, c, ...

<u>Case 2</u> Suppose unbounded ascension or descension occurs. Without loss of generality let it be descension. Then for some word w and cyclic permutation of R_0, R_0 is a word in $w_0, w_1, \ldots, w_\mu$. This implies that R is a word $\overline{R}$ in a, w(b, c, ...). Define

$$H = gp(a, w \mid \overline{R}(a, w)).$$

Then $G = \{H * gp(b, c, \ldots \mid .) ; w = w(b, c, \ldots)\}$. There are two cases to consider now.

First suppose some letter b or c, say b occurs in $w(b, c, \ldots)$ separated by some letter other than b. Then one can determine an integer m such that

$$b^{\alpha}, c, \ldots, w(b, c, \ldots)$$

freely generate a free subgroup in the group generated freely by b, c, ... This implies that

$$a, b^{\alpha}, c, \ldots$$

freely generate a free group in G. For no word in $b^{\alpha}, c, \ldots$ will be a power of $w(b, c, \ldots)$, hence will not lie in the amalgamated subgroup. So any non-trivial word in $a, b^{\alpha}, c, \ldots$ will be of length $\geqslant 1$, in the generalized free product above.

Secondly, suppose b and c occur in $w(b, c, \ldots)$ only once, that is

$$w(b, c, \ldots) = V_1 b^{\beta} V_2 c^{\gamma} V_3$$

where V_1, V_2, V_3 do not involve b or c. If $\beta = 1$ let

$$\begin{aligned} G &= gp(a, b, c, \ldots \mid R(a, w(b, c, \ldots))) \\ &= gp(a, b, c, \ldots, x \mid R(a, w(b, c, \ldots)), x = bV_2c^{\gamma}) \\ &= gp(a, x, c, \ldots \mid R(a, V_1xV_3)) \\ &= gp(c) * gp(a, x, \ldots \mid R(a, V_1xV_3)). \end{aligned}$$

It will now be shown that $a, b, c^{\alpha}, \ldots$ freely generate a free group for $\alpha > m = 2\gamma$. For suppose

$$W(a, b, c^{\alpha}, \ldots) = 1.$$

Then $W(a, xc^{-\gamma}V_2^{-1}, c^{\alpha}, \ldots) = 1$. But it is impossible to eliminate c from this equation for between powers of c will occur words of the following types

$$x^{-1}W_i(a, \ldots)x,\ W_i(a, \ldots),\ W_i(a, \ldots)x,\ x^{-1}W_i(a, \ldots).$$

These words are non-trivial elements in the second factor. This follows immediately from the Freiheitssatz for the first two words. For the last two, a relation such as

$$W_i(a, \ldots)x = 1$$

would imply that $R(a, V_1xV_3)$ is $W_i(a, \ldots)x$, using Lemma 1.3.6*. But R, being a word in a, V_1xV_3, must have only one syllable a power of V_1xV_3, hence R has only one occurrence of a, contradicting the hypothesis.

Thus one has that $W(a, xc^{-\gamma}V_2^{-1}, c^{\alpha}, \ldots) = 1$ is impossible in G for non-trivial W.

If $\beta \neq 1$ let

$$H = gp(a, y, c, d, \ldots \mid R(a, V_1yV_2c^{\gamma}V_3)).$$

Then $G = \{H * gp(b) ; y = b^{\beta}\}$. Now if $W(a, b, c^{\alpha}, d, \ldots) = 1$ for α defined as above, one may rewrite this replacing b^{β} by y, to obtain

$$W_1b^{\beta_1}W_2b^{\beta_2}\ldots W_rb^{\beta_r} = 1 \qquad (10)$$

where the words W_i are non-trivial in a, y, c^{α}, d, ... and β_i are integers $0 < \beta_i < \beta$. Now no factor W_i is a power of y by the case just considered. Since no b^{β_i} lies in the amalgamated subgroup, the length of the left hand side of (10) is strictly greater than zero, unless $W(a, b, c^\alpha, d, \ldots)$ is freely equal to the empty word. This completes the proof of the theorem.

In proving case 2 above we did not use the fact that $\sigma_a(R) = 0$. This result may be stated as

Corollary 1.3.12 Let $G = gp(a, b, c, \ldots \mid \bar{R}(a, w(b, c, \ldots)))$ where a, b, c occur non-trivially in $\bar{R}$, and where powers of a occur in $\bar{R}$ separated by words in b, c, ... for every cyclic permutation of $\bar{R}$. Then one can choose a generator b or c (say b) and an integer m such that for all integers $\alpha > m$

$$a, b^\alpha, c, \ldots$$

freely generate a subgroup of G.

Lemma 1.3.11 Let $G = gp(a, b, c, \ldots \mid R)$, $\sigma_a(R) = 0$. Then G has trivial Frattini subgroup if

(i) no element in $N = gp_G(b, c, \ldots)$, ascends or descends more than a bounded number of steps,

or (ii) G has more than two generators.

Using the Spelling Theorem proved in the next chapter, one has in particular

Theorem 1.3.12 A one-relator group has trivial Frattini subgroup if

(a) it is torsion-free with more than two generators,

or (b) it has torsion with more than one generator.

<u>Proof of Lemma 1.3.11</u> Let $\Phi(G)$ be the Frattini subgroup of G, and let $h \in \Phi(G)$. Then $h \in N$ since G/N is infinite cyclic. Define

$$k = a^{-m}ha^{-m}ha^{-m}ha^{m}ha^{2m}, \quad m \text{ large}.$$

Since $\Phi(G)$ is normal in G, then $k \in \Phi(G)$. Let $g = a^{-1}k$. Then $\{k, g, b, c, \ldots\}$ generates G, hence $\{g, b, c, \ldots\}$ generates G, for k is a non-generator, being in $\Phi(G)$. Thus there exists a word W in g, b, c, ... with

$$W(g, b, c, \ldots) = a^{-1}.$$

Premultiply by a and rewrite this relation in N. A typical segment will be, for some words V_1, V_2, ...

$$gV_1(b, c, \ldots)gV_2(b, c, \ldots)g^{-1}, \tag{1}$$

that is, with the usual notation for translated words,

$$(h_m h_{2m} h_{3m} h_{2m})V_1(b_0, c_0, \ldots)(h_{m+1}h_{2m+1}h_{3m+1}h_{2m+1}) \times$$
$$V_2(b_1, c_1, \ldots)(h^{-1}_{2m+1}h^{-1}_{3m+1}h^{-1}_{2m+1}h^{-1}_{m+1}).$$

If no element in N ascends or descends unboundedly then choosing m sufficiently large, no cancellation or reduction can occur on the left hand side, assuming of course $V_2(b_1, c_1, \ldots) \neq 1$. This proves part (1).

Suppose G has more than two generators. Since the Frattini subgroup of a non-trivial free product is trivial, without loss of generality assume that more than two generators (say for simplicity a, b, c only) occur non-trivially in R. Then it will be shown that, by taking a suitable conjugate of h, one can ensure that h will not ascend nor descend unboundedly. For suppose $\underline{W}(b_0,\ldots,b_{\mu-1},c_0,\ldots,c_{\mu-1})$ ascends unboundedly. Then one has a natural trisection of N_0 and R_0 is a word in

$$w(b_0,\ldots,b_{\mu-1},c_0,\ldots,c_{\mu-1}),b_1,\ldots,b_{\mu-1},c_1,\ldots,c_{\mu-1}$$
$$z(b_1,\ldots,b_\mu,c_1,\ldots,c_\mu),$$

and $\underline{W}$ must be a word in $w,b_1,\ldots,b_{\mu-1},c_1,\ldots,c_{\mu-1}$. Take $\overline{W} = c_0^{-r}W(b_0,\ldots,b_{\mu-1},c_0,\ldots,c_{\mu-1})c_0^{\,r}$, and $\overline{W}$ is not a word in

$$w(b_0,\ldots,b_{\mu-1},c_0,\ldots,c_{\mu-1}),b_1,\ldots,b_{\mu-1},c_1,\ldots,c_{\mu-1}.$$

Hence $\overline{W}$ does not ascend. Without loss of generality one may assume h neither ascends nor descends.

If h neither ascends nor descends then reduction of the left hand side of (1) can occur only if

(a) $V_1(b_0,c_0,\ldots)h_{m+1}$ ascends unboundedly

or (b) $h_{2m+1}V_2(b_1,c_1,\ldots)h_{2m+1}^{-1}$ ascends unboundedly.

But if (a) holds then by choosing m large enough, V_1 must ascend unboundedly. Hence h_{m+1} must be a word in $w_k(b_k,c_k),\ldots,w_{k+\mu-1}(b_{k+\mu-1},c_{k+\mu-1})$ and a suitable conjugate of h has just been chosen so that this does not occur. Similarly one proves (b), thus proving the lemma.

Chapter 2

One-relator groups with torsion

The fundamental result used in the theory of one-relator groups with torsion is the Spelling Theorem proved in Section 2.1. In that section are also included an assortment of easy consequences of this theorem, one of which vastly improves the result on the freeness of the generators proved in Section 1.3. In Section 2.2 we introduce the concept of malnormal subgroups and prove the appropriate theorems as outlined in the introduction. In Section 2.3 we use malnormal subgroups to determine the Abelian structure of one-relator groups with torsion.

Section 2.1 The Spelling Theorem.

Theorem 2.1.1 (The Spelling Theorem). Let G be the group

$$G = gp(a, b, \ldots \mid R^n) \quad n > 1,$$

where R is cyclically reduced. Suppose that two words $W(a, b, \ldots)$, $V(b, \ldots)$, where W is a freely reduced word containing a non-trivially and V does not contain a, define the same element of G. Then W contains a subword which is identical with a subword of $R^{\pm n}$ of length greater than $(n - 1)/n$ times the length of R^n.

Proof The theorem will be proved by induction on $\lambda(R^n)$.

If $\lambda(R^n) < 4$ or if R contains only one generator, then the theorem is obvious, since G is a free product of cyclic groups. Suppose therefore that R contains two or more generators when cyclically reduced, and assume the theorem is true for all groups with relator length < $\lambda(R^n)$. There are three cases to consider. For simplicity of notation one may assume that at most generators a, b, c, t occur in R, and further that a, b occur non-trivially in R.

Case 1 Suppose $\sigma_a(R) = 0$. This implies $\sigma_a(W) = 0$ and so W, V belong to $N = gp_G(b, c, t)$. Construct N in the usual way from

$$N_0 = gp(b_0, \ldots, b_\mu, c_i, t_i \text{ (all integers i)} \mid R_0{}^n).$$

Rewriting the relation W = V as a relation in N one has

$$W_1(b_{i_1}, \ldots, b_{i_2}, c_i, t_i) = V(b_0, c_0, t_0) \qquad (1)$$

where, without loss of generality one assumes b_i, for some $i \neq 0$, occurs non-trivially in W_1. The word W_1 is freely reduced. But in order to continue one requires a more refined result (cf. Hauptform of Magnus 1930) than Theorem 2.1.1; namely that the relation (1) in N implies W_1 contains a subword identical with a subword of $R_i^{\pm n}$ of length > (n - 1)/n times the length of $R_i{}^n$ for some integer i. Actually one may prove more than this, namely the following Lemma 2.1.2. Note that one is still working within the proof of Theorem 2.1.1 so the induction hypothesis of Theorem 2.1.1 is still applicable.

<u>Lemma 2.1.2</u> In the notation above, let

$$W_1(b_0,\ldots,b_{\mu+k},c_i,t_i) = V_1(b_1,\ldots,b_{\mu+k},c_i,t_i) \qquad \mu + k \geq 0$$

or

$$W_1(b_0,\ldots,b_{\mu+k},c_i,t_i) = V_2(b_0,\ldots,b_{\mu+k-1},c_i,t_i) \qquad \mu + k > 0$$

where W_1 is freely reduced and contains in $W_1 = V_1$ the generator b_0 non-trivially, or in $W_1 = V_2$ the generator $b_{\mu+k}$ non-trivially. Then W_1 contains a subword identical with a subword of $R_i^{\pm n}$ of length > $(n - 1)\lambda(R_i^n)/n$ for some integer i.

<u>Proof</u> The lemma will be proved by induction on k. Since the proof for $W_1 = V_2$ is similar to that for $W_1 = V_1$ one need consider only $W_1 = V_1$. If $k \leq 0$ then $W_1 = V_1$ is an equation involving only generators from $K_0(= N_0)$. Hence, using the induction hypothesis of Theorem 2.1.1, W_1 contains as a subword more than $(n - 1)/n$ of $R_0^{\pm n}$. Assume now that Lemma 2.1.2 has been established for all $K_i = gp(N_0,\ldots,N_i)$ with $0 \leq i < k$, and let

$$K_k = \{N_0 * gp(N_1,\ldots,N_k) \ ; \ J_1\}.$$

Now $V_1 \in gp(N_1,\ldots,N_k)$, hence so must W_1. Let W_1 be written as

$$W_1 \equiv P_0 b_0^{\alpha_1} P_1 b_0^{\alpha_2} \ldots b_0^{\alpha_m} P_m, \quad \alpha_i \neq 0, \ m \geq 1,$$

where each P subword is a word in $b_1, \ldots, b_{\mu+k}, c_i, t_i$ and P_0, P_m are possibly empty.

Firstly note that $b_0^{\alpha_i} \in N_0$ but $b_0^{\alpha_i} \notin J_1$ by the induction hypothesis of Lemma 2.1.2. Secondly, any P which contains b_j with $j > \mu$ as the maximum subscripted b-letter, and is in J_1 will imply

$$P(b_1, \ldots, b_j, c_i, t_i) = V(b_1, \ldots, b_\mu, c_i, t_i)$$

or conjugating

$$P(b_0, \ldots, b_{j-1}, c_{i-1}, t_{i-1}) = V(b_0, \ldots, b_{\mu-1}, c_{i-1}, t_{i-1})$$

and by the present induction hypothesis P contains as a subword more than $(n - 1)/n$ of $R_i^{\pm n}$. Hence W_1 has the same property. Thus one may assume that any P representing an element in J_1 is actually a word in the generators of J_1. Let $P_{i_1}, \ldots, P_{i_2}$ be consecutive P subwords which are in J_1. Then if the subword

$$W_2 = b_0^{\alpha_{i_1}} P_{i_1} b_0^{\alpha_{i_1+1}} \ldots P_{i_2} b_0^{\alpha_{i_2+1}}$$

is in J_1 then W_2 contains as a subword more than $(n - 1)/n$ of $R_0^{\pm n}$ by the present induction hypothesis. Hence W_1 also has as a subword more than $(n - 1)/n$ of $R_i^{\pm n}$. This proves that $|W_1| > 1$ in the generalized free product or that W_1 contains as a subword more than $(n - 1)/n$ of $R_i^{\pm n}$. Since $|W_1| \leq 1$, the lemma is proved.

We return to the

Proof of Theorem 2.1.1. From Lemma 2.1.2, the equation

$$W_1(b_0,\ldots,b_{\mu+k},c_i,t_i) = V(b_0,c_0,t_0)$$

immediately implies W_1 has a subword identical with a subword of $R_i^{\pm n}$ of length $> (n-1)\lambda(R_i^{\,n})/n$. Hence W has the required subword consisting of more than $(n-1)/n$ of $R^{\pm n}$ by the following lemma.

Lemma 2.1.3 Suppose R is cyclically reduced with $\sigma_a(R) = 0$, and let W be a freely reduced word with $\sigma_a(W) = 0$. In the usual way rewrite R, W as R_0, W_0. If W_0 is identical with a subword of $R_i^{\pm n}$ of length $>$ $(n-1)\lambda(R_i^{\,n})/n$ for some integer i, then W contains a subword identical with a subword of $R^{\pm n}$ of length $>$ $(n-1)\lambda(R^n)/n$, even if one disregards any a terms which might occur at the beginning or end of W.

Proof Let $R_i^{\,n} = (b_jV_1)^{n-1}b_jV_1$ and

$$W_0 = (b_jV_1)^{n-1}b_j, \text{ (taking the worst possible case!)}.$$

Rewriting one has

$$a^{-i}R^na^i = (a^{-j}ba^ja^{-s_1}Va^{s_2})^n$$

where V is assumed freely reduced, and V neither begins nor ends with $a^{\pm 1}$, and

$$W = (a^{-j}ba^ja^{-s_1}Va^{s_2})^{n-1}a^{-j}ba^j.$$

Here $\lambda(R^n) = \lambda(ba^ja^{-s_1}Va^{s_2-j})^n$

$$= n[1 + |j - s_1| + |j - s_2| + \lambda(V)].$$

Now disregarding any a terms at the beginning and end of W one has

$$\lambda(W) = \lambda[(ba^j a^{-s_1} V a^{s_2} a^{-j})^{n-1} b]$$
$$= (n - 1)[1 + |j - s_1| + |j - s_2| + \lambda(V)] + 1$$
$$= 1 + (n - 1)\lambda(R^n)/n.$$

Hence W contains the required subword. This completes the proof of Lemma 2.1.3.

We now return to Theorem 2.1.1 and consider the remaining cases.

Proof of Theorem 2.1.1

Case 2 Suppose $\sigma_b(R) = 0$. Let $N = gp_G(a,c,t)$ and construct N from

$$N_0 = gp(a_0,\ldots,a_\mu, c_i, t_i \text{ (all integers } i) \mid R_0{}^n).$$

By premultiplying W, V by a suitable power of b, say b^r, one may ensure $\sigma_b(b^r W) = 0$, whence $\sigma_b(b^r V) = 0$, and $b^r W$ when freely reduced contains a. Rewriting $b^r W = b^r V$ as an equation in N one obtains

$$W_1(a_{i_1},\ldots,a_{i_2}, c_i, t_i) = V_1(c_i, t_i).$$

Clearly W_1 is freely reduced and contains some a_i non-trivially. By Lemma 2.1.2 W_1 contains as a subword more than $(n - 1)/n$ of $R_i^{\pm n}$ for some integer i. Hence $b^r W$ contains a subword consisting of more than $(n - 1)/n$ of $R^{\pm n}$, this subword neither beginning nor ending with b. Hence W has as a subword more than $(n - 1)/n$ of $R^{\pm n}$.

Case 3 Suppose $\sigma_a(R) = \alpha \neq 0$, $\sigma_b(R) = \beta \neq 0$. As usual one considers the group

$$H = gp(a, x, c, t \mid R^n(a, x^\alpha, c, t)).$$

Using Tietze transformations

$$H = gp(y, x, c, t \mid R^n(yx^{-\beta}, x^\alpha, c, t))$$

Let $R^n(yx^{-\beta}, x^\alpha, c, t)$ when cyclically reduced be $\underset{\sim}{R}^n(x, y, c, t)$. Now rewrite the equation $W = V$ as an equation in H under the monomorphism of G into H mapping

$$a \to yx^{-\beta},\ b \to x^\alpha,\ c \to c,\ t \to t.$$

Suppose on freely reducing $W(yx^{-\beta}, x^\alpha, c, t) = V(x^\alpha, c, t)$ one obtains

$$W_1(y, x, c, t) = V_1(x, c, t).$$

The process of freely reducing W to obtain W_1 will remove at most x terms, hence W_1 contains y non-trivially. From case 2 one may conclude that W_1 contains as a subword more than $(n - 1)/n$ of $\underset{\sim}{R}^{\pm n}$, and this subword neither starts nor ends with an $x^{\pm 1}$. One easily shows that this implies $W(a, b, c, t)$ contains a subword identical with a subword of $R^{\pm n}$ with length $> (n - 1)\lambda(R^n)/n$.

This completes the proof of the Spelling Theorem.

Corollary 2.1.4 The word problem and the extended word problem are solvable in a one-relator group with torsion.

Proof Given $G = gp(a, b, c, \ldots \mid R^n)$ $n > 1$ and two words W_1 and W_2 in the letters $a, b, c, \ldots$, one must be able to determine algorithmically if $W_1 = W_2$ as elements of G. This is equivalent to deciding if $W_1W_2^{-1} = 1$ in G. But in order for $W_1W_2^{-1}$ to be the trivial element in G, $W_1W_2^{-1}$ must, when freely reduced, be the empty word, or contain a subword identical with more than $(n - 1)/n$ of $R^{\pm n}$. Replacing any such subword by the obviously shorter complementary word, one easily determines if $W_1W_2^{-1} = 1$.

For the extended word problem one must determine algorithmically if an arbitrary word W in the letters a, b, c, ... can be equal in G to a word V in some given subset of the generators of G. Clearly one may use almost the same algorithm.

<u>Corollary 2.1.5</u> Let $G = gp(a, b, c, \ldots \mid R^n)$ $n > 1$ and let $\mathbf{W}$, $\mathbf{Z}$ be subsets of the generators. Then there is an algorithm to determine for an arbitrary element $g \in G$ if $g = w(\mathbf{W})z(\mathbf{Z})$ for some words w, z.

<u>Proof</u> If $g = w(\mathbf{W})z(\mathbf{Z})$ then $w^{-1}(\mathbf{W})g = z(\mathbf{Z})$. Without loss of generality one may assume $w^{-1}(\mathbf{W})$ begins with a generator a, say, of $\mathbf{W}$ which is not in $\mathbf{Z}$, for if $\mathbf{W}$ coincides with $\mathbf{Z}$ the problem degenerates to the extended word problem. It will now be shown that in order for a to be removed from $w^{-1}g$, $\lambda(w)$ must be small.

Firstly, let w, g be words of minimal length representing elements of G; because the word problem is solvable in G one can always obtain algorithmically a word of minimal length representing an element of G. By a reduction of a word will be meant the process of replacing a largest possible subword which is identical with a subword of $R^{\pm n}$ of length $> (n - 1)\lambda(R^n)/n$, by its complementary subword, or deleting a trivial relator $x^{\epsilon}x^{-\epsilon}$, $\epsilon = \pm 1$, x a generator. Then in reducing $w^{-1}g$ at most $2\lambda(g)$ reductions are possible. For suppose one can perform k reductions. Let L = length of $w^{-1}g$ when no further reductions are possible. Now

$$L \leq \lambda(g) + \lambda(w) - k,$$

since every reduction will decrease the length of a word by at least 1. But $w^{-1} = (w^{-1}g)g^{-1}$ hence

$$\lambda(w^{-1}) \leq L + \lambda(g^{-1}).$$

That is

$$\lambda(w) \leq L + \lambda(g)$$

or $\quad \lambda(w) \leq \lambda(g) + \lambda(w) - k + \lambda(g)$

hence $\quad k \leq 2\lambda(g)$.

Secondly, the final segment of w^{-1} that can be altered by reductions of $w^{-1}g$ is of length at most $2\lambda(g)\lambda(R^n)$. For since g, w^{-1} are reduced and w^{-1} is minimal then the reductions of $w^{-1}g$ can involve at most $2\lambda(g)$ reductions and each reduction after the first must be possible because of the effect of the previous reductions. Thus the first reduction can alter at most the last $\lambda(R^n)$ letters of w^{-1}. The second reduction can alter at most the next $\lambda(R^n)$ letters of w^{-1}, and so on. Altogether there can be at most $2\lambda(g)\lambda(R^n)$ letters altered.

Since the first letter of w^{-1} is not in Z, all of w^{-1} must be affected by reductions of $w^{-1}g$. Thus $\lambda(w) \leqslant 2\lambda(g)\lambda(R^n)$. Although G may be infinitely generated it suffices to have w involving at most the generators occurring in g and R. Thus the algorithm is simply, given g, test for the finitely many words $w(\underset{\approx}{W})$ if

$$w^{-1}(\underset{\approx}{W})g = z(\underset{\approx}{Z})$$

(the extended word problem!).

In 1962 Lyndon posed the problem (Problem 3.6): Let F be free on a set X of generators and let $R \in F$ and $N = gp_F(R)$. Let $\underset{\approx}{W}$ and $\underset{\approx}{Z}$ be subsets of X. Is

$$gp(\underset{\approx}{W})gp(\underset{\approx}{Z})N$$

a recursive subset of F? This question is answered in the affirmative for R a proper power, by the corollary above. Lyndon points out that a solution of this problem enables one to extend both the Magnus solution of the extended word problem and the Hauptform of the word problem.

Although our proof of the Spelling Theorem uses the Freiheitssatz it should be pointed out that our induction hypothesis is stronger than the Freiheitssatz and the proof could easily be presented so as to avoid using the Freiheitssatz. The Spelling Theorem allows one to generalize the Freiheitssatz as follows.

Corollary 2.1.6 Let $G = gp(a, b, c, \ldots \mid R^n)$, $n > 1$, where R is cyclically reduced, involving a, b non-trivially, and suppose β is any integer which does not divide the a-exponents in R^n. Then

$$a^\beta, b, c, \ldots$$

freely generate a subgroup of G.

Proof Consider any freely reduced word W in $a^\beta, b, \ldots$. If W represents the identity element in G, then W contains a subword identical to a subword of $R^{\pm n}$ of length $> (n - 1)\lambda(R^n)/n$. This can occur only if R^n is $[a^\alpha X(b, c, \ldots)]^2$ for some integer α and word X in b, c, ... Without loss of generality one may assume the relator is

$$[a^\alpha X(b, c, \ldots)]^2 \qquad \alpha > 0.$$

The corollary will be proved by induction on $\lambda(R^n)$. The corollary is clearly true if $\lambda(R^n) \leqslant 4$. Inductively assume the result is true for all groups with length $< \lambda(R^n)$.

Case 1 Suppose R involves two generators a and b only. Here the relator is $(a^\alpha b^r)^2$, and without loss of generality assume $r > 0$, and G involves only a, b. First embed

G in $H = gp(x, y \mid (x^{-r\alpha}(x^{\alpha}y)^{r})^{2})$ by mapping

$$a \to x^{-r},\ b \to x^{\alpha}y.$$

One now looks at $N = gp_H(y)$ and as usual takes

$$N_0 = gp(y_0, y_1, \ldots, y_{(r-1)\alpha} \mid (y_{(r-1)\alpha} \cdots y_{\alpha} y_0)^2).$$

Now let $W(a^{\beta}, b)$ be rewritten in H as $W(x^{-\beta r}, x^{\alpha}y)$. For $W = 1$, one requires that this equation takes place in N, and written as a relation in y_i, it must contain as a subword

$$(y_{t\alpha+k} \cdots y_k y_{(r-1)\alpha+k} \cdots y_{(t+1)\alpha+k} y_{t\alpha+k})^{\pm 1}$$

for some $t \in \{0, \ldots, r-1\}$. But this subword when rewritten in H is

$$(x^{-k}x^{-t\alpha}yx^{t\alpha+k} \ldots x^{-k}yx^{k}x^{-(r-1)\alpha-k}yx^{(r-1)\alpha+k} \ldots x^{-t\alpha-k} yx^{t\alpha+k})^{\pm 1}$$

or

$$x^{u}[(x^{\alpha}y)^{t+1}x^{-r\alpha}(x^{\alpha}y)^{r-t}]^{\pm 1}x^{v}$$

for some integers u, v. But this can occur as a subword of $W(x^{-\beta r}, x^{\alpha}y)$ only if $W(a^{\beta}, b)$ contains as a subword

$$b^{t+1}a^{\alpha}b^{r-t}.$$

This is impossible since β does not divide α.

Case 2 Suppose R involves more than two generators, say a, b, c, t. By using the usual embedding procedure one may, without loss of generality assume $\sigma_b(R) = 0$ and generators of G are a, b, c, t. Let $N = gp_G(a, c, t)$ and take

$$N_0 = gp(a_0, c_i, t_i, \text{ (all integers } i) \mid [a_0^{\alpha} X(c_i, t_i)]^2)$$

and

$$N = \{\prod_{i=-\infty}^{\infty} {}^{*} N_i \ ; \ gp(c_i, t_i)\} .$$

On rewriting $W(a^{\beta}, b, c, t) = 1$ as a relation in N one obtain

$$\prod_{\lambda=1}^{m} a_{\lambda}^{r_{\lambda}\beta} V_{\lambda}(c_i, t_i) = 1, \quad \lambda, m, r_{\lambda} \text{ integers.}$$

But if the word on the left-hand side is to be 1, then one can subdivide it into words in the components N_i, and some such subword must lie in the amalgamated subgroup. Thus, for example, a subword

$$V_{\nu} a_{\nu+1}^{r_{\nu+1}\beta} V_{\nu+1} \ldots a_{v}^{r_v \beta} V_v$$

where $a_{\nu+1} = a_{\nu+2} = \ldots = a_v$, must be a word in c_i, t_i. But this immediately implies a non-trivial relation in $gp(a_v^{\beta}, c_i, t_i)$ which contradicts the induction hypothesis. Thus case 2 is proved and so the corollary.

An important conjecture in the theory of one-relator groups which is still unconfirmed is that one-relator groups with torsion are residually finite. One can show that such groups need not be residually torsion-free nilpotent, for example take

$$G = gp(a, b \mid (a^{-1}b^{-1}ab^2)^2).$$

Here $N = gp_G(a, b^2)$ has a presentation

$$gp(x, y, z \mid [y, z^{-1}][x^{-1}, y^{-1}]y)$$

under the mapping $x \to a$, $y \to b^2$, $z \to bab^{-1}$, and clearly y is a non-trivial element contained in all terms of the lower central series.

In the direction of residual properties we can easily prove the following.

Corollary 2.1.7. Let $G = gp(a, b, c, \ldots \mid R^n(a, b, c, \ldots))$ $n > 1$. Then G is residually a two-generator one-relator group with torsion.

Proof Without loss of generality assume G has three generators a, b, c each non-trivial in R and $\sigma_a(R) = 0$. Let $v(a, b, c)$ be any non-trivial element of G, and without loss of generality assume $v(a, b, c)$ is a minimal word. Put

$$w = x^m y x^{-2m} y x^m$$

where m is large compared to the lengths of R and v. Define

$$H = gp(x, y \mid R^n(x, y, w)),$$

and denote $R^n(x, y, w)$ by $\bar{R}^n(x,y)$ when cyclically reduced. Note that $\sigma_x(\bar{R}) = 0$. The mapping

$$a \to x, \; b \to y, \; c \to w$$

defines a homomorphism of G onto H. The corollary is proved if it can be shown that

$$v \to \bar{v}(x,y) \neq 1, \; \bar{v}(x,y) \text{ freely reduced.}$$

If $\sigma_x(\bar{v}) \neq 0$ then $\bar{v} \neq 1$ in H. Hence suppose $\sigma_x(\bar{v}) = 0$. Construct $N = gp_H(y)$ in the usual way from

$$N_0 = gp(y_0, \ldots, y_\mu \mid \bar{R}_0{}^n(y_0, \ldots, y_\mu)).$$

Let $\bar{v}$ be rewritten in terms of the generators of N

$\bar{v} = \bar{\bar{v}}(y_{i,1},\ldots,y_{i,2})$. If $\bar{\bar{v}} = 1$ then $\bar{\bar{v}}$ contains as a subword more than $(n - 1)/n$ of $\bar{\bar{R}}_i{}^n$ by Lemma 2.1.2. Hence by Lemma 2.1.3 more than $(n - 1)/n$ of $\bar{R}^n$ occurs as a subword of $\bar{v}$, even if one disregards x terms at the beginning or end of the subword, (and this is the key observation). Hence, because in freely reducing $v(x, y, w)$ only x terms can cancel, v contains as a subword more than $(n - 1)/n$ of R^n, contradicting the supposed minimality of v.

Section 2.2 The theory of malnormal subgroups.

Let H be a subgroup of a group G. Then H is a malnormal subgroup of G if for all $g \in G$

$$g^{-1}Hg \cap H \neq 1 \text{ implies } g \in H.$$

It is clear from the definition that no element outside a malnormal subgroup H can commute with a non-trivial element of H. In particular no non-trivial element of H can have a root outside H, so a malnormal subgroup is a π-pure subgroup for any set π of primes.

Lemma 2.2.1 Let $C = \{A * B\ ;\ J\}$ where J is a malnormal subgroup of the factors A and B. Then A and B are malnormal subgroups of C.

Proof From the symmetry between A and B in C it will suffice to prove that A is a malnormal subgroup of C. Suppose

$$g^{-1}a_1g = a_2,\ g \in G,\ 1 \neq a_1,\ a_2 \in A.$$

Let coset representatives of J in A and B be chosen and suppose the normal forms for g, a_1, a_2 are

$$g = s_1 s_2 \ldots s_n \hat{j}, \quad a_1 = t_1 j_1, \quad a_2 = t_2 j_2$$

where $t_1, t_2 \in A$, $\hat{j}, j_1, j_2 \in J$ and $s_1, s_2, \ldots, s_n$ are coset representatives alternating from A and B. It is required to prove that $g \in A$. This will be done by induction on $|g|$, the length of g in normal form. If $|g| = 0$, then $g \in J$ and so $g \in A$. Suppose that for all elements $g \in G$ it has been shown that $g^{-1}a_1 g = a_2$ implies $g \in A$ if $|g| < n$, a_1, a_2 non-trivial elements of A. Let $|g| = n > 0$. Then

$$t_1 j_1 s_1 s_2 \ldots s_n \hat{j} = a_1 g = g a_2 = s_1 s_2 \ldots s_n \hat{j} t_2 j_2.$$

If $t_1 = 1$ then

$$(s_1^{-1} j_1 s_1) s_2 \ldots s_n \hat{j} = s_2 \ldots s_n \hat{j} t_2 j_2$$

and this implies $s_1 \in J$, which is absurd since s_1 is a non-trivial coset representative of J in A or B. If $t_1 \neq 1$ then $(s_1^{-1} t_1 j_1 s_1) s_2 \ldots s_n \hat{j} = s_2 \ldots s_n \hat{j} t_2 j_2$. This implies s_1, t_1 belong to the same factor and

$$s_1^{-1} t_1 j_1 s_1 = j_3 \in J.$$

Hence

$$j_3 s_2 \ldots s_n \hat{j} = s_2 \ldots s_n \hat{j} t_2 j_2$$

and by induction

$$s_2 \ldots s_n \hat{j} \in A.$$

Hence $s_1 s_2 \ldots s_n \hat{j} \in A$. This covers all possibilities and so proves the lemma.

Clearly a malnormal subgroup of a malnormal subgroup is a malnormal subgroup, hence any malnormal subgroup of A is a malnormal subgroup of C, in the notation above.

Lemma 2.2.2 Let $C = \{A * B \; ; \; J\}$ where A and B have all soluble subgroups cyclic and J is a malnormal subgroup of A and B. Then C has all its soluble subgroups cyclic.

Proof Let P be any non-trivial soluble subgroup of C, of soluble length r. Let $\delta(G)$ denote the commutator subgroup $[G, G]$ and define inductively

$\delta^{i}(G) = \delta(\delta^{i-1}(G))$, $i = 1, 2, 3, \ldots$, where $\delta^{0}(G) = G$. Then $\delta^{r-1}(P) \neq 1 = \delta^{r}(P)$. The proof will be divided into two cases.

Case 1 Suppose $\delta^{r-1}(P) \cap A \neq 1$. Let $a \in \delta^{r-1}(P) \cap A$, $a \neq 1$, and let g be any non-trivial element of P. Then every element of $\delta^{r-1}(P)$ commutes with a and so lies in A, since A is a malnormal subgroup of C. Hence $\delta^{r-1}(P) \subset A$, so $\delta^{r-1}(P) \cap A$ is a normal subgroup of P. Again malnormality implies $P \subset A$ and so P is a cyclic group.

Without loss of generality one may now assume that $\delta^{r-1}(P)$ has trivial intersection with A and B.

Case 2 Suppose $\delta^{r-1}(P) \cap A = 1 = \delta^{r-1}(P) \cap B$. Let g and u be any non-trivial elements of $\delta^{r-1}(P)$, where in normal form

$$g = s_1 s_2 \dots s_n j_1, \quad u = t_1 t_2 \dots t_m j_2,$$

where $n, m > 1$, $j_1, j_2 \in J$, and s_i, t_i are coset representatives of J in A or B lying alternatively in A or B. By taking a suitable conjugate of P one can, without loss of generality, assume that g is cyclically reduced of length $n > 1$. Then u is cyclically reduced of length $m > 1$, since g and u commute, and $|u| \neq 0$. Suppose u is chosen so that $|u|$ is the minimal length of any non-trivial element of $\delta^{r-1}(P)$. Without loss of generality let u have its coset representatives t_1, t_m belonging to A, B respectively. It will be shown that $\delta^{r-1}(P) = gp(u)$. For

$$\begin{aligned} g &= ugu^{-1} \\ &= ug_1 \text{ or } g_1 u^{-1} \end{aligned}$$

for some element g_1 with $|g_1| < |g|$. Using the natural induction on $|g|$ one proves g is a power of u. This proves $\delta^{r-1}(P)$ is a cyclic group generated by u.

Suppose v is any non-trivial element of $\delta^{r-2}(P)$ and suppose v has normal form

$$v = r_1 r_2 \dots r_q \bar{j}$$

where $\bar{j} \in J$ and r_i are coset representatives in A or B alternatively. First it will be shown that v is cyclically reduced of length $q > 1$. For suppose $r_1, r_q \in A$. Then

$$vuv^{-1} = u^s$$

for some integer s, from the preceeding work. If s is positive then

$$|vu| \leq |v| + |u|$$

and $|u^s v| = |v| + s|u|$.

This implies $s = 1$, hence $vu = uv$. This implies that v is cyclically reduced of length > 1 or $v \in J$. But J is a malnormal subgroup of C so v cannot belong to J since u does not belong to J. Hence v is cyclically reduced of length > 1. If s is negative then $v^2 u = u^{s^2} v^2$ and so the case where the exponent of u is positive is applicable, and thus v^2 is cyclically reduced of length > 1. If r_1, $r_q \in B$, one may repeat the argument using u^{-1} instead of u.

It has now been proved that every non-trivial element of $\delta^{r-2}(P)$ is cyclically reduced with length > 1. Suppose w is a non-trivial element of $\delta^{r-2}(P)$ with minimal length in $\delta^{r-2}(P)$. It will now be shown that

$$\delta^{r-2}(P) = gp(w).$$

For $w^{-1}uw = u^s$ for some integer, hence

$$\begin{aligned} u &= wu^s w^{-1} \\ &= wu_1 \text{ or } u_1 w^{-1}, \end{aligned}$$

for some element u_1 with $|u_1| < |u|$. Again using the natural induction on $|u|$ one proves that u is a power of w. The argument may be repeated to prove that not only is every element of $\delta^{r-1}(P)$ a power of w but that every element of $\delta^{r-2}(P)$ is a power of w. For

$$w^{-1}vwv^{-1} = w^t,$$

for some integer t, and so

$$\begin{aligned} v &= w^{t+1}vw^{-1} \\ &= w^{t+1}v_1 \text{ or } v_1w^{-1} \end{aligned}$$

for some element v_1 with $|v_1| < |v|$. Once again one may use induction to prove v is a power of w. Thus $\delta^{r-2}(P)$ is cyclic. This implies P is cyclic and completes the proof of the lemma.

Lemma 2.2.3 Let $C = \{A * B \,;\, J\}$ and suppose $g^{\alpha} = h^{\beta}$, $|g| \geq |h|$ and g^{α} is cyclically reduced of length > 1. Then $g = hg_1$, where $|g_1| < |g|$, and g_1 is cyclically reduced.

Proof Let $g = s_1s_2\ldots s_nj_1$ in normal form. Now h must be cyclically reduced of length > 1, say

$$h = t_1t_2\ldots t_mj_2 \qquad \text{where } 1 < m \leq n.$$

Then

$$g^{\alpha} = (s_1s_2\ldots s_n)j_1g^{\alpha-1} = (t_1t_2\ldots t_m)j_2h^{\beta-1} = h^{\beta}.$$

Now this relation implies that

$$s_1 = t_1,\ s_2 = t_2,\ \ldots,\ s_m = t_m.$$

Thus
$$\begin{aligned} g &= t_1t_2\ldots t_ms_{m+1}\ldots s_nj_1 \\ &= h(j_2^{-1}s_{m+1}\ldots s_nj) \\ &= hg_1 \qquad \text{where } g_1 = j_2^{-1}s_{m+1}\ldots s_nj \text{ is cyclic-} \end{aligned}$$
ally reduced with $|g_1| < |g|$.

Lemma 2.2.4 Let $C = \{A * B \,;\, J\}$ where J is a malnormal subgroup of A and B. If A and B are groups in which the centralizer of every non-trivial element is cyclic then the same is true of C.

Proof Let w be any element of C. Without loss of generality one may assume w is cyclically reduced. From Lemma 2.2.2, C has all Abelian subgroups cyclic. If w lies in A or B then the centralizer of w in C lies in the same subgroup since A and B are malnormal subgroups of C. Hence it suffices to assume that w has length > 1, and in fact no element of length < 2 is in the centralizer of w. Suppose w commutes with two elements $\underset{\sim}{g}$, $\underset{\sim}{h}$. Then there exist elements g, h such that

$$\underset{\sim}{g},\ w \in gp(g),$$

and

$$\underset{\sim}{h},\ w \in gp(h).$$

Thus $g^{\alpha} = h^{\beta}$ for some integers α, β, and clearly g, h are cyclically reduced with $|g| \geq |h| > 1$ or $|h| \geq |g| > 1$. Without loss of generality suppose $|g| \geq |h|$. By Lemma 2.2.3 $g = hg_1$ with $|g_1| < |g|$ and g_1 is cyclically reduced of length > 1 or else is the identity element. Since

$$|g_1| + |h| < |g| + |h|$$

one may use an induction argument on $|g| + |h|$ to conclude that g and h commute. Hence $\underset{\sim}{g}$, $\underset{\sim}{h}$ commute.

Thus the centralizer of w is Abelian, and so is cyclic.

The literature on malnormal subgroups is small. In fact it was not until after the completion of this work that it was brought to my attention that Benjamin Baumslag 1965 had used them in his doctoral thesis and named them malnormal. Tekla Lewin 1967 used malnormal subgroups to derive certain results on D-groups and the final lemma above is contained in her work. Malnormal subgroups are used by Driscoll 1967, in her work on the conjugacy problem, and A. Whitmore 1967 in her work on the Frattini subgroup.

Section 2.3 The Abelian subgroups of one-relator groups with torsion.

Lemma 2.3.1 Let $G = gp(a,b,\ldots,c,t \mid R^n)$ $n > 1$ where R is cyclically reduced.

I. Any subset of the generators of G generates a malnormal subgroup of G.

II. Suppose $w_1(a,b,\ldots,c)$ is a word which when cyclically reduced involves a non-trivially, and $w_2(b,\ldots,c,t)$ is a word which when cyclically reduced involves t non-trivially. Then w_1 and w_2 are not conjugates in G if R involves a, t non-trivially.

Proof The lemma will be proved by induction on $\lambda(R^n)$. If $\lambda(R^n) < 4$ then the lemma is trivially true, as it is if R involves only one generator. Without loss of generality assume R involves all the generators non-trivially. To simplify the notation assume G involves at most generators a, b, c, t.

Firstly consider I. Here it suffices to prove that $H = gp(b, c, t)$ is a malnormal subgroup of G for one may easily prove a lemma analagous to Lemma 1.2.1.

Case 1 Suppose G has two generators a, b and $\sigma_a(R) = 0$, and $g^{-1}b^m g = b^r$, $g \in G$, m, r integers $r \neq 0$. It is required to prove that g is a power of b. Let $N = gp_G(b)$ and construct N from copies of

$$N_0 = gp(b_0,\ldots,b_\mu \mid R_0{}^n)$$

where $\lambda(R_0{}^n) < \lambda(R^n)$. Now by induction

$$J_1 = gp(b_1,\ldots,b_\mu)$$

is a malnormal subgroup of N_0 and N_1 and so N_0 is a malnormal subgroup of $K_1 = gp(N_0, N_1)$ by Lemma 2.2.1. One continues as in the proof of Lemma 1.2.2 to prove that $K_i = gp(N_0,\ldots,N_i)$ is a malnormal subgroup of N for all integers i. In particular N_0 is malnormal in N.

Assume $\sigma_a(g) = s \geq 0$, $g = a^s\bar{g}$, and

$$\bar{g}^{-1}b_s{}^m\bar{g} = b_0{}^r \tag{1}$$

If $s = 0$, then $\bar{g} \in N_0$ since N_0 is a malnormal subgroup of N. Since by induction $gp(b_0)$ is malnormal in N_0, then $\bar{g} = b_0{}^p$ for some integer p, whence g is a power of b. If $0 < s \leq \mu$ then again $\bar{g} \in N_0$. But the equation

(1) is impossible in N_0 by the induction hypothesis. If $s = \mu + k$, $k > 0$, then $\bar{g} \in K_k$ since K_k is malnormal in N. Write

$$K_k = \{N_0 * gp(N_1,\ldots,N_k) ; J_1\}.$$

Then $b_0{}^r \in N_0$, $b_s{}^m \in gp(N_1,\ldots,N_k)$, and equation (1) is possible only if $b_0{}^r$ is conjugate in N_0 to an element of J_1. This is impossible by the induction hypothesis.

<u>Case 2</u> Suppose G has two generators a and b with $\sigma_a(R) \neq 0$. Without loss of generality one may assume $\sigma_b(R) = 0$ (using the usual embedding if necessary). Here

$$g^{-1}b^m g = b^m \qquad (2)$$

and without loss of generality one may assume $\sigma_b(g) = 0$. One may assume m is large and $m > 0$. Let $N = gp_G(a)$ and construct N from

$$N_0 = gp(a_0,\ldots,a_\mu \mid R_0{}^n).$$

Let g be denoted by g_0 when rewritten as an element of N. One may assume that of all words representing the same element as g_0, g_0 has minimal length. If g_0 contains some a_i without loss of generality assume g_0 is a word in

$$a_0,\ldots,a_\lambda,\ \lambda \geq 0.$$

Then equation (2) becomes

$$g_0(a_m,\ldots,a_{\lambda+m}) = g_0(a_0,\ldots,a_\lambda).$$

This relation in N implies g_0 has the generator a_0 removable, hence by Lemma 2.1.2, g_0 has its length reducible, contradicting the choice of g_0. Thus g_0 involves no a_i term and so g is a power of b.

Case 3 Suppose G has more than two generators. Without loss of generality one may assume $\sigma_b(R) = 0$. Let $N = gp_G(a, c, t)$ and construct N from

$$N_0 = gp(a_0, \ldots, a_\mu, c_i, t_i \text{ (all integers i) } \mid R_0^n).$$

As before one proves that $gp(c_i, t_i$ (all integers i)) is a malnormal subgroup of N. Suppose now

$$g^{-1}w_1(b,c,t)g = w_2(b,c,t)$$

where

$$g = b^r g_0, \quad w_1 = b^s \bar{w}_1, \quad w_2 = b^s \bar{w}_2$$

for $g_0, \bar{w}_1, \bar{w}_2 \in N$. Then

$$b^{-s}g_0^{-1}b^s b^{-r}\bar{w}_1 b^r g_0 = \bar{w}_2.$$

Rewriting in N one has

$$g_0^{-1}(a_{i+s}, c_{j+s}, t_{k+s}) = \bar{w}_2(c_i, t_i)g_0^{-1}(a_i, c_j, t_k) \bar{w}_1^{-1}(c_i, t_i) \qquad (3)$$

Now if s = 0 then

$$g_0^{-1}(a_i, c_j, t_k)\bar{w}_1(c_i, t_i)g_0(a_i, c_j, t_k) = \bar{w}_2(c_i, t_i)$$

whence $g_0 \in gp(c_i, t_i$ (all i)). Hence $g \in gp(b, c, t)$. Thus one may assume $s \neq 0$, and that of all words representing the same element as g_0, g_0 has minimal length. But equation (3) shows that some a_i term can be removed from g_0, contradicting the choice of g_0.

Thus g_0 does not involve a_i terms and so $g \in gp(b,c,t)$.

This completes the proof of I. One now proves II.

Case 4 Suppose G involves only two generators a and b and $g^{-1}a^m g = b^r$. It is clear that $\sigma_a(R) \neq 0$, $\sigma_b(R) \neq 0$ otherwise there is nothing to prove. Without loss of generality assume r is large. Let $\sigma_a(R) = -\alpha$, $\sigma_b(R) = \beta$. As usual embed G in

$$H = gp(x,y \mid R^n(x^\beta, x^\alpha y)),$$

by mapping $a \to x^\beta$, $b \to x^\alpha y$. The equation $g^{-1}a^m g = b^r$ becomes in H

$$g_1{}^{-1}x^{m\beta}g_1 = (x^\alpha y)^r$$

where g maps into g_1. Without loss of generality assume $\alpha r > 0$, and $\sigma_x(g_1) = 0$. Let $N = gp_H(y)$ and construct N from

$$N_0 = gp(y_0,\ldots,y_\mu \mid R_0{}^n).$$

Since $m\beta = \alpha r$ the equation above may be written as a relation in N,

$$g_0{}^{-1}(y_{i+\alpha r})g_0(y_i) = y_{\alpha r-r}y_{\alpha r-2r}\cdots y_0 \tag{4}$$

where g_0 is g_1 rewritten. One may assume that $g_0(y_i)$ is the word of minimal length representing the element g_1. Now either the minimal or maximal y_i in $g_0{}^{-1}(y_{i+\alpha r})g_0(y_i)$ can be removed. Hence this word has as a subword xx^{-1} or more than half of $R_i{}^{\pm n}$ for some integer i. By choosing r sufficiently large one may assume that such a subword occurs entirely within

$g_0^{-1}(y_{i+\alpha r})$ or $g_0(y_i)$. In either case this contradicts the minimality of $g_0(y_i)$. Thus g_1 is a power of x, hence g is a power of a, in which case $g^{-1}a^m g = b^r$ is impossible by the Spelling Theorem.

<u>Case 5</u> Suppose G involves more than two generators. Without loss of generality assume $\sigma_b(R) = 0$. Let $N = gp_G(a,c,t)$ and construct N from

$$N_0 = gp(a_0,\ldots,a_\mu,c_i,t_i \text{ (all integers i)} \mid R_0{}^n).$$

Let $g^{-1}w_1 g = w_2$ and suppose $w_1 = b^r\bar{w}_1$, $w_2 = b^r\bar{w}_2$ where $\bar{w}_1, \bar{w}_2 \in N$.

Suppose r = 0. The relation now is

$$g^{-1}\bar{w}_1(a_{i_1},\ldots,a_{i_2},c_i)g = \bar{w}_2(c_i,t_i) \qquad (5)$$

where $\bar{w}_1$, $\bar{w}_2$ when cyclically reduced contain some a_i, t_i terms, respectively. If $i_1,\ldots,i_2$ lies in the range $0,\ldots,\mu$ then $\bar{w}_1, \bar{w}_2 \in N_0$, whence $g \in N_0$. By the induction hypothesis this equation is impossible. Suppose that the relation (5) takes place in $K_k = gp(N_0,\ldots,N_k)$, $k > 0$. Without loss of generality assume that of all words representing conjugates of the element $\bar{w}_1$, the word $\bar{w}_1(a_0,\ldots,a_{\mu+k},c_i)$ has minimal length, and also a_0, $a_{\mu+k}$ occur non-trivially in $\bar{w}_1$. Inductively assume that the relation (5) is impossible in K_{k-1}. Put

$$K_k = \{N_0 * gp(N_1,\ldots,N_k)\ ;\ J_1\}.$$

Now $\bar{w}_2 \in J_1$. If $\bar{w}_1$ is cyclically reduced then it must be in one of the factors. This however implies

$$\bar{w}_1(a_0,\ldots,a_{\mu+k},c_i) = W(a_0,\ldots,a_\mu,c_i,t_i)$$

or

$$\bar{w}_1(a_0,\ldots,a_{\mu+k},c_i) = W(a_1,\ldots,a_{k+\mu},c_i,t_i)$$

for some word W. In either case $\bar{w}_1$ has a generator removable which implies $\bar{w}_1$ may be reduced in length contradicting the choice of $\bar{w}_1$. Hence assume $\bar{w}_1$ is not cyclically reduced. But by a trivial modification of g one may assume

$$\bar{w}_1 = u_1(a_1,\ldots,a_{k+\mu},c_i)a_0^{\alpha_1}u_2(a_1,\ldots,a_{k+\mu},c_i)\ldots a_0^{\alpha_m}.$$

where $\alpha_i \neq 0$, $a_{k+\mu}$ is non-trivial in u_1. If now $\bar{w}_1$ is not cyclically reduced then $\bar{w}_1$ can be shortened, again contradicting the choice of $\bar{w}_1$.

Suppose $r \neq 0$. Let $g^{-1}b^r\bar{w}_1g = b^r\bar{w}_2$. Take the m-th power of both sides, m an integer,

$$g^{-1}(b^r\bar{w}_1)^m g = (b^r\bar{w}_2)^m.$$

Since $b^r\bar{w}_1$ does not involve t and when cyclically reduced contains a non-trivially, then $(b^r\bar{w}_1)^m$ does not involve t and when cyclically reduced contains a non-trivially. Similarly $(b^r\bar{w}_2)^m$ does not involve a and when cyclically reduced contains t non-trivially. Hence $(b^r\bar{w}_1)^m$ and $(b^r\bar{w}_2)^m$ satisfy the same hypothesis as for w_1 and w_2. Thus one may assume r is large. Let

$$w_1 = b^r\bar{w}_1(a_{i_1},\ldots,a_{i_2},c_i)$$
$$w_2 = b^r\bar{w}_2(t_i,c_i)$$

$$g = g(a_{i_3},\ldots,a_{i_4},c_i,t_{j_3},\ldots,t_{j_4})$$

and every free reduction of a cyclic permutation of $b^r\overline{w}_1$ contains an a-term. Assume $i_3 + r > i_4 + \mu$. The equation now is

$$\overline{w}_1(a_{i_1},\ldots,a_{i_2},c_i)g(a_{i_3},\ldots,a_{i_4},c_i,t_{j_3},\ldots,t_{j_4})$$
$$= g(a_{i_3+r},\ldots,a_{i_4+r},c_{i+r},t_{j_3+r},\ldots,t_{j_4+r})\overline{w}_2(t_i,c_i). \tag{6}$$

Now the a_i with $i < i_3 + r$ appearing on the left hand side of (6) are removable. Without loss of generality one may assume the left hand side of (6) is freely reduced, hence $\overline{w}_1 g$ contains a subword identical with a subword of $R_i^{\pm n}$ of length $> \frac{1}{2}\lambda(R_i^{\,n})$. One may assume $\overline{w}_1$, g are written as words of minimal length. Since every subword which is more than half $R_i^{\pm n}$ contains some t_j for each t_j appearing in R_i, one is restricted in the choice of such R_i, namely to those R_i with t_j in the range $t_{j_3},\ldots,t_{j_4}$. When such a subword is replaced to shorten $\overline{w}_1 g$ no new letters will be introduced, certainly no new t_j. Since no free cancellation can take place, the only a_i that are removable are those a_i appearing in this restricted set of R_i. If there is some a_i not in this range with $i < i_3 + r$ then one is finished. Suppose there are no a_i other than the a_i which are removable together with a_i in the range $a_{i_3+r},\ldots,a_{i_4+r}$. Then all the removable a_i in $\overline{w}_1$ must occur at the end of $\bar{w}_1$ so take

$$\bar{w}_1 = W_1X, \; g = X^{-1}h$$

where X involves only a_i, c_i, and h involves c_i, t_i. The relation $g^{-1}b^r\bar{w}_1 g = b^r\bar{w}_2$ may now be rewritten

$$h^{-1}Xb^rW_1XX^{-1}h = b^r\bar{w}_2$$

or

$$h^{-1}(Xb^rW_1)h = b^r\bar{w}_2.$$

Now a is removable from the left hand side, hence a is removable from Xb^rW_1. Since no t occurs in Xb^rW_1 the a is removable by free reduction. But this implies a is removable from a cyclic permutation of $b^r\bar{w}_1$ ($= b^rW_1X$) by free reduction, contradicting an earlier remark.

This completes the proof of Lemma 2.3.1.

We now have all the results needed to determine the Abelian subgroups of a one-relator group with torsion.

<u>Theorem 2.3.2</u> The Abelian subgroups of a one-relator group with torsion are cyclic.

<u>Proof</u> Let $G = gp(a,b,c,\ldots \mid R^n)$ $n > 1$. The theorem will be proved by induction on $\lambda(R^n)$. The theorem is true if $\lambda(R^n) \leq 4$, as it is if R contains only one generator. Assume therefore that R contains more than one generator when cyclically reduced. By means of the usual embedding process one may, without loss of

generality, assume $\sigma_a(R) = 0$. Inductively assume the theorem has been proved for all groups with relator length $< \lambda(R^n)$. Let $N = gp_G(b,c,\ldots)$ and construct N using

$$N_0 = gp(b_0,\ldots,b_\mu,c_i,\ldots \text{ (all integers } i) \mid R_0{}^n).$$

Since $\lambda(R_i{}^n) < \lambda(R^n)$ the only Abelian subgroups of N_i are cyclic. The amalgamated subgroup J_1 is a malnormal subgroup of both N_0 and N_1. From Lemma 2.2.2 the only Abelian subgroups of $K_1 = gp(N_0,N_1)$ are cyclic. Similarly one may show that the only Abelian subgroups of $K_k = gp(N_0,\ldots,N_k)$ are cyclic. Since K_k is a malnormal subgroup of N no locally cyclic non-cyclic Abelian subgroups are contained in N. Hence the only Abelian subgroups of N are cyclic.

Let A be an Abelian subgroup of G not contained in N. Then

$$A/A \cap N = AN/N \subset G/N = \text{infinite cyclic group.}$$

Now A, being an Abelian infinite cyclic extension of a cyclic group is either cyclic or the direct product of a cyclic group and an infinite cyclic group, the latter infinite cyclic factor not in N. Suppose A is not cyclic. Let x, y be generators of A where $1 \neq y \in N$, $x = a^r\bar{x}$, r an integer and $\bar{x} \in N$. Without loss of generality assume r is positive and large. Let y when written as an element of N be $y_0(b_0,\ldots,b_{\mu+k},c_i,\ldots)$

and assume this is the shortest possible word representing the element y, and that $b_0, b_{\mu+k}, \mu + k \geq 0$ are non-trivial in y. One further assumes that of all possible non-cyclic Abelian subgroups of G one has chosen that for which $\mu + k$ is least. This implies that no conjugate of y has smaller $\mu + k$ value than y_0 has. Now

$$x^{-1}y^{-1}xy = 1$$

implies

$$\bar{x}^{-1}a^{-r}y^{-1}a^{r}\bar{x}y = 1$$

or

$$\bar{x}^{-1}y_r^{-1}\bar{x}y_0 = 1 \tag{1}$$

where $y_r = y_0(b_r, \ldots, b_{r+\mu+k}, c_{i+r}, \ldots)$. Since r is large take $r > \mu + k$. One now proves that equation (1) is impossible.

By Lemma 2.3.1 this relation cannot take place in N_0. Hence suppose $r + \mu + k > \mu$. Then y_0 and y_r are in K_{k+r} and since K_{k+r} is malnormal in N then $\bar{x} \in K_{k+r}$. Thus equation (1) takes place in K_{k+r}. Let

$$K_{k+r} = \{gp(N_0, \ldots, N_k) * gp(N_{k+1}, \ldots, N_{k+r}) \; ; \; J_{k+1}\}.$$

Now y_0 is in the first factor and y_r is in the second factor, hence y_0 and y_r can be conjugates only if y_0 is conjugate to an element of J_{k+1}. By the choice of y this implies that $0 \leq \mu + k \leq \mu - 1$ and $y_0 \in N_0$ and is conjugate in N_0 to an element of J_1. But again this is impossible by Lemma 2.3.1. Thus equation (1) is impossible, and the theorem is proved.

Corollary 2.3.3 The soluble subgroups of a one-relator group with torsion are cyclic.

Proof Suppose S is a metabelian subgroup of G where the notation of Theorem 2.3.2 is used. If $S \subset N$ then by Lemma 2.2.2 and the usual argument, S is cyclic. Suppose therefore $S \not\subset N$ and let $x \in S$ but $x \notin N$. Clearly $\delta S \subset N$, and so δS is cyclic generated by y, say. Then for some integer s

$$x^{-1}yxy^{-1} = y^s.$$

Then the elements

$$y,\ xyx^{-1}, x^2yx^{-2}, \ldots$$

generate a non-cyclic locally cyclic subgroup of G, unless for some integer r

$$x^r y x^{-r} = y,$$

in which case x^r and y generate a non-cyclic Abelian subgroup of G, contradicting the result of the previous theorem. Thus the only metabelian subgroups are cyclic proving the corollary.

Corollary 2.3.4 The centralizer of every non-trivial element of a one-relator group with torsion is cyclic.

Proof Using the notation of Theorem 2.3.2 one first shows that the centralizer in N of every non-trivial element of N is cyclic. This is proved by the usual induction argument together with Lemma 2.2.4. Since by the theorem above the Abelian subgroups of G are cyclic, no element outside N can commute with a non-trivial element inside N. Hence the centralizer in G of every element in N is cyclic. Let $w \notin N$ and suppose w commutes with g and h. Then w commutes with $g^{-1}h^{-1}gh$ which lies in N, hence $g^{-1}h^{-1}gh = 1$. Thus g and h generate a cyclic group, and the centralizer of w is cyclic. This proves the corollary.

Chapter 3

The conjugacy problem for one-relator groups with torsion

Algorithmic solutions of the word problem, the conjugacy problem and the isomorphism problem were formulated and investigated by Dehn 1912. He showed all these problems were solvable for the one-relator groups

$$G_k = gp(a_1, b_1, \ldots, a_k, b_k \mid [a_1, b_1] \ldots [a_k, b_k]),$$

the fundamental groups of closed, orientable, two dimensional surfaces. In 1954 Novikov proved the conjugacy problem is unsolvable in general, even for a class of finitely presented groups having a solvable word problem. In this chapter we prove that the conjugacy problem, and the extended conjugacy problem in certain cases, are solvable for one-relator groups with torsion.

We describe precisely what is meant by these problems. Let G be a group with a given presentation, that is

$$G = gp(x_1, x_2, \ldots \mid R_1(x_1, x_2, \ldots), R_2(x_1, x_2, \ldots), \ldots)$$

where the x_i are a possibly infinite but recursively enumerable set of generators, and the $R_i(x_1, x_2, \ldots)$ are a recursive set of defining relators of G.

The conjugacy problem is as follows: For any pair of words $w_1(x_1,x_2,\ldots)$, $w_2(x_1,x_2,\ldots)$ in the generators of G, give an effective procedure for determining in a finite number of steps whether or not w_1 and w_2 define conjugate elements of G.

The extended conjugacy problem for G relative to a subgroup H is as follows: Let

$$H = gp(w_1(x_1,x_2,\ldots),\ w_2(x_1,x_2,\ldots),\ldots)$$

where the w_i are words in the generators of G. For an arbitrary element $w(x_1,x_2,\ldots)$ of G, give an effective procedure to determine in a finite number of steps whether or not w is conjugate to an element w* of H, and if it is, to express w* in terms of the given generators of H.

In Section 3.1 we develop the basic theory of strongly-malnormal subgroups which plays a role similar to that of p-pure subgroups in Chapter 1, and malnormal subgroups in Chapter 2. In Section 3.2 we use strongly-malnormal subgroups to solve the conjugacy problem for one-relator groups with torsion, and solve the extended conjugacy problem relative to any subgroup generated by a subset of the generators.

In Section 3.3 we consider the problem of determining the roots of an element in a one-relator group with torsion.

Section 3.1 The theory of strongly-malnormal subgroups.

Let J and A be groups with given presentations, and let there be given an isomorphism of J into A. We say J is strongly-malnormal in A if

(i) J is a malnormal subgroup of A,

(ii) the word problem is solvable in J for the given presentation of J,

and (iii) given any pair of elements $g, h \in A$ as words in the generators of A, one can

(a) algorithmically determine if there exist elements $j, j_1 \in J$ such that $jh = gj_1$,

and (b) algorithmically determine j, j_1 when they exist, as words in the given generators of J.

Note that j, j_1 will be unique if $g \notin J$. For, suppose there exists another pair $k, k_1 \in J$ with $kh = gk_1$. Then

$$k^{-1}gk_1 = h = j^{-1}gj_1,$$

hence $$g^{-1}(jk^{-1})g(k_1j_1^{-1}) = 1.$$

Since J is malnormal in A and $g \notin J$, then

$$jk^{-1} = k_1j_1^{-1} = 1,$$

thus proving the uniqueness of j, j_1.

Note also that if J is a strongly-malnormal subgroup of A, then we can solve the extended word problem of A with respect to J, and hence we can solve the word problem for A since the word problem for J is solvable.

To see this, let g be any element of A given as a word in the generators of A. Then one can decide if there exist elements $j_1, j_2 \in J$ such that

$$j_1 g = 1 j_2,$$

and if so determine j_1, j_2. The element $g \in J$ if and only if j_1, j_2 exist, in which case

$$g = j_1^{-1} j_2,$$

and not only can we determine algorithmically if g is in J but we can write g as an element of J.

If H is a strongly-malnormal subgroup of G then it will be understood that presentations for H and G are given and there is some effective procedure for writing an arbitrary word in the generators of H, as a word in the generators of G.

<u>Lemma 3.1.1</u> A strongly-malnormal subgroup K of a strongly-malnormal subgroup H of G is a strongly-malnormal subgroup of G.

<u>Proof</u> Firstly K is a malnormal subgroup of G. Secondly the word problem is solvable in K since K is strongly-malnormal in H. Thirdly, let $g_1, g_2 \in G$. Since H is strongly-malnormal in G one can determine if there exist $h_1, h_2 \in H$ such that

$$h_1 g = g_2 h_2,$$

and if so, determine h_1, h_2. If $g_1 \notin H$ then h_1, h_2 are unique if they exist, and so it suffices to determine if h_1, $h_2 \in K$. This can be done since the extended word problem of H with respect to K is solvable.

If $g_1 \in H$ then $g_2 \in H$ if h_1, h_2 exist. Hence g_1, g_2 can be written as words in the generators of H. Then one can determine if k_1, k_2 exist such that k_1, $k_2 \in K$ and

$$k_1 g_1 = g_2 k_2,$$

and if so, determine k_1, k_2. Thus, given g_1, $g_2 \in G$ one can determine if there exist k_1, $k_2 \in K$ such that $k_1 g_1 = g_2 k_2$, and if so, find k_1, k_2.

In writing a generalized free product as

$$C = \{A * B \ ; \ J\}$$

we assume throughout this chapter

(i) presentations for A, B, J are given,

(ii) isomorphisms of J into A, and into B are given by a specific process which allows one to write an element of J as an element of A or B,

(iii) the presentation for C has as generating symbols those of A and B, as relators those of A and B together with the elements of A in J identified with those of B in J.

In the case where J is strongly-malnormal in A and B, one can decide if an element of C is in A or B, and if it is, to write it as an element of A or B. This implies that an element g of C can be written in reduced form

$$g = g_1 g_2 \dots g_n$$

where the g_i alternate from the factors A and B, and $g_i \notin J$.

<u>Lemma 3.1.2</u> Let $C = \{A * B ; J\}$ where J is a strongly-malnormal **subgroup of** A **and** B. Then A and B are strongly-malnormal subgroups of C.

<u>Proof</u> By symmetry it will suffice to prove that A is strongly-malnormal in C. Firstly A is malnormal in C by Lemma 2.2.1. Secondly the word problem is solvable in A since J is strongly-malnormal in A. Thirdly, let g, h be any elements of C and write g, h in reduced form

$$g = g_1 g_2 \dots g_n, \ h = h_1 h_2 \dots h_m$$

where the g_i, $h_i \in A$ or B. It is necessary to decide if there exist elements a_1, $a_2 \in A$ such that

$$a_1 g_1 g_2 \dots g_n = h_1 h_2 \dots h_m a_2 .$$

Without loss of generality assume $|g| \leq |h|$. Suppose $|g| \geq 2$. There are various cases to consider. Suppose $g = g_A g_B \dots$, $h = h_A h_B \dots$. (This is to be interpreted as $g_A \in A$, $g_B \in B$ and $h_A \in A$, $h_B \in B$ and so on). Then $a_1 g_A = h_A j_1$ for some $j_1 \in J$, and $j_1^{-1} h_B = g_B j_2$, $j_2 \in J$. But there is an algorithm to determine if j_1, j_2 exist and if so, to determine j_1, j_2. Hence there is an algorithm to determine if a_1 exists and if so, to find a_1. If a_1 exists it is unique so one can determine $a_2 = h^{-1}a_1g$ and decide if $a_2 \in A$. The remaining cases are similar.

Suppose $|g| < 2$. If $g \in A$ then $a_1g = ha_2$ implies $h \in A$. Therefore there is nothing to prove since $1 \cdot g = h(h^{-1}g)$. If $g \in B$ and $g \notin J$ then h is (i) h_B or (ii) h_Bh_A or (iii) $h_Ah_B\ldots$. Consider these in turn:

(i) $a_1g = h_Ba_2$ implies $a_1 \in J$ and $a_2 \in J$ so one can decide if a_1, a_2 exist and if so determine them.

(ii) $a_1g = h_Bh_Aa_2$ implies $a_1 \in J$ and $a_1g = h_Bj_1$ for $j_1 \in J$, so one can decide if a_1 exists and if so determine a_1, and hence a_2 if it exists.

(iii) $a_1g = h_Ah_B\ldots a_2$ implies $a_1 = h_Aj_1$ for $j_1 \in J$ and $j_1^{-1}h_B = gj_2$ for $j_2 \in J$. Hence one can again decide if a_1, a_2 exist and if so determine a_1, a_2. This proves Lemma 3.1.2.

Lemma 3.1.3 Let $C = \{A * B \; ; \; J\}$ where J is a strongly-malnormal subgroup of A and B. Then given $g, h \in C$ one can

(i) determine algorithmically if there exist elements a, b in A, B respectively such that

$$ag = hb,$$

and (ii) determine a, b algorithmically if they exist.

Moreover if a, b $\notin$ J then a and b are unique unless g and h lie in different factors or one (say g) is of the form $g_A g_B$ (in the notation above) while the other, h, is not of the form $h_B h_A$.

<u>Proof</u> The proof is similar to that of Lemma 3.1.2.

We now come to one of the key lemmas.

<u>Lemma 3.1.4</u> Let $C = \{A * B ; J\}$ where J is a strongly-malnormal subgroup of A and B. If the conjugacy problem and the extended conjugacy problem relative to J are solvable in the factors A and B, then the conjugacy problem is solvable in C, and the extended conjugacy problem relative to A and B.

<u>Proof</u> Since J is a strongly-malnormal subgroup of A and B then the extended word problem with respect to J is solvable in both A and B, and the extended word problem for C with respect to A and B. Let g, h be any pair of elements of C, and write them in reduced form

$$g = g_1 g_2 \dots g_n, \quad h = h_1 h_2 \dots h_m.$$

Without loss of generality assume g and h cyclically reduced.

<u>Case 1</u> Suppose $|g| > 1$. If g and h are conjugates then h must have the same length as g and a cyclic permutation of one must be conjugate to the other by

an element of J. Thus it suffices to consider if there exists an element $j \in J$ such that

$$j^{-1}(g_1 g_2 \ldots g_n)j = h_1 h_2 \ldots h_m.$$

This implies $j^{-1}g_1 = h_1 j_1$ for some $j_1 \in J$, which uniquely determines j. Thus there is an algorithm to determine j. Finally one checks if $j^{-1}gj = h$.

Case 2 Suppose $|g| = 1$, and g is not conjugate to an element of J. Suppose $g \in A$. Then h lies in A and one can write g, h as elements of A. But by hypothesis one can determine if g, h are conjugates in A, hence one can determine if g and h are conjugates in C.

Case 3 Suppose $|g| \leqslant 1$ and g is conjugate to an element of J. Then if g and h are conjugates, h must lie in one of the factors and g and h are each conjugate within one of the factors to an element of J. But the extended conjugacy problem relative to J is solvable by hypothesis in both A and B, so one can determine $j_1, j_2 \in J$ such that g and h are conjugate to j_1, j_2 respectively. Now if g and h are conjugates, then j_1, j_2 are conjugates in C and since J is malnormal in C, then j_1, j_2 are conjugates in J. But the conjugacy problem is solvable in J, so the conjugacy problem is solvable in C.

The solution of the extended conjugacy problem for C relative to the factor A and B is easy. For one can cyclically reduce an element $g \in C$, and if it is conjugate to an element of A say, then $|g| \leq 1$. If it lies in A one is finished. If it lies in B it must be conjugate in B to an element of J, and one can determine if this is so, and also determine some such element of J, since the extended conjugacy problem in A and B relative to J is solvable by hypothesis.

Lemma 3.1.5 If K is a malnormal subgroup of H, and H is a malnormal subgroup of G, and G has the extended conjugacy problem solvable relative to H, and H has the extended conjugacy problem solvable relative to K, then G has the extended conjugacy problem relative to K solvable.

Proof Let $g \in G$. One can determine if g is conjugate to an element $h \in H$ and if so determine h. If g is not conjugate to an element of H then g is not conjugate to an element of K. Now one can determine if h is conjugate in H to an element k of K, and if so determine k. If h is not conjugate in H to an element $k \in K$ then h is not conjugate in G to an element $k \in K$, for H is malnormal in G.

Section 3.2 The Conjugacy Problem.

In the previous section we proved the appropriate results as outlined in the introduction. We now follow the usual routine and apply these results to one-relator groups. First we prove a lemma to help simplify the work which follows.

Lemma 3.2.1 In order to prove that any subset of the generators of a one-relator group H with torsion generates a subgroup, which, with the obvious presentation, is a strongly-malnormal subgroup of H, it suffices to prove that in all groups

$$G = gp(a,b,\ldots,t \mid R^n) \quad n > 1,$$

where R is a cyclically reduced word in $a,b,\ldots,t$ involving $a,b,\ldots,t$ non-trivially, the

$$gp(b,\ldots,t)$$

is a strongly-malnormal subgroup of G.

Proof Let $H = gp(x_1,x_2,\ldots \mid R^n)$ $n > 1$, be any one-relator group with torsion. Without loss of generality assume R is a cyclically reduced word. Let $\{y_1,y_2,\ldots\}$ be any subset of the generators of H, and let $Y = gp(y_1,y_2,\ldots)$. If the set $\{y_1,y_2,\ldots\}$ is not a proper subset of the generators then $Y = H$, and Y is strongly-malnormal in H. If the subset is empty, Y is again strongly-malnormal in H.

Let $\{y_1,y_2,\ldots\}$ be a proper non-empty subset of the generators of H. Firstly suppose that every generator in R is in the set $\{y_1, y_2, \ldots\}$. A presentation for Y is

$$Y = gp(y_1, y_2, \ldots \mid R^n).$$

Then the generators of H may be split into two disjoint subsets,

$$\{y_1, y_2, \ldots\}, \ \{z_1, z_2, \ldots\}$$

where the z_i do not appear in R. These two subsets generate free factors Y, Z respectively such that

$$H = Y * Z.$$

Hence Y is a strongly-malnormal subgroup of H by Lemma 3.1.2, since the amalgamated subgroup (here trivial) is a strongly-malnormal subgroup of Y and Z.

Secondly suppose there exists a generator say x_i which is not in $\{y_1, y_2, \ldots\}$ but is in R. Let

$$X = gp(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots)$$

and X is a free group on these generators by the Freiheitssatz. It is clear that $Y \subset X$ and is free and a free factor of X. Hence Y is strongly-malnormal in X. It will suffice therefore to prove that X is strongly-malnormal in H. Let

$$G = gp(a, b, \ldots, t \mid R^n) \quad n > 1$$

be obtained from H by deleting those generators which do not occur in R and writing $x_i = a$ and the remaining generators in R as $b, \ldots, t$. Then

$$H = \{G * X \; ; \; gp(b,\ldots,t)\}.$$

In order to prove X is strongly-malnormal in H it will suffice, by Lemma 3.1.2, to prove that the amalgamated subgroup is strongly-malnormal in both G and X. But it is strongly-malnormal in X. Hence the problem is reduced to proving that $gp(b,\ldots,t)$ is a strongly-malnormal subgroup of G, thus proving the lemma.

Lemma 3.2.2 Any subset of the generators of a one-relator group G with torsion generates a subgroup which, with the obvious presentation, is a strongly-malnormal subgroup of G.

Proof Let $G = gp(a,b,\ldots,t \mid R^n)$ $n > 1$. In view of Lemma 3.2.1 it suffices to prove that $H = gp(b,\ldots,t)$ is strongly-malnormal in G. Firstly H is malnormal in G by Lemma 2.3.1. Secondly H is freely generated by $b,\ldots,t$ and so the word problem is solvable in H. Thirdly let g_1, g_2 be any pair of elements of G. Without loss of generality assume g_1, g_2 are words of minimal length and involve the generator a non-trivially. If for $i = 1, 2$ and elements $h_i \in H$,

$$\lambda(h_i) > l = 2\lambda(R^n)[\lambda(g_1) + \lambda(g_2)]$$

then

$$h_2g_2 \neq g_1h_1$$

or, equivalently

$$g_1h_1g_2^{-1} \neq h_2.$$

For suppose $\lambda(h_1) > 1$. Then the middle of h_1 is too far removed from its ends to be altered in reducing $g_1h_1g_2^{-1}$, assuming of course h_1 is minimal (see proof of Corollary 2.1.5). Thus the middle of h_1 acts as a barrier to ensure that reduction of g_1 and g_2 do not affect one another. Hence if a can be removed from $g_1h_1g_2^{-1}$ then a can be removed from g_1h_1 which implies $g_1 \in H$, contradicting the choice of g_1 as a minimal word involving a. Thus there is an algorithm to determine h_1, h_2; simply determine for the finitely many h_i with $\lambda(h_i) \leq 1$ and h_i minimal, if

$$g_1h_1g_2^{-1}h_2^{-1} = 1.$$

If for some h_1, h_2 this equation holds then h_1, h_2 are the required elements of H. Otherwise no elements h_1, h_2 exist.

<u>Theorem 3.2.3</u> Let G be a one-relator group with torsion. Then the conjugacy problem and the extended conjugacy problem relative to the subgroup generated by any subset of the generators are solvable in G.

<u>Proof</u> The theorem will be proved by induction on the length of the relator. Let

$$G = gp(a,b,c,\ldots \mid R^n) \quad n > 1.$$

If $\lambda(R^n)$ is 2 or 3 the result is well known, as it is if R involves only one-generator non-trivially. Assume R is cyclically reduced and involves a, b non-trivially.

Inductively assume the theorem proved for all groups with relator length $< \lambda(R^n)$.

Firstly consider the conjugacy problem. Let g and h be any given elements of G, and without loss of generality let them be cyclically reduced. To simplify the notation assume at most a, b, c are the generators of G.

<u>Case 1</u> Suppose $\sigma_a(R) = 0$, $\sigma_a(g) = 0$. Let $N = gp_G(b,c)$ and construct N from

$$N_0 = gp(b_0,\ldots,b_\mu,c_0,\ldots,c_\mu \mid R_0{}^n)$$

where not all of b_0, c_0, b_μ, c_μ need occur in R_0. Then for g and h to be conjugate $\sigma_a(h) = 0$ so $g, h \in N$. Suppose $w^{-1}gw = h$ where $w = a^r\underset{\sim}{w}$, $\underset{\sim}{w} \in N$. Then

$$\underset{\sim}{w}^{-1}a^{-r}ga^r\underset{\sim}{w} = h$$

or

$$\underset{\sim}{w}^{-1}g_r\underset{\sim}{w} = h$$

where g_r is the element $a^{-r}ga^r$ written as a word in N.

But it is easy to show that there is a bound on r. This will be established as a separate lemma.

<u>Lemma 3.2.4</u> In N defined as above let $g(b_0,\ldots,b_r,c_0,\ldots,c_r) \neq 1$ and $h(b_m,\ldots,b_{k+\mu},c_m,\ldots,c_{k+\mu})$ be elements of N written as words of minimal length. If $r + \mu < m$ then g and h are not conjugates in N.

<u>Proof</u> If g and h are conjugates in N they are conjugates within K_k. Construct K_k as

$$K_k = \{gp(N_0,\ldots,N_{m-1}) * gp(N_m,\ldots,N_k) \; ; \; J_m\}.$$

Now $g \in$ first factor K_{m-1}, and $h \in$ second factor. They will be conjugates only if g is conjugate to an element of J_m. This implies g is conjugate to an element of N_{m-1}. Construct K_{m-1} as

$$K_{m-1} = \{gp(N_0,\ldots,N_{m-2}) * N_{m-1} \; ; \; J_{m-1}\}.$$

Now g is conjugate to an element of N_{m-1} only if g is conjugate to an element of J_{m-1}. One continues in this fashion proving that g is conjugate to an element of $J_{m-\mu},\ldots,J_m$. Hence there is an element $j \in J_{m-\mu}$ conjugate to elements in $J_{m-\mu+1},\ldots,J_m$. If $j_0 (\in J_0)$ is cyclically reduced and is conjugate to an element of J_1 then $j_0 \in J_0 \cap J_1$. For let $j_0 = j_0(b_0,\ldots,b_{\mu-1}, c_0,\ldots,c_{\mu-1})$ and $j_1 = j_1(b_1,\ldots,b_\mu,c_1,\ldots,c_\mu)$ be conjugates where b_0, c_0 are not both trivial in j_0. Clearly j_0, j_1 are conjugates in N_0. If all the generators involved in R_0 do not occur in j_0, j_1 then j_0, j_1 lie in a free group and are conjugates within this free group. Hence by cyclic reduction b_0, c_0 can be removed from j_0, contradicting the fact that j_0 is cyclically reduced. Hence suppose all the generators in R_0 do occur in j_0, j_1. By Lemma 2.3.1 the elements j_0, j_1 are not conjugate in N_0. Hence b_0, c_0 can be removed from j_0 implying

$$j_0 \in J_0 \cap J_1.$$

Similarly if j_0 is conjugate to an element in $J_1,\ldots,J_r$ then $j_0 \in J_0 \cap J_1 \cap \ldots \cap J_r$. Thus if j is in $J_{m-\mu}$ and is conjugate to elements in $J_{m-\mu+1},\ldots,J_m$ then

$$j \in J_{m-\mu} \cap \ldots \cap J_m$$

and so $j = 1$. Hence g is trivial, contradicting the hypothesis.

<u>Proof of Theorem 3.2.3</u> (continued). The equation $\underset{\sim}{w}^{-1} g_r \underset{\sim}{w} = h$ implies a bound on r, so it will suffice to solve the conjugacy problem for N. Suppose $g, h \in K_k = gp(N_0,\ldots,N_k)$. If g, h are conjugates in N, then they must be conjugate in K_k since K_k is malnormal in N. Hence it suffices to solve the conjugacy problem for K_k. This one does by induction on k. If $k = 0$ then $K_0 = N_0$ and from the original induction hypothesis N_0, and in fact N_i for any integer i, has a solvable conjugacy problem. Now

$$K_1 = \{N_0 * N_1 ; J_1\}$$

and (i) J_1 is strongly-malnormal in N_0 and N_1,
and (ii) the conjugacy problem is solvable in N_0 and N_1
and(iii) the extended conjugacy problem in N_0, N_1 relative to J_1 is solvable.

Therefore, by Lemma 3.1.4 the conjugacy problem is solvable in K_1, and also the extended conjugacy problem relative to N_0 and N_1. One continues in the usual fashion to prove K_k has a solvable conjugacy problem.

Case 2 Suppose $\sigma_a(R) = 0$, $\sigma_a(g) = r \neq 0$. If g and h are conjugates then $\sigma_a(h) = r$. Let

$$g = a^r g_1(b_0,\ldots,b_l,c_0,\ldots,c_l),\ h = a^r h_1(b_0,\ldots,b_l,\\ c_0,\ldots,c_l),$$

and suppose $x^{-1}gx = h$ where x, g, h are words of minimal length. Then

$$x_r^{-1} g_1(b_0,\ldots,b_l,c_0,\ldots,c_l)x_0 = h_1(b_0,\ldots,b_l,c_0,\ldots,c_l)$$

where x_r is $a^{-r}xa^r$ written as a word in N. Without loss of generality assume $r \geqslant 0$. It is clear from this that x_0 does not involve b_i for $i < 0$; for if it did, the equation $h_1^{-1}x_r^{-1}g_1 = x_0^{-1}$ shows that b_i, $i < 0$ can be eliminated from the right hand side, contradicting the minimality of x. Similarly x_r does not involve b_i, $i > l$. Thus x_0 is a word $x_0(b_0,\ldots,b_{l-r},c_0,\ldots,c_{l-r})$. Choose an integer m such that

$$(m - 1)r > l.$$

Then $g^m = a^{mr}g_2(b_0,\ldots,b_{l+(m-1)r},c_0,\ldots,c_{l+(m-1)r})$ and $h^m = a^{mr}h_2(b_0,\ldots,b_{l+(m-1)r},c_0,\ldots,c_{l+(m-1)r})$. Note that in reducing g_2 and h_2, the bottom and the top generators cannot be removed, otherwise they could be removed from g_1 or h_1. Thus g_2, h_2 if not trivial involve letters from $\{b_0,\ldots,b_l,c_0,\ldots,c_l\}$ and from $\{b_{(m-1)r},\ldots,b_{l+(m-1)r},c_{(m-1)r},\ldots,c_{l+(m-1)r}\}$, which cannot be removed by reduction. Consider the equation

$$x^{-1}(b_{mr},\ldots,b_{l+(m-1)r},c_{mr},\ldots,c_{l+(m-1)r}) \times$$

$$g_2(b_0,\ldots,b_{l+(m-1)r},c_0,\ldots,c_{l+(m-1)r})x(b_0,\ldots,b_{l-r},c_0,\ldots,c_{l-r})$$
$$= h_2(b_0,\ldots,b_{l+(m-1)r},c_0,\ldots,c_{l+(m-1)r}).$$

Suppose this takes place in K_k,

$$K_k = \{gp(b_0,\ldots,b_{l+\mu},c_0,\ldots,c_{l+\mu}) * gp(b_{l+1},\ldots,b_{l+(m-1)r}, c_{l+1},\ldots,c_{l+(m-1)r}) \; ; \; gp(b_{l+1},\ldots,b_{l+\mu},c_{l+1},\ldots, c_{l+\mu})\}$$
$$= \{A * B \; ; \; J\} \text{ say.}$$

Let $g_2 = s_1s_2\ldots s_u$, $h_2 = t_1t_2\ldots t_v$, $u, v > 0$.
Then $x^{-1}(b_{mr},\ldots,b_{l+(m-1)r},c_{mr},\ldots,c_{l+(m-1)r}) \in B$
and $x(b_0,\ldots,b_{l-r},c_0,\ldots,c_{l-r}) \in A$.

But apart from a few exceptional cases (see Lemma 3.1.3) one can determine algorithmically if x exists, and if so determine x uniquely. Thus one can determine if $x^{-1}gx = h$. The exceptional case when g_2, h_2 lie in different factors is eliminated here, since g_2, h_2 if not trivial involve letters from A and B not in J. Thus $|g_2|$, $|h_2| > 1$, unless g_2, h_2 are trivial. Thus the only exceptional case to consider is

$$g_2 = s_Bs_A,\ h_2 = t_Bt_A,\ s_A, t_A \in A,\ s_B, t_B \in B,$$

and s_A, t_A are not both trivial and s_B, t_B are not both trivial. Then

$$x_{mr}^{-1}s_Bs_A = t_Bt_Ax_0^{-1}.$$

This implies

$$x_{mr}^{-1}s_B = t_Bj(b_{l+1},\ldots,b_{l+\mu},c_{l+1},\ldots,c_{l+\mu}) \qquad (1)$$

and

$$j^{-1}(b_{l+1},\ldots,b_{l+\mu},c_{l+1},\ldots,c_{l+\mu})t_A x_0^{-1} = s_A. \quad (2)$$

Now translate (1) by $-mr$ obtaining

$$x_0^{-1}s_B(-mr) = t_B(-mr)j(b_{l+1-mr},\ldots,b_{l+\mu-mr},c_{l+1-mr},\ldots,c_{l+\mu-mr}) \quad (3)$$

where $s_B(-mr)$, $t_B(-mr)$ denotes s_B, t_B translated by $-mr$. Multiplying (2) and (3) together to eliminate x_0, one obtains

$$s_A s_B(-mr) = j^{-1}t_A t_B(-mr)j(-mr),$$

or

$$t_A^{-1}js_A = t_B(-mr)j(-mr)s_B^{-1}(-mr).$$

Examining the generators which occur on each side of this equation one sees that both sides can be reduced to a word in $b_0,\ldots,b_{l-r},c_0,\ldots,c_{l-r}$. This implies that all the letters of j on the left hand side can be removed, since j is a word in $b_{l+1},\ldots,b_{l+\mu},c_{l+1},\ldots,c_{l+\mu}$. This implies that j is not too long a word since, assuming j is minimal, there is a limit on the number of reductions possible in $t_A^{-1}js_A$. Also j will surely involve only letters contained in $R_0,\ldots,R_l$, t_A or s_A. Hence there are only finitely many possibilities for j. Thus one can decide algorithmically if there exists an element j satisfying the above, and hence determine if there exists an element x with $x^{-1}gx = h$.

Case 3 Suppose $\sigma_a(R) \neq 0$, $\sigma_b(R) \neq 0$. Here one may embed G in a larger one-relator group $\underset{\sim}{H}$ in the usual way and solve the conjugacy problem in $\underset{\sim}{H}$. Now

$$\underset{\sim}{H} = \{G * gp(\underset{\sim}{b}) \ ; \ b = \underset{\sim}{b}^s\} \quad \text{(as usual)}.$$

If g_1, g_2 are elements of G and are conjugates in $\underset{\sim}{H}$ then they are conjugates in G. For if g_1, g_2 are not conjugates in G they must be conjugate to elements of gp(b), say j_1, j_2. Now j_1, j_2 are conjugates in H. But conjugating j_1, j_2 by elements of the second factor will not alter j_1, j_2 since this factor is cyclic. Hence it is clear that if j_1, j_2 are conjugates in $\underset{\sim}{H}$, they are conjugates in G. This proves g_1, g_2 are conjugates in G if they are conjugates in $\underset{\sim}{H}$. This proves the conjugacy half of the theorem.

Consider now the extended conjugacy problem for G relative to a subgroup H generated by a subset of the generators of G. It suffices to prove the result for $G = gp(a,b,c,\ldots,t \mid R^n)$ and $H = gp(b,c,\ldots,t)$, where $a,b,c,\ldots,t$ are the generators occurring non-trivially in R. Let $g \in G$ and suppose without loss of generality that g is a minimal word. It is necessary to prove that one can decide if g is conjugate to an element of H, and if so, to find such an element of H. If g does not involve a there is nothing to prove. Assume therefore that g involves the generator a non-trivially, and that b occurs non-trivially in R. For convenience again assume a, b, c are at most the generators in G.

Case 4 Suppose $\sigma_a(R) = 0$. If g is conjugate to an element of H then $\sigma_a(g) = 0$. It is necessary to decide if there exist elements $x \in G$ and $w \in H$ such that

$$x^{-1}gx = w,$$

and if so, to find some such w. Let $x = a^r y$, where $y \in N = gp_G(b, c)$. Then

$$y^{-1}(a^{-r}ga^r)y = w.$$

Now it has been shown that there is a bound on r, hence it suffices to solve the extended conjugacy problem for N with respect to $gp(b_0, c_0)$. In fact it will suffice to prove $K_k = gp(N_0, \ldots, N_k)$ (where N_0 has any of the usual definitions) has a solvable extended conjugacy problem relative to $gp(b_0, c_0)$.

Let $g \in K_k$, and without loss of generaltiy assume g is cyclically reduced as a word in the generalized free product

$$K_k = \{N_0 * gp(N_1, \ldots, N_k) \; ; \; J_1\}.$$

If $|g| > 1$, then g is not conjugate to an element of N_0. If $|g| = 1$ and $g \in N_0$, one is finished by the induction hypothesis, If $g \in gp(N_1, \ldots, N_k)$ then an easy induction argument on k shows that one can determine if an element in $gp(N_1, \ldots, N_k)$ is conjugate to an element of J_1, and if so, determine a conjugate $j \in J_1$.

Case 5 Suppose G has only two generators a, b, and

$\sigma_a(R) \neq 0$. Here one must decide if $g(a,b)$ is conjugate to a power of b, and if so determine that power. Let

$$x^{-1}gx = b^r.$$

Then $x^{-1}gxb^{-r} = \Pi_{i=1}^{m} S_i^{-1} R^{\epsilon_i} S_i$, for some words S_i in a,b. Taking exponent sums

$$\sigma_a(g) = (\Sigma\,\epsilon_i)\,\sigma_a(R)$$

$$\sigma_b(g) - r = (\Sigma\,\epsilon_i)\,\sigma_b(R).$$

If $\sigma_b(R) = 0$ or if $\Sigma\epsilon_i = 0$ then $r = \sigma_b(g)$. Otherwise

$$r = (\,\sigma_a(R)\sigma_b(g) - \sigma_a(g)\sigma_b(R))\,\sigma_a^{-1}(R).$$

Thus one need only determine if g is conjugate to this particular power of b. This has been shown to have an algorithmic solution.

Case 6 Suppose G has more than two generators and $\sigma_a(R) \neq 0$, $\sigma_b(R) = 0$, $\sigma_b(g) = 0$. Suppose that

$$x^{-1}gx = w(b,c).$$

Without loss of generality assume $\sigma_b(x) = 0$. Clearly $\sigma_b(w) = 0$. Let $N = gp_G(a,c)$. Then $g \in N$ and is conjugate in N to $w(c_i)$. If $g \in N_0 = gp(a_0,\ldots,a_\mu,c_i \mid R_0{}^n)$, one is finished by the induction hypothesis. If $g \in K_k$, cyclically reduce g as a word in

$$K_k = \{N_0 * gp(N_1,\ldots,N_k)\ ;\ J_1\},$$

and if it lies in $gp(N_1,\ldots,N_k)$ one uses an induction argument on k to show that w may be algorithmically determined. If g, when cyclically reduced does not lie in $gp(N_1,\ldots,N_k)$ then g is not conjugate to an element w.

Case 7 Suppose G has more than two generators, $\sigma_a(R) \neq 0$, and the exponent sums in R and g of b, c are not both zero. Let

$$\sigma_a(R) = \alpha_1 \qquad \sigma_b(R) = \beta_1 \qquad \sigma_c(R) = \gamma_1$$
$$\sigma_a(g) = \alpha_2 \qquad \sigma_b(g) = \beta_2 \qquad \sigma_c(g) = \gamma_2.$$

If $\alpha_2\gamma_1 - \alpha_1\gamma_2 \neq 0$, map $G \to \underset{\sim}{H}$ as follows,

$$a \to \underset{\sim}{a}\underset{\sim}{b}^{\beta_1\gamma_2-\beta_2\gamma_1}$$
$$b \to \underset{\sim}{b}^{\alpha_2\gamma_1-\alpha_1\gamma_2}$$
$$c \to \underset{\sim}{c}\underset{\sim}{b}^{\alpha_1\beta_2-\alpha_2\beta_1}.$$

Here $g \to \underset{\sim}{g}$ and $R \to \underset{\sim}{R}$ with $\sigma_{\underset{\sim}{b}}(\underset{\sim}{g}) = \sigma_{\underset{\sim}{b}}(\underset{\sim}{R}) = 0$. Let $\underset{\sim}{H} = gp(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c} \mid \underset{\sim}{R}^n)$. One may now solve the extended conjugacy problem in $\underset{\sim}{H}$ relative to $gp(\underset{\sim}{b}, \underset{\sim}{c})$. Suppose $\underset{\sim}{g}$ is conjugate in $\underset{\sim}{H}$ to an element $h \in gp(\underset{\sim}{b}, \underset{\sim}{c})$. Write $\underset{\sim}{H}$ as

$$\underset{\sim}{H} = \{G * gp(\underset{\sim}{b}) \ ; \ b = \underset{\sim}{b}^{\alpha_2\gamma_1-\alpha_1\gamma_2}\}$$

Then $|h| \leq 1$ when h is cyclically reduced. This implies h lies in G or in $gp\{\underset{\sim}{b}\}$. If the latter occurs then g is trivial since $\sigma_{\underset{\sim}{b}}(\underset{\sim}{g}) = 0$, so assume $h \in G$. It remains to show that if $\underset{\sim}{g}$ and h are conjugates in $\underset{\sim}{H}$, they are conjugates in G. But this is obvious, for if g is conjugate to a power of b, then $\underset{\sim}{g}$ is conjugate to a power of $\underset{\sim}{b}$ which, as remarked above, is impossible.

If $\alpha_2\gamma_1 - \alpha_1\gamma_2 = 0$ then $\alpha_1 \neq 0$ implies $\gamma_1 \neq 0$. Here embed G in H by mapping

$$a \to \underset{\sim}{a}\underset{\sim}{c}^{-\gamma_1}$$

$b \to \underset{\sim}{b}$

$c \to \underset{\sim}{c}^{\alpha_1}$

and $\underset{\sim}{H} = gp(\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c} \mid R^n(\underset{\sim}{a}\underset{\sim}{c}^{-\gamma_1},\underset{\sim}{b},\underset{\sim}{c}^{\alpha_1}))$.
If $g \to \underset{\sim}{g}$, $R \to \underset{\sim}{R}$ then $\sigma_{\underset{\sim}{c}}(\underset{\sim}{g}) = \sigma_{\underset{\sim}{c}}(\underset{\sim}{R}) = 0$. As before one can solve the extended conjugacy problem in $\underset{\sim}{H}$ relative to $gp(\underset{\sim}{b},\underset{\sim}{c})$, and then show that it is solvable in G relative to $gp(b,c)$.

This completes the proof of the theorem.

Section 3.3 The roots of an element.

In this section we show there is an algorithm to determine the roots of an element g in a one-relator group with torsion. The problem of finding an algorithm to determine whether or not an arbitrary element of a group is a power has been investigated by Reinhart 1962 and by Lipschutz 1965 and 1967.

Theorem 3.3.1 Let $G = gp(a,b,c,\ldots \mid R^n)$ $n > 1$. Given $g \in G$ there is an algorithm to determine the roots of g.

Proof The theorem is proved by induction on $\lambda(R^n)$. If $\lambda(R^n)$ is 2 or if R involves only one generator then the problem is solvable. Assume the theorem proved for all groups with length less than $\lambda(R^n)$. Without loss of generality assume R is cyclically reduced.

Case 1 Suppose $\sigma_a(R) = 0$, $\sigma_a(g) = 0$. Let $N = gp_G(b,c,\ldots)$. Then $g \in N$, and without loss of generality let $g \in K_k$

where

$$K_k = \{K_{k-1} * N_k \ ; \ J_k\}.$$

(Any of the usual definitions for N_0 will do!) Since K_k is malnormal in N, the roots of g must lie in K_k. If $g \in N_0$ then by the induction hypothesis one can find the roots of g. If $g \in N_k$ or $g \in K_{k-1}$ proceed by induction on k. Hence suppose $g \in K_k$ but $g \notin N_k$ and $g \notin K_{k-1}$. Let g (without loss of generality) be cyclically reduced,

$$g = s_1 s_2 \dots s_n \qquad n > 1$$

where s_i alternate from K_{k-1} and N_k. Then a root of g will be

$$s_1 s_2 \dots s_t j \qquad j \in J_k, \quad t < n,$$

and $\quad js_1 = s_{t+1} j_1 \qquad j_1 \in J_k.$

But since J_k is strongly-malnormal in N_k and K_{k-1}, j can be determined. Thus one can algorithmically determine all possible roots and check if any are indeed roots of g.

<u>Case 2</u> Suppose $\sigma_a(R) = 0$, $\sigma_a(g) = r$. Let $g = a^r \tilde{g}$ where $\tilde{g} \in N$. Let $a^s h$, $h \in N$ be a p-th root of g, $p > 1$. Assume $\tilde{g}$, h minimal words. Then without loss of generality assume $r > 0$ and in the usual notation for translated words,

$$\tilde{g} = h_{(p-1)s} \dots h_{2s} h_s h_0.$$

Now the minimum $b_i, c_i, \dots$ in h_0 must coincide with those of $\underset{\sim}{g}$, otherwise, writing h_0 on one side it is clear that a minimal generator of h_0 is removable, contradicting the minimality of h. Similarly the maximum letters in $h_{(p-1)s}$ must coincide with those of $\underset{\sim}{g}$. Hence for some integer q

$$h_0 = h_0(b_0,\ldots,b_{q+s},c_0,\ldots,c_{q+s},\ldots)$$

and let

$$g_0 = \underset{\sim}{g}(b_0,\ldots,b_{q+ps},c_0,\ldots,c_{q+ps},\ldots).$$

Now $\qquad g_0 = h_{(p-1)s}\cdots h_s h_0$

whence $\qquad g_s = h_{ps}g_0h_0^{-1}$

and so $\qquad g_{ps+s} = h_{2ps}g_{ps}h_{ps}^{-1}$

therefore $\qquad g_{ps+s}g_s = h_{2ps}g_{ps}g_0h_0^{-1}$.

Also $\qquad g_{2ps+s} = h_{3ps}g_{2ps}h_{2ps}^{-1}$

therefore $\qquad g_{2ps+s}g_{ps+s}g_s = h_{3ps}g_{2ps}g_{ps}g_0h_0^{-1}$.

Continue until

$$g_{nps+s}g_{(n-1)ps+s}\cdots g_{ps+s}g_s = h_{(n+1)ps}g_{nps}\cdots g_0h_0^{-1},$$

where $(n+1)ps > t + s$. Then

$$(g_s^{-1}\cdots g_{nps+s}^{-1})h_{(n+1)ps}(g_{nps}\cdots g_0) = h_0.$$

Now on the left hand side of this equation every letter in $h_{(n+1)ps}$ can be removed. This implies that there is an upper bound on $\lambda(h_0)$. Since the only generators that can appear in h_0 are those in the range $b_0,\ldots,b_{t+s}$, $c_0,\ldots,c_{t+s},\ldots$ it is easy to see that one has a bounded number of possibilities for h. Thus an algorithm is obtainable in this case.

<u>Case 3</u> Suppose $\sigma_a(R) \neq 0$, $\sigma_b(R) \neq 0$. As usual embed G in H and solve the problem in H. Suppose $H = \{G * gp(\underset{\sim}{b}) \; ; \; b = \underset{\sim}{b}^m\}$. If $g = h^p$ then g, h commute and so g, h lie in G or else g lies in G and is conjugate to a power of b. But by the previous theorem one can determine such a power of b. By the malnormality of $gp(b)$, any root of b^r is a power of b. Thus one can determine the roots of g.

This completes the proof of the theorem.

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