Symmetry improvement techniques for non-perturbative quantum field theory

By
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Thesis submitted by Michael Jonathan Brown in partial fulfillment of the requirements for the Degree of Doctor of Philosophy (Physics) in College of Science and Engineering at James Cook University.

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5 April 2017

Declaration

I declare that this thesis is my own work and has not been submitted in any form for another degree or diploma at any university or other institute of tertiary education. Information derived from the published and unpublished work of others has been acknowledged in the text and a list of references is given.

Michael Jonathan Brown
5 April 2017

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Acknowledgements

I would like to thank my primary advisor, Prof. Ian Whittingham, and secondary advisor A/Prof. Daniel Kosov for their endless patience, encouragement, feedback and support.

I would also like to thank my friends and family for their love and support. Special thanks go to my wife, without whose regular encouragement this project would have been impossible, and to my parents for encouraging me in the sciences (though I ended up changing my thesis topic from my father's suggestion of basket weaving).

Thanks to Elwood, who is a glorious bastard.

Numerous open source tools were used to build this thesis: LyX, LaTeX 2$_\varepsilon$, Python, numpy/scipy/matplotlib, git, and more. It is good to live in a world with such a vibrant open source community. Thanks to all who participate.

This research was supported by an Australian Postgraduate Award 2013-2016.

 All source code for this thesis document and related programs (MATHEMATICA notebooks etc.) will be made available online at https://github.com/mikejbrown/thesis. This document has been built from the following version of the source: v1.0.0

Abstract

Quantum field theory is the mathematical language of nature at the deepest levels currently known. Viewed in this light, much of the last century of theoretical physics can be seen as quantum field theory calculations performed in a variety of approximations. However, despite having a very successful standard model of physics and a phenomenally useful perturbation theory for extracting predictions from it, persistent problems remain. Many of these are due to the inability to calculate in practice what is calculable in principle. The difficulty of these problems comes from the importance of a large number of degrees of freedom, or a strong coupling between degrees of freedom, or both. Only in the case of very simple systems or systems close to thermal equilibrium can their properties be determined with confidence. The remaining cases require a resummation, or re-organisation, of perturbation theory. However, *ad hoc* resummations are problematic because perturbation series are asymptotic in nature and the re-organisation of the terms of an asymptotic series is mathematically questionable. Systematic resummation schemes are required to guarantee consistency with the original non-perturbative theory.

n-particle irreducible effective actions (nPIEAs; $n = 1, 2, 3, \cdots$) are a proven method of resummation which can be thought of as generalisations of mean field theory which are (a) elegant, (b) general, (c) in principle exact, and (d) have been promoted for their applicability to non-equilibrium situations. These properties make the nPIEAs compelling candidates for filling the gap between theory and experiment anywhere the problem is the inability to compute the behaviour of strongly interacting degrees of freedom or degrees of freedom which are out of thermal equilibrium. Unfortunately, nPIEAs are known to violate the symmetries of a field theory when they are truncated to a form that can be solved in practice. This can lead to *qualitatively* wrong physical predictions as an artefact of the method of calculation. If one takes the nPIEA predictions at face value one may reject a physically correct model on the basis of an incorrect prediction. Therefore it is critical to gain a better understanding of the symmetry problem in order for nPIEAs to be useful in the future.

This is where this thesis aims to make a contribution. This thesis examines the theory of global symmetries in the nPIEA formalism for quantum field theory with the goal of improving the symmetry properties of solutions obtained from practical approximation schemes. Following pedagogical reviews of quantum field theory and nPIEAs, symmetries are introduced by considering a scalar $(\phi^2)^2$ field theory with $\mathrm{O}(N)$ symmetry and spontaneous symmetry breaking. The symmetries are embodied by Ward identities which relate the various correlation functions of the theory. When truncated, the nPIEAs for $n > 1$ violate these identities with phenomenological consequences which are reviewed. The symmetry improvement method (SI) of Pilaftsis and Teresi [1] addresses this by imposing the 1PI Ward identities directly onto the solutions of nPI equations of motion through the use of Lagrange multipliers. In novel work, SI is extended to the 3PIEA. The SI-3PIEA is then renormalised in the Hartree-Fock and two loop truncations in four dimensions and in

the three loop truncation in three dimensions. The Hartree-Fock equations of motion are solved and compared to the unimproved and SI-2PI Hartree-Fock solutions. In both the SI-2PI and SI-3PI methods Goldstone's theorem is satisfied. However, the SI-2PI solution correctly predicts that the phase transition is second order while the SI-3PI solution predicts a weak (weaker than the unimproved 2PIEA) first order transition. Further checks of the formalism are performed: it is shown that the SI-3PIEA obeys the Coleman-Mermin-Wagner theorem and is consistent with unitarity to $\mathcal{O}(\hbar)$, but only if the full three loop truncation is kept.

Following this a novel method of soft symmetry improvement (SSI) is introduced. SSI differs from SI in that the Ward identities are imposed softly in the sense of least squared error. A new *stiffness* parameter controls the strength of the constraint. The motivation for this method is that the singular nature of the constraint in SI leads to pathological behaviour such as the non-existence of solutions in certain truncations and the breakdown of the theory out of equilibrium. In order to regulate the IR behaviour, a box of finite volume is used and the limit of infinite volume taken carefully. Three limiting regimes are found. Two are equivalent to the unimproved 2PIEA and SI-2PIEA respectively. The third is a novel limit which has pathological behaviour including a strongly first order phase transition and a regime where solutions cease to exist for a range of temperatures below the critical temperature. As the stiffness increases this range increases and at a finite value of the stiffness the solution ceases to exist even at zero temperature. This loss of solution is confirmed both analytically and numerically.

Finally, the linear response of a system in equilibrium subject to external perturbations is studied for the first time in the SI-2PIEA formalism. It is shown that, beyond equilibrium, the scheme is inconsistent in that the equations of motion are generically over-constrained. Thus, the SI-2PIEA predicts qualitatively incorrect physics out of equilibrium. This result can be understood as a result of the decoupling of the propagator fluctuation from the field fluctuation in the 2PIEA formalism and the failure of higher order Ward identities. This argument generalises so that, as a result, SI-nPIEA are invalid out of equilibrium, at least within the linear response approximation in generic truncations.

Contents

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List of Publications

This thesis is largely based on work published by the author (M.J. Brown) in the following peer reviewed journal articles:

- M.J. Brown and I. B. Whittingham, Symmetry improvement of 3PI effective actions for O(N) scalar field theory, Phys. Rev. D 91, 085020 (2015). DOI: 10.1103/PhysRevD.91.085020, arXiv: 1502.03640, [2],

  Statement of author contributions: The authors jointly developed the research question. All initial calculations, figures and the first draft of the manuscript were produced by Brown. The manuscript was revised with checking of results and editorial input from Whittingham.

- M. Brown and I. Whittingham, Two-particle irreducible effective actions versus resummation: Analytic properties and self-consistency, Nucl. Phys. B900, 477 (2015). DOI: 10.1016/j.nuclphysb.2015.09.021, arXiv: 1503.08664, [3],

  Statement of author contributions: The authors jointly developed the research question. All initial calculations, figures and the first draft of the manuscript were produced by Brown. The manuscript was revised with checking of results and editorial input from Whittingham.

- M.J. Brown, I. B. Whittingham and D. S. Kosov, Linear response theory for symmetry improved two particle irreducible effective actions, Phys. Rev. D 93, 105018 (2016). DOI: 10.1103/PhysRevD.93.105018, arXiv: 1603.03425, [4],

  Statement of author contributions: The authors jointly developed the research question. All initial calculations and the first draft of the manuscript were produced by Brown. The manuscript was revised with checking of results and editorial input from Whittingham and Kosov.

- M.J. Brown and I. B. Whittingham, Soft symmetry improvement of two particle irreducible effective actions, Phys. Rev. D 95, 025018 (2017). DOI: 10.1103/PhysRevD.95.025018 arXiv: 1611.05226, [5],

  Statement of author contributions: The authors jointly developed the research question. All initial calculations and the first draft of the manuscript were produced by Brown. The manuscript was revised with checking of results and editorial input from Whittingham.

Notation

This thesis uses high-energy physics (HEP) conventions. That is $\hbar = c = k_B = 1$ and the metric is $\eta_{\mu\nu} = \mathrm{diag}\left(+1, -1, -1, -1\right)$. Loop counting factors of $\hbar$ will be kept explicit. The repeated index summation convention is used. Often the deWitt summation convention is used. That is, for quantities with internal symmetry indices accompanied by spacetime arguments, repeated indices imply integration over the corresponding spacetime arguments. So for example

$$\phi_a \phi_a = \sum_{a=1}^{N} \int \mathrm{d}^4 x \; \phi_a\left(x\right) \phi_a\left(x\right),$$

and

$$V_{abc} \phi_a \Delta_{bc} = \sum_{a,b,c=1}^{N} \int \mathrm{d}^4 x \; \mathrm{d}^4 y \; \mathrm{d}^4 z \; V_{abc}\left(x, y, z\right) \phi_a\left(x\right) \Delta_{bc}\left(y, z\right).$$

The final expression can always be reconstructed uniquely from the short version of it. Where the deWitt convention is not used the intended meaning will be obvious from context. Often functional convolutions are implied, so for example

$$\mathrm{Tr}\left(\Delta_0^{-1} \Delta\right) = \Delta_{0ab}^{-1} \Delta_{ba} = \sum_{a,b=1}^{N} \int \mathrm{d}^4 x \; \mathrm{d}^4 y \; \Delta_{0ab}^{-1}\left(x, y\right) \Delta_{ba}\left(y, x\right),$$

and

$$\varphi K \varphi = \varphi_a K_{ab} \varphi_b = \sum_{a,b=1}^{N} \int \mathrm{d}^4 x \; \mathrm{d}^4 y \; \varphi_a\left(x\right) K_{ab}\left(x, y\right) \varphi_b\left(y\right),$$

etc.

Time variables can be chosen to exist on different contours depending on the application being considered. Most manipulations are agnostic about the contour chosen. Where relevant the contour in use is specified. The choices are:

- *Zero temperature, Real time* or *Minkowski time* contours (Figure 0.1): t runs from $-\infty + i\epsilon$ to $+\infty - i\epsilon$ where $\epsilon \to 0^+$ is an infinitesimal convergence parameter,

- *Equilibrium, Imaginary time* or *Matsubara* contours (Figure 0.2): t runs from 0 to $-i\beta$ with periodic boundary conditions,

- *Non-equilibrium, Closed time path* or *Schwinger-Keldysh* contours (Figure 0.3): t runs on a multi-branch contour from t_0 up to $+\infty$, then back down from $+\infty - i\epsilon$ to $t_0 - i\epsilon$ then down the imaginary axis to $t_0 - i\beta$ which is identified with t_0.

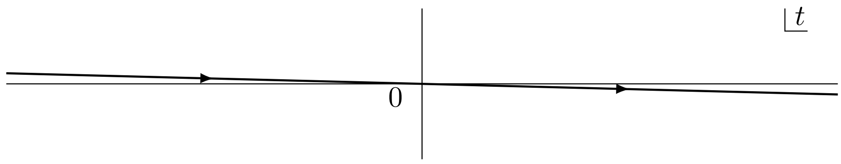


Figure 0.1: The Minkowski contour in the complex time plane.

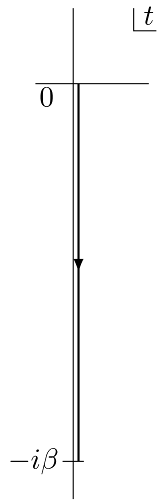


Figure 0.2: The Matsubara contour in the complex time plane. 0 and $-i\beta$ are identified.

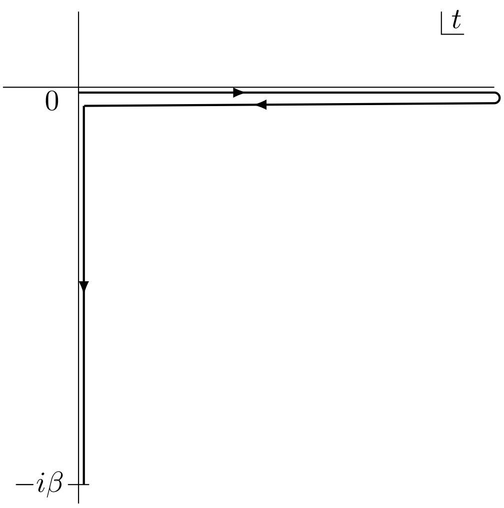


Figure 0.3: The Schwinger-Keldysh contour in the complex time plane. The initial time is taken as $t_0 = 0$. 0 and $-i\beta$ are identified.

Table 0.1: Commonly used symbols and their meanings

Symbol	Meaning
QFT	Quantum Field Theory
nPI	n-Particle Irreducible ($n = 1, 2, 3, \cdots$)
nPIEA	nPI Effective Action
SI-nPIEA	Symmetry Improved nPIEA
SSI-nPIEA	Soft Symmetry Improved nPIEA
CTP	Closed Time Path
WI	Ward Identity
SSB	Spontaneous Symmetry Breaking
SM, BSM	Standard Model, Beyond Standard Model
MSSM, NMSSM	Minimal Supersymmetric extension of the SM, Next to Minimal ...
2HDM	Two Higgs Doublet Model
MS($\overline{\text{MS}}$)	(Modified) Minimal Subtraction
x, x^μ	Spacetime coordinates of event x
$\boldsymbol{x}$	Spatial part of x^μ projected in some implied reference frame
d^4x	Spacetime measure $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$
$\int_x$	Integration over all spacetime $\int_x = \int_{-\infty}^{+\infty} \mathrm{d}^4x$
$\mathrm{d}^3\boldsymbol{x}$	Spatial measure $\mathrm{d}^3\boldsymbol{x} = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$
$\int_{\boldsymbol{x}}$	Integration over all space at a moment of time $\int_{\boldsymbol{x}} = \int_{-\infty}^{+\infty} \mathrm{d}^3\boldsymbol{x}$
x^4	Euclidean time coordinate. $x^0 = -ix^4$ after Wick rotation.
∂_μ	Spacetime derivative $\partial_\mu = \partial/\partial x^\mu$
$\partial_\mu^{(x)}$, $\partial_\mu^{(y)}$	Derivative w.r.t. specified spacetime variable $\partial_\mu^{(y)} = \partial/\partial y^\mu$ etc.
$\overleftrightarrow{\partial_\mu}$	Right-left derivative: $f\overleftrightarrow{\partial_\mu}g \equiv f\left(\partial_\mu g\right) - \left(\partial_\mu f\right)g$
p, p_μ	Momentum conjugate to x^μ. As an operator $p_\mu = i\partial_\mu$.
$\boldsymbol{p}$	Spatial part of p^μ
d^4p, $\mathrm{d}^3\boldsymbol{p}$	Standard measures for 4- and 3-momentum
$\int_p$	Integration over all momenta *with* 2πs in measure $\int_p = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4p}{(2\pi)^4}$
$\int_{\boldsymbol{p}}$	$\int_{\boldsymbol{p}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}$
$p \cdot x$	4-vector dot product $p \cdot x = p_\mu x^\mu$, so $p^2 = m^2$ is the mass shell condition
$\mathrm{T}\left(\tilde{\mathrm{T}}\right)$	(Anti-)Time ordered product
T_C	Contour time ordered product
β	Inverse temperature $\beta = 1/T$
ω_n	Matsubara frequencies $\omega_n = 2\pi n/\beta$
a, b, c, ...	Fundamental $\mathrm{O}(N)$ group indices ranging from 1 to N
A, B, C, ...	Adjoint $\mathrm{O}(N)$ group indices ranging from 1 to $N(N-1)/2$
T^A	(Hermitian) $\mathrm{O}(N)$ generators
ϕ_a	Quantum scalar field

Symbol	Meaning
φ_a	Expectation value of the scalar field $\varphi_a = \langle\phi_a\rangle$
v	Vacuum expectation value of the scalar field, i.e.
	$\varphi_a = (0, \cdots, 0, v)$ in vacuum
m	Lagrangian mass parameter for the scalar field
λ	Lagrangian self-coupling parameter for the scalar field
μ	Renormalisation subtraction scale
Λ	Ultraviolet cutoff scale
m_G	Mass of Goldstone bosons
m_H	Mass of Higgs boson (i.e. the radial mode)
Δ_{0ab}	Free propagator or connected two-point correlation function
Δ_{ab}	nPI propagator
Δ_G, Δ_H	nPI Goldstone and Higgs propagators
Δ_{0ab}^{-1}, etc.	Inverses in the sense of kernels, i.e.
	$\int_y \Delta_{0ab}^{-1}(x,y)\,\Delta_{0bc}(y,z) = \delta_{ac}\delta^{(4)}(x-z)$
V_0, W	Bare three and four point vertex functions
V_{abc}, $\bar{V}$, V_N	Proper vertex or three point correlation functions
$V^{(n)}$	Generic proper n-point vertex functions
$\mathcal{W}$, $\mathcal{W}_a^A$, $\mathcal{W}_{ab}^A$, ...	1PI Ward identities
$S[\phi]$	Classical action functional for the scalar field theory
J, $K = K^{(2)}$, $K^{(3)}$	One-, two- and three-point source functions
$Z\left[J, K^{(2)}, \cdots, K^{(n)}\right]$	Partition functional
$W\left[J, K^{(2)}, \cdots, K^{(n)}\right]$	Connected generating functional
$\Gamma\left[\varphi, \Delta, V, \cdots, V^{(n)}\right]$	nPIEA, the Legendre transform of $W\left[J, K^{(2)}, \cdots, K^{(n)}\right]$
Σ_{ab}	Self-energy of the scalar field
$\mathcal{T}_{G/H}$, $\mathcal{T}_{G/H}^{\text{fin}}$, $\mathcal{T}_{G/H}^{\text{th}}$	Tadpole integrals (bare, finite part of, thermal part of)
j_μ^A	Conserved $\mathrm{O}(N)$ currents
$T_{\mu\nu}$	Energy-momentum tensor

Table 0.2: Summary of $\mathrm{O}(N)$ model calculations performed or discussed in the thesis.

Calculation	Location	Broken phase Goldstone mass	Order of phase transition	Critical temperature (leading order)	Higgs decay rate (leading order)	Comments
Benchmark		0	2$^{\text{nd}}$	$\sqrt{\frac{12\bar{v}^2}{\hbar(N+2)}}$	$\frac{N-1}{2}\frac{\hbar}{16\pi m_H}\left(\frac{\lambda v}{3}\right)^2$	
2PI Hartree-Fock	Section 3.4	> 0	1$^{\text{st}}$	correct	0	
2PI External propagator	Section 3.5	0	1$^{\text{st}}$	correct	incorrect (violations of unitarity)	not fully self-consistent
Large N leading order	Section 3.6	0	2$^{\text{nd}}$	incorrect by $\times\sqrt{1+2/N}$	$\frac{N}{2}\frac{\hbar}{16\pi m_H}\left(\frac{\lambda v}{3}\right)^2$ plus s-channel resummation at $\mathcal{O}\left(\hbar^{\geq 2}\right)$	Goldstone theorem lost at higher orders
SI-2PI Hartree-Fock	Section 4.3 and [1]	0	2$^{\text{nd}}$	correct	0	
SI-2PI Two loop (dim reg)	Section 4.4 and [1]	0	2$^{\text{nd}}$	correct	correct	full self-energies, not just pole positions, in [1]
SI-2PI Two loop (cutoff)	Section 4.4 and [6]	0	2$^{\text{nd}}$	correct	?	existence of solutions depends upon cutoff
SI-3PI Hartree-Fock	Section 4.8 and [2]	0	1$^{\text{st}}$	correct	0	not fully self-consistent
SI-3PI Two loop	Section 4.7.2 and [2]	?	?	?	incorrect by unitarity (Section 4.10)	calculation set up but not performed
SI-3PI Three loop	Section 4.7.3 and [2]	?	?	?	perturbative correct to $\mathcal{O}(\hbar)$	calculation set up but not performed; renormalisation only valid in $D < 4$ dimensions
SSI-2PI Hartree-Fock	Section 5.4 and [5]	0 for zero mode; > 0 for non-zero modes	1$^{\text{st}}$	correct when solutions exist	0	results listed for novel limit

Chapter 1

Introduction

1.1 Background and motivation

There has been remarkable progress in physics in the last century. The revolutions of relativity and quantum mechanics in the first quarter of the 20th century led inevitably to the establishment of what is now known as "the standard model" (SM) – the mathematical formulation of the laws of nature at the most fundamental level currently known. This model successfully explains phenomena on scales ranging from a few thousands of a proton radius[1] to the size of the observable universe[2]. The outstanding puzzles where the standard model is known to fall short can be grouped into roughly three areas: questions about the character of the model itself which, although consistent, is arguably peculiar (e.g. the various fine tuning and hierarchy problems); questions about the behaviour of matter at extremely high energies which are mainly relevant to the early universe (e.q. questions about quantum gravity and the origin of matter); and questions about the behaviour of particles that may be abundant today, but are very weakly interacting with ordinary matter (neutrinos and dark matter). All of these questions are far removed from ordinary experience. The general consensus is that in the (limited) sense of knowing the underlying laws, all ordinary phenomena from atoms to solar systems are completely understood.

This statement should be incredible to anyone without a physics education, but it can be quickly supported by reading the published literature on fundamental physics (as conveniently summarised in, e.g., the annual Review of Particle Physics [8]) or by trawling the daily postings on the physics arXiv[3]: the research topics of interest to modern theoretical physicists lie very far from everyday experience. Physics students are taught to identify the relevant scales of length, time and energy in a problem and use the set of laws appropriate for that scale. However, they are also taught that this procedure is purely one of convenience, and that the laws governing phenomena at one scale can always be

[1] Corresponding to energy scales of a few hundreds of GeV (giga-electron-volts).

[2] For the cognoscenti: I am including classical general relativity with a cosmological constant in the "standard model" for rhetorical purposes. This does not exactly match standard particle physics usage of the term though, despite the famous difficulties of quantum gravity, it is consistent to include general relativity as a low energy effective field theory (see, e.g. [7]). The cosmological model is taken as ΛCDM, i.e. the "dark" components of the universe are a cosmological constant and cold dark matter.

[3] The openly accessible repository for preprint papers in physics and other quantitative fields, operated on a not-for-profit basis by the Cornell University Library and available at https://arxiv.org/.

connected to the laws at neighbouring scales so that, for example, the laws of chemistry can be derived from the laws governing atoms which can in turn be derived from the laws governing sub-atomic particles and so on. In the language of particle physics and statistical mechanics this intuitive idea has been refined into the technically precise notion of effective field theories: a mathematical framework that allows one to mechanically derive the laws governing phenomena at larger distance scales from those governing smaller distance scales.

There are some embarrassing omissions in this tidy picture, however, all of which have to do with the *inability to calculate in practice what is calculable in principle*. For instance, it is extremely difficult to compute, from the basic quantum mechanics of atoms, how a given protein molecule (consisting of many atoms) will fold into a biologically significant three dimensional shape. More relevant to this thesis, it is difficult to connect the "fundamental" theory of the standard model of sub-nuclear quarks and gluons (which are strongly interacting) with the very different looking effective theory of protons, neutrons and various mesons that is valid at nuclear scales. Similarly, it is difficult to accurately predict the behaviour of the Higgs field in the early universe which is filled with a very hot and dense plasma of many particles which may also be strongly interacting. In all cases the difficulty of the problem comes from the importance of a large number of degrees of freedom, or a strong coupling between degrees of freedom, or both. Only in the case of very simple systems or systems close to thermal equilibrium can their properties be determined with confidence.

This inability to compute is not simply a philosophical difficulty for those seeking to shore up a reductionistic world-view. Neither is it simply a practical difficulty for those pure theoreticians who only want to better understand their mathematical models. It can also hinder efforts to learn real physics. For example consider the scenario of *electroweak baryogenesis*, which is discussed in detail in the next section as a motivating physical example, though the methods discussed in the remainder of the thesis are broadly applicable. Briefly, it is believed that the Higgs field undergoes a phase transition at high temperatures which were reached in the early universe. The nature of this phase transition (e.g. whether it was first or second order, how much was the latent heat etc.) can have a profound impact on the amount of matter surviving to the present day universe. Hence, cosmological observations of the matter (baryon) density, as well as gravitational waves and primordial magnetic fields, have the potential to constrain this phase transition.

However, it is difficult to reliably tell whether a given particle physics model has a phase transition with given properties. For example, it is known that the standard model possesses a second order phase transition, but this finding was the result of a focused effort and could only be established using an expensive computational method (lattice Monte Carlo; see, e.g. [9]). It is practically impossible to scan the parameters of a theory of beyond standard model (BSM) physics using this method. In the BSM context one largely has to make do with a patchwork of semi-empirical curve fitting methods to connect the small scale physics (the fundamental parameters of the BSM model) with the large scale physics (phase transition thermodynamics and baryon yield). All of these methods require approximations which are somewhat suspect. Thus, such cosmological observations are of limited usefulness for fundamental physics. New methods of calculation are required to bridge the gap between theory and experiment in this case.

The standard model and the various BSM models are all examples of *quantum field theories* (QFTs). Quantum field theory is the mathematical language of nature. Viewed in this light, much of the last century of theoretical physics can be seen as quantum field theory calculations performed in a variety of approximations. The most fruitful scheme so far is clearly perturbation theory in small couplings, which has seen wide use and great success. The classic example of a successful perturbation theory is the theory of quantum electrodynamics (QED): the quantum field theory of electrons and photons. In this theory the small parameter is the electric charge $e \approx 0.3$ of the electron, and observables are computed in a series expansion in e of the form $a_0 + a_1 e + a_2 e^2 + a_3 e^3 + \cdots$. For example, the anomalous magnetic moment a_e of the electron has been computed to $\mathcal{O}\left(e^{10}\right)$ with the result being in fantastic agreement with experiment [10]:

$$a_e = \begin{cases} 1159652180.73\,(0.28) \times 10^{-12} & \text{experiment,} \\ 1159652181.643\,(25)\,(23)\,(16)\,(763) \times 10^{-12} & \text{theory,} \end{cases} \tag{1.1}$$

where the numbers in parenthesis represent the uncertainties in the last digits (the theoretical prediction has uncertainties coming from several sources). These high order computations are typically laborious. For instance, the contribution of [10] was to recompute a subset of the $\mathcal{O}\left(e^{10}\right)$ terms with greater accuracy using Monte Carlo integration, which took 65 days on a 128 core computer cluster. Perturbation theory is widely useful in practice despite the difficulty of improving its accuracy to high orders.

However, there are important physical situations where naive perturbation theory fails. Electroweak baryogenesis is discussed in the next section as a concrete example of a real physics proposal where perturbation theory breaks down in several important steps of the computation. In such cases perturbation theory must be enhanced by resummation if it can be used at all. Resummation works by re-organising the terms of the perturbation series so that each term of the resummed series contains contributions from all orders of the original series. This re-organisation can manifest qualitatively important physical effects which are invisible to the original perturbation theory. For example, massless particles in thermal plasmas produce large loop corrections which must be resummed, causing thermal mass generation and non-analyticities in thermodynamic functions [11]. Similarly, large logarithms can invalidate naive perturbation theory in problems involving disparate length or energy scales. Resummation of these logarithms leads to the renormalisation group running of coupling constants [12].

Although the need to go beyond naive perturbation theory is clear in many cases, *ad hoc* resummations are problematic because perturbation series are asymptotic in nature [13] and the re-organisation of the terms of an asymptotic series is mathematically questionable at best [14]. Systematic resummation schemes are required to guarantee consistency with the original non-perturbative theory. There are three such schemes which can claim to be widely studied and successful: the renormalisation group (RG) [12], large N expansion

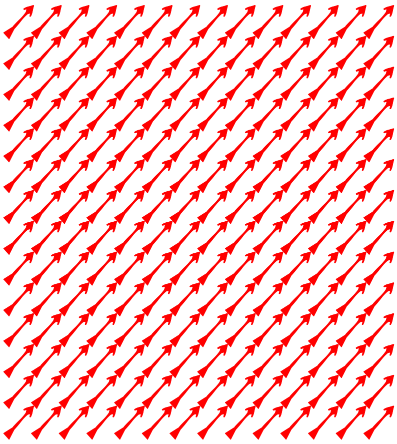
[15], and n-particle irreducible effective actions (nPIEAs) [16]. This thesis focuses on the latter.

nPIEAs are a set of techniques (one for each $n = 1, 2, 3, \cdots$) which can be thought of as generalisations of mean field theory which are (a) elegant, (b) general, (c) in principle exact, and (d) have been promoted for their applicability to non-equilibrium situations (see, e.g., [17] and references therein for extensive discussion of all these points). Non-perturbative methods are essential in non-equilibrium QFT because secular terms (i.e. terms which grow without bound over time) in the time evolution equations invalidate perturbation theory. nPIEAs with $n > 1$ achieve the required non-perturbative resummation in a manifestly self-consistent way which can be derived from first principles. "In principle exact" here means that the nPIEA equations of motion are exactly equivalent to the original non-perturbative definition of the quantum field theory. The only necessary approximation is in the numerical solution of these equations. The resulting equations of motion are also useful in equilibrium because many-body effects are included self-consistently. "General" means that the methods are applicable in principle to any quantum field theory whatsoever (although in a theory with many fields or with large n the resulting nPIEA could be very bulky). Finally, "elegant" here means that few conceptually new elements are needed in the formulation of nPIEAs in addition to the usual terms of textbook quantum field theory. The complication is mainly of a technical, not conceptual, nature. To the author's knowledge no other techniques satisfy all of these criteria.

These properties make the nPIEAs compelling candidates for filling the gap between theory and experiment anywhere the problem is the inability to compute the behaviour of strongly interacting degrees of freedom or degrees of freedom which are out of thermal equilibrium. As such, they form the basis of this thesis. Chapter 2 of this thesis reviews nPIEAs and then shows, in novel work, how they achieve a re-organisation of perturbation theory which is competitive with or exceeding more traditional resummation methods (using the 2PIEA and a toy model where exact results are available for the comparison). This success is explained in terms of the sensitivity of the nPIEA to analytic features of the exact solution which are invisible to perturbation theory. The remainder of the thesis focuses on one of the major shortcomings of nPIEAs – their poor handling of symmetries – and efforts to overcome it.

Symmetries are central to modern physics. The standard model is not a generic quantum field theory. It is a theory whose structure is determined by a large symmetry group[4]. The symmetry group of a model determines what types ("representations") of particles may appear, as well as governing the form of the interactions and the conservation laws of the model. In the simplest case symmetries are *global*, i.e. the same symmetry transformation applies everywhere in spacetime. In more complicated theories such as the standard model the symmetries are *gauge symmetries*, meaning that independent symmetry transformations can be applied at each event in spacetime without altering the

[4]Usually written as $\mathrm{P}\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ where P is the Poincaré group of spacetime transformations and (S)U (d) are the (special) unitary groups of $d \times d$ unitary matrices respectively. Strictly speaking however, the group is $\mathrm{Spin}(3,1) \times (\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$, where $\mathrm{Spin}(3,1)$ is the double cover of the Poincaré group and the quotient by the discrete group $\mathbb{Z}_6$ removes transformations which act trivially on the standard model fields.


(a)

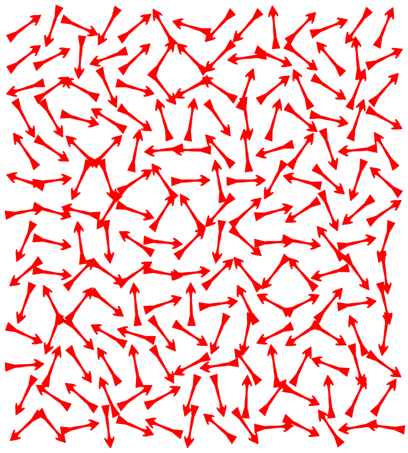

(b)

Figure 1.1: Schematic depiction of the spin orientation of atoms in a ferromagnet in the ordered phase (1.1a) and disordered phase (1.1b).

physics. The most important situation phenomenologically is the case of *spontaneously broken continuous symmetries*. In this case the symmetry is not actually broken, merely hidden because there is an interaction which causes a ground state which behaves non-trivially under the symmetry to be energetically preferred. The classic example is the ferromagnet: the underlying dynamics is rotationally symmetric, but there is an interaction which tends to align the spins of neighbouring atoms in the ferromagnet, causing the ground state to be one with all spins aligned. This ground state violates the underlying rotational symmetry and a hypothetical microscopic physicist living inside the ferromagnet may take a long time to notice that there is no such thing as a preferred direction in space. This situation is schematically depicted in Figure 1.1.

This simple picture is easily generalised and has become central to constructing realistic models of particle physics such as the standard model. In the standard model the symmetry involved is the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge symmetry[5], which is spontaneously broken in the ground state by an ordering interaction of the Higgs field. The result is that only the $\mathrm{U}(1)_Q$ symmetry of electromagnetism is manifest at low energies and the remaining gauge bosons (the $W^{\pm}$ and Z^0) gain large masses by the Higgs mechanism. This technique is essential to model building in particle and condensed matter physics, as the Higgs mechanism is the only known method to make a consistent theory of massive non-abelian gauge bosons[18][6]. Thus, for any technique meant to be applicable to the problems of modern physics, a decent handling of symmetries is of paramount importance.

Unfortunately, nPIEAs are known to violate symmetries when they are truncated to a form that can be solved in practice (see, e.g. [1] and references therein). This thesis focuses on theories with global symmetries because the issues faced by nPIEAs are similar for global

[5] The subscripts on $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ imply that the $\mathrm{SU}(2)$ gauge bosons only couple to left chiral fermions and the $\mathrm{U}(1)$ gauge bosons couples to the *hypercharge* quantum number Y rather than the electric charge Q.

[6] Another method by Stückelberg can be used to construct theories of massive $\mathrm{U}(1)$ gauge bosons, but *only* $\mathrm{U}(1)$ gauge bosons. See, e.g., [19]

and gauge symmetries, the main difference being one of complexity. The symmetries of a quantum field theory are embodied in a set of *Ward identities*[7]. These identities relate, for each n, the n-point Green functions of the theory (which can be thought of as scattering amplitudes for n particles) to the $n+1$-point Green functions in the limit that the incoming momentum of the "extra" particle goes to zero. From this description it is not obvious that the simplest ($n = 1$) Ward identity also embodies the famous *Goldstone theorem*, which states that *in any quantum field theory with a spontaneously broken global symmetry, each broken symmetry has a corresponding massless ("Goldstone") boson* [20]. This has profound phenomenological consequences as the existence of massless bosons generates long range interactions which do not exist in a purely massive theory.

The essential problem with symmetries in the nPIEA formalism is that the rearrangement of the perturbation series accomplished by nPIEAs violates the Ward identities order by order (this is shown in Chapter 3). The correct identities are only recovered for the *exact* solution of the nPIEA (which is just as intractable as solving the quantum field theory exactly in the first place). The result is that practical approximations to nPIEAs often have pathologies such as massive Goldstone bosons, which leads to *qualitatively* wrong physical predictions. It is important to note that the problem is with the method of calculation, not the underlying theory itself. If one takes the nPIEA predictions at face value one may reject a physically correct model on the basis of an incorrect prediction. Therefore it is critical to gain a better understanding of the symmetry problem in order for nPIEAs to be useful in the future.

This is where this thesis aims to make a contribution. Chapter 4 examines a method recently proposed by Pilaftsis and Teresi [1] to improve the symmetry properties of nPIEA. Their *symmetry improvement* method is generalised and extended, then applied to a number of case studies in equilibrium. It is found to be an effective method with some limitations. Chapter 5 then introduces a novel technique called *soft symmetry improvement* which aims to address these limitations. This method is shown to possess as limits both the unimproved and symmetry improved effective actions, as well as a novel limit which is examined in detail. Unfortunately the novel limit is pathological, but the nature of the pathology hopefully sheds some light on the difficulty of handling symmetries in nPIEA more generally. Then Chapter 6 extends the consideration of symmetry improvement beyond equilibrium for the first time, finding new pathologies with the method. Chapter 7 brings everything together, drawing conclusions and pointing out directions for future work. Finally, there are three appendices which contain intermediate results and code for reference purposes, but which would otherwise interrupt the flow of the main text.

1.2 A case study: Electroweak baryogenesis

Electroweak baryogenesis (EWBG) is a hypothetical process which may have taken place in the very early universe, when the temperature of the universe was of the order of ~ 100 GeV. EWBG creates an asymmetry in the abundance of baryons (protons, neutrons

[7] The analogous identities for abelian gauge theories are called Ward-Takahashi identities and for non-abelian gauge theories Slavnov-Taylor identities [18].

and related particles which also have the quantum numbers of three bound quarks) and anti-baryons. This asymmetry persists after the universe cools down and is ultimately responsible for the abundance of matter in the universe. EWBG is considered in this section as a case study of a real physics proposal intimately involving both non-perturbative and non-equilibrium aspects. As such, it is difficult to compute accurate and reliable EWBG predictions from fundamental physics. The purpose of the following discussion is to highlight these aspects of EWBG, not to give a comprehensive review of the current state of the field. For more detailed treatments the reader is encouraged to consult the original literature cited below.

It is well known that for every type of matter particle (electron, proton, neutron, quarks etc.) there is a corresponding antiparticle with the same mass and opposite charge. This is now understood to be a consequence of combining the principles of special relativity and quantum mechanics, which requires that antiparticles exist and have closely related properties to the corresponding particles[8]. This is a very robust result, requiring only basic physical principles, and it is satisfied by the SM. Further, the standard cosmological model of the universe with a cosmological constant and cold dark matter (ΛCDM) predicts that the early universe was in a very hot and dense state, where the maximum temperature that the universe reaches could be as high as 10^{12} GeV [21]. At extremely high temperatures the universe contains a plasma where particle (ψ)/antiparticle ($\bar{\psi}$) pair creation and annihilation reactions such as $\gamma\gamma \leftrightarrow \psi\bar{\psi}$ are in equilibrium.

As the universe expands it cools, and when the temperature drops below $\sim m_\psi c^2/k_B$ the creation reactions $\gamma\gamma \to \psi\bar{\psi}$ become Boltzmann suppressed. However, the annihilation reaction $\psi\bar{\psi} \to \gamma\gamma$ remains efficient as long as the (thermally averaged) reaction rate $\Gamma \sim \langle n\sigma v\rangle$, where n is the number density, σ is the annihilation cross section and v is the relative velocity, is more rapid than the Hubble expansion rate of the universe $H = \dot{a}/a$, where $a\,(t)$ is the cosmic scale factor. In this regime particle/antiparticle pairs are annihilated, depleting the number densities of both and temporarily slowing the cooling of the plasma. Once $\Gamma \lesssim H$ the annihilation reactions shut off and the $\psi, \bar{\psi}$ effectively decouple from the plasma, a phenomenon known as *freeze out*. After freeze out the *relic densities* $n_{\psi(\bar{\psi})} \propto a^{-3}$ simply track the expansion of the universe. (This story assumes that ψ particles are stable for simplicity.) Thus one should expect the number density of any species to be determined by the mass, reaction rates and expansion history in the way described, allowing one to derive from a model of particle physics a prediction for the amount of matter and antimatter in the universe.

The relic densities derived this way for the SM lead immediately to two puzzles: first, they are too small by about ten orders of magnitude [22], far too small for galaxies to form, and the density of matter and antimatter should be comparable. Yet astronomical observations have convincingly shown that the universe is made of matter. The abundance of antimatter is negligible compared to that of matter [23]. The baryon asymmetry of the

[8] It is sometimes said that neutral particles such as the photon and Higgs boson have no anti-particles, or that they are their own anti-particles. This is a matter of semantics, not physics, and the two descriptions are equivalent. The convention of this thesis is that *all* particle species have a corresponding anti-particle species (which is sometimes identical to the particle species).

universe (BAU) has been measured at the time of *big bang nucleosynthesis* (BBN), about three minutes after the big bang, using the abundances of light elements as [24, 25]

$$\eta \equiv \frac{n_B - n_{\bar{B}}}{n_\gamma} = \begin{cases} (5.9 \pm 0.5) \times 10^{-10}, & \text{Kirkman } et\ al.\text{, cited in [24],} \\ (5.8 \pm 0.27) \times 10^{-10}, & \text{Steigman [25],} \end{cases} \tag{1.2}$$

where $n_{B(\bar{B})}$ are the number densities of (anti-)baryons and n_γ is the number density of photons. This ratio should be conserved with the expansion of the universe since $n_{B,\bar{B},\gamma}$ all scale the same way with $a(t)$[9]. Thus an independent confirmation can be obtained from the cosmic microwave background, and the latest Planck data [26] is in agreement[10]:

$$\eta = (6.05 \pm 0.09) \times 10^{-10}. \tag{1.3}$$

The simplest possible explanation for the observed BAU is that it stems from an asymmetry of the initial condition of the universe. However a net baryon number $B \neq 0$ initial condition for the universe is implausible as a solution to this problem for several reasons. First, if cosmic inflation is indeed the solution to the horizon, flatness and monopole problems in cosmology this requires at least about 60 e-folds of expansion which would greatly dilute any initial baryon asymmetry. Second, if baryon number conservation is violated by any beyond standard model (BSM) physics at high energies then any initial $B \neq 0$ would be washed out when the universe comes into chemical equilibrium at a temperature $T \gg T_{EW}$, where $T_{EW} \sim 100$ GeV is the electroweak scale of the SM. Third, the SM itself violates B conservation and lepton number L conservation through non-perturbative *sphaleron* processes [27]. The only exactly conserved flavour quantum number of the SM is the combination $B - L$. Any initial asymmetry in the orthogonal channel $B + L$ will be washed out by sphaleron processes at $T > T_{EW}$. Note that at temperatures $T < T_{EW}$ sphalerons are exponentially suppressed and unobservable, hence this scenario is compatible with constraints on unobserved processes like proton decay $p \to e^+\gamma$.

If the BAU cannot be the result of an asymmetric initial condition it must be generated dynamically by some new BSM particle physics. This is called *baryogenesis*. Numerous recent reviews of popular scenarios are available [22, 28–33]. In 1967, Sakharov [28, 34] laid out three general conditions which are required to dynamically generate $B \neq 0$ from a symmetrical initial state[11]:

1. Baryon number violation: Obviously if B is conserved then $B \neq 0$ cannot be generated from a state with $B = 0$.

2. Out of equilibrium dynamics: If the universe is in thermal equilibrium then every B-violating reaction is balanced by the reverse reaction and no net asymmetry can be generated.

[9] Up to a threshold correction due to electron/positron freeze out $e\bar{e} \to \gamma\gamma$ at temperatures $T \sim m_e c^2/k_B$, which is straightforward to correct for.

[10] Planck measured the parameter $\Omega_b h^2 = 0.02207 \pm 0.00033$ which is converted to η via $\eta = \Omega_b h^2 / \left(3.65 \times 10^7\right)$. See [23, 24] for further details.

[11] Note that, although the Sakharov's original paper dealt with a particular model of baryogenesis, his three conditions are completely general.

3. C and CP violation: If charge conjugation (C) or charge-parity conjugation (CP) is preserved then for every B-violating reaction there is a C or CP conjugate reaction that has the same rate but opposite B asymmetry. Hence every B-violating reaction is balanced by another reaction with the opposite effect and no net B can be created.

The SM has all three of the necessary ingredients: B violating sphaleron processes come from the non-perturbative electroweak sector of the SM, C is maximally violated by the chiral fermion couplings of the weak interactions, CP is violated by the complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix in the quark sector, and the SM plasma could be driven out of equilibrium by the electroweak phase transition (EWPT).

For sufficiently small Higgs mass the EWPT is a first order transition which proceeds by bubble formation, analogous to boiling water, a violent process that locally departs from thermal equilibrium at the bubble walls. It was within the realm of possibility then, before the late 1990s (when the LEP2 experiment began placing limits on the Higgs mass), that the SM alone could account for baryogenesis. However, it is now known that the Higgs mass $m_H = 126$ GeV > 72 GeV, where 72 GeV is the critical point of the EWPT. For values of m_H greater than the critical value the EWPT is a smooth crossover transition which does not depart from equilibrium. Thus m_H is too large for the EWPT to depart from equilibrium, violating the second Sakharov condition. Further, detailed computations show that the CP violating phase of the CKM matrix is many orders of magnitude too small to account for the observed BAU [28].

Since the time of Sakharov, numerous models have been proposed that satisfy the Sakharov criteria. A recent review counted 44 scenarios in the literature for generating the BAU [31]. Of those only a handful are well motivated and provide decent prospects for experimental confrontation in the foreseeable future. These are the models that propose new particles at or below the electroweak scale. One prominent option is to include additional scalar fields that couple to the SM Higgs and strengthen the EWPT, and may also provide new sources of CP violation. Such scalars can come from supersymmetric (SUSY) extensions of the SM, such as the well studied *minimal SUSY extension of the SM* (MSSM), or from grand unified theories (GUTs). These models are classified as *electroweak baryogenesis* (EWBG) scenarios since they all involve the EWPT in an extended electroweak/Higgs sector.

EWBG was first investigated in the SM [35], however, with the measured Higgs mass, modern EWBG scenarios are all formulated in more or less minimal extensions of the SM such as the MSSM, the *next to minimal SUSY extension* (NMSSM) with a single extra gauge singlet chiral superfield, and non-SUSY extensions such as the *two Higgs doublet models* (2HDM). In all of these models the EWBG scenario relies on non-perturbative processes very similar to those of the original SM. The SM is briefly reviewed here, without going into the formal field theory, to recall the physics and discuss EWBG at first in a familiar context. This review is by no means comprehensive, nor suitable for a first introduction. Up to date topical reviews are available from the Particle Data Group [8].

The SM is a relativistic quantum gauge field theory. The gauge group is $\mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. $\mathrm{SU}(3)$ is the gauge group of quantum chromodynamics (QCD). The

Table 1.1: SM bosonic representations. Y is weak hypercharge. The physical photon and Z^0 bosons are both mixtures of the neutral weak boson and the hypercharge boson.

Particle(s)	Spin	$SU(3)$ rep	$SU(2)$ rep	Y
Gluons	1	**8**	**1**	0
Weak bosons	1	**1**	**3**	0
Hypercharge boson	1	**1**	**1**	0
Higgs boson	0	**1**	**2**	+1

Table 1.2: SM fermionic representations. All are left chiral spin 1/2. Y, Q, B, and L are weak hypercharge, electric charge, baryon number and lepton number respectively.

Particle(s)	$SU(3)$ rep	$SU(2)$ rep	Y	Q	B	L
Quarks $\begin{pmatrix} \text{up type: up, charm, top} \\ \text{down type: down, strange, bottom} \end{pmatrix}$	**3**	**2**	$\frac{1}{3}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$\frac{1}{3}$	0
Leptons $\begin{pmatrix} e-,\ \mu-,\ \tau-\text{neutrinos} \\ \text{electron, muon, tau} \end{pmatrix}$	**1**	**2**	-1	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	0	1
Up-type antiquarks	$\mathbf{3}^\star$	**1**	$-\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0
Down-type antiquarks	$\mathbf{3}^\star$	**1**	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
Anti-leptons	**1**	**1**	2	+1	0	-1

QCD gauge bosons are the eight *gluons*. The electroweak gauge group is $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. $\mathrm{SU}(2)_L$ describes the weak *isospin* and acts only on left chiral fermions, hence the subscript L. The $\mathrm{U}(1)_Y$ is the weak *hypercharge* group. The groups $\mathrm{SU}(3)$, $\mathrm{SU}(2)_L$, $\mathrm{U}(1)_Y$ have coupling constants g_S, g and g' respectively. There is also a Higgs field which is a scalar (i.e. spinless) and a doublet under $\mathrm{SU}(2)_L$ with hypercharge $+1$. There are three generations of fermions which are identical apart from mass. The fermions fall into *quarks*, which are charged under $\mathrm{SU}(3)$ and bind into *baryons*, and *leptons* which are not charged under $\mathrm{SU}(3)$. The standard model particles are summarised in Tables 1.1 and 1.2.

Due to an ordering interaction of the Higgs field, the electrically neutral component of the Higgs doublet develops an expectation value v. When this happens the electroweak symmetry $SU(2)_L \times U(1)_Y$ is no longer manifest. Instead the weak bosons $W^\pm$ and Z^0 gain mass by the Higgs mechanism, while the remaining electroweak boson A becomes the massless photon. The masses of the weak gauge bosons are[12] $m_W = gv/\sqrt{2}$ and $m_Z = m_W/\cos\theta_W$, where θ_W is the Weinberg angle given by $\cos\theta_W = g/\sqrt{g^2+g'^2}$. The fundamental electric charge is identified as $e = g\sin\theta_W$. The fermions couple to the Higgs doublet with *Yukawa* couplings y_i which, after symmetry breaking, result in fermion masses $m_i = y_i v$. All of the SM fermions gain their mass in this way.

Finally notice that the *classical* field equations of the SM conserve baryon number B and lepton number L as defined by Table 1.2. These generate a *global* (not gauged) $\mathrm{U}(1)_B \times \mathrm{U}(1)_L$ symmetry of the theory. This is an accidental symmetry in that it was

[12]Note that the relation between m_W and m_Z is modified by radiative corrections.

not put in by design, but is rather a result of the fact that the only terms possible to write down with the given fields *happen* to obey it. However, symmetries of theories with chiral fermions coupled to gauge bosons are spoiled *chiral anomalies* [36, 37] and indeed quantum effects reduce the global symmetry group to $\mathrm{U}(1)_{B-L}$. Thus $B+L$ is *not* a conserved quantum number of the SM at a quantum level, but $B-L$ *is*.

The mechanism of B and L violation is topological in nature and as such is invisible to perturbation theory. At zero temperature it proceeds by quantum tunnelling between gauge vacua with different topologies. A potential barrier separates these field configurations and the leading contribution to the transition rate at zero temperature comes from the semi-classical tunnelling trajectory called an *instanton* (see, e.g. [38, 39]). Kuzmin et al. [40] argued that at finite temperature the transition proceeds by classical fluctuations *over the barrier* at a rate increasing exponentially with temperature, finally becoming unsuppressed entirely at sufficiently high temperature. The height of the barrier can be computed by finding the (unstable) configuration of the gauge and Higgs fields that maximises the energy atop the barrier. These configurations are called *sphalerons* and were discovered by Klinkhamer and Manton [41]. Further early analysis of these configurations in the SM were carried out in [27]. Pedagogical discussions can be found in [37, 42].

The weak gauge fields possess a topological invariant called the *Chern-Simons* number $\mathcal{K}$ which, effectively, measures how many times the gauge field configuration "winds" around the sphere at infinity. In vacuum $\mathcal{K}$ is an integer. Transitions from one gauge vacuum to another are possible and when this happens the chiral anomaly is responsible for changing B and L by $\Delta B = \Delta L = 3\Delta\mathcal{K}$. More generally $\Delta B = \Delta L = n_g \Delta\mathcal{K}$ where n_g is the number of fermion generations. One can sketch the potential energy along the $\mathcal{K}$ "direction" in field space as shown in Figure 1.2. The sphaleron is the minimal energy configuration interpolating between two adjacent vacua (i.e., the smallest barrier). Note that this figure is only illustrative. To find an honest expression for the potential energy one needs to develop a one parameter ansatz for the gauge and Higgs fields which passes through the vacuum and sphaleron field configurations at appropriate parameter values and compute $\mathcal{K}$ and the energy of the field configuration as a function of the parameter. The resulting figure is somewhat arbitrary, depending on the choice of ansatz. Physically, there are many paths over the ridge in field space represented by the sphaleron.

The height of the barrier and the sphaleron solution are found from a static configuration of the gauge and Higgs fields that extremises the energy subject to the boundary conditions imposed by the required Chern-Simons number. The resulting field profiles must be found numerically [41], but in the $g' \to 0$ limit they depend only on the ratio λ^2/g^2 where λ is the Higgs self-coupling. The energy can then be written

$$E = \frac{4\pi v}{g} C\left(\frac{\lambda^2}{g^2}\right), \tag{1.4}$$

where the numerical factor $C\left(\lambda^2/g^2\right)$ has been estimated in the original literature. Its precise value is not important here, but $C \sim \mathcal{O}(1-3)$ for the whole range of λ^2/g^2 from 0 to $+\infty$. Thus the height of the barrier at zero temperature is of the order of 10 TeV. One

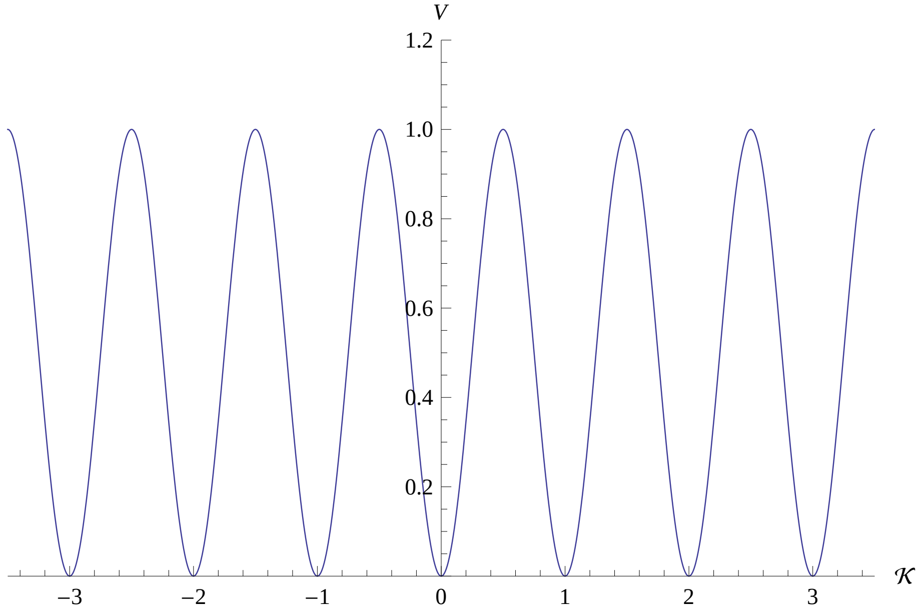


Figure 1.2: Potential energy along the Chern-Simons number $\mathcal{K}$ "direction" in (infinite dimensional) field space for an $SU(2)$ gauge theory. Degenerate vacua correspond to integer values of $\mathcal{K}$, which is then equal to the winding number of the gauge field. Vertical scale is arbitrary.

can thus think of the sphaleron, very roughly, as an unstable bound configuration with of the order of 30 each of W^+, W^-, Z and 10 Higgs particles in a region $\sim 10^{-17}$ m across. In reality the sphaleron is a coherent state, not a particle number eigenstate, so this picture is at best qualitative. However, this illustrates that, while 10 TeV might not sound like a terribly high energy for modern colliders, sphalerons are difficult if not impossible to produce at present day collider experiments [42].

The second Sakharov criterion requires that the early universe depart from thermal equilibrium at some point. This is achieved in EWBG by a first order phase transition associated with electroweak symmetry breaking. Recall that if the Higgs field(s) has a non-vanishing expectation value the electroweak symmetry is said to be broken, while if the expectation value is zero the symmetry is manifest or unbroken. The behaviour of the Higgs field ϕ in equilibrium can be determined by minimising the effective potential $V(\phi)$. $V(\phi)$ is determined from $\Gamma[\phi]$, which is the (Euclidean) *one particle irreducible effective action* of the Higgs field. This formalism will be discussed in detail in Chapter 2. The effective potential can be computed in a semi-classical expansion in powers of $\hbar$. The lowest order term is just the classical Higgs potential $V^{(0)}(\phi) = \mu^2\phi^2 + \frac{\lambda^2}{2}\phi^4$. The first correction is due to Feynman diagrams with one loop, which can be summed up to $V^{(1)}(\phi) = V^{(1)}_{\text{vac}}(\phi) + V^{(1)}(\phi, T)$, where $V^{(1)}_{\text{vac}}(\phi)$ is the contribution present at $T = 0$ which serves to renormalise[13] the parameters in $V^{(0)}(\phi)$, and $V^{(1)}(\phi, T)$ is the temperature

[13] This ignores the logarithmic Coleman-Weinberg terms, which can be justified by choosing an appropriate renormalisation scale.

dependent correction [11, 43, 44]:

$$V^{(1)}(\phi, T) = \sum_i \Delta V_i^{(1)}, \tag{1.5}$$

where the sum runs over all species. One can assume that the universe as a whole is neutral and that $B - L = 0$ because the baryon asymmetry $\eta \sim 6 \times 10^{-10}$ is small and is expected to have only a small impact on the thermal properties of the early plasma. For illustrative purposes it is good enough to perform a high temperature expansion of the $\Delta V_i^{(1)}$ for $m \ll T$ of the form [11]

$$\Delta V^{(1)}_{\text{fermions}} = -\frac{7\pi^2}{720}T^4 + \frac{1}{48}m^2T^2 + \mathcal{O}\left(m^4\right), \tag{1.6}$$

$$\Delta V^{(1)}_{\text{bosons}} = -\frac{\pi^2}{90}T^4 + \frac{1}{24}m^2T^2 - \frac{1}{12\pi}m^3T + \mathcal{O}\left(m^4\right). \tag{1.7}$$

For a species obtaining mass from the Higgs, $m \sim g\phi$ or $m \sim y\phi$ for gauge bosons and fermions respectively. Thus the terms independent of the mass m give a ϕ independent contribution to the potential, thus having no impact on the derivative $\mathrm{d}V/\mathrm{d}\phi$ or the location of the extrema, hence these can be ignored for present purposes.

To illustrate a first order phase transition assume (incorrectly) that $m_H < m_W, m_Z, m_t$, i.e. that the Higgs boson is lighter than the $W^\pm$, Z and top quarks, and ignore the contributions of all particles lighter than the $W^\pm$ bosons since the heavy particles have a correspondingly greater coupling to the Higgs field. The assumption that the Higgs is light allows the use of perturbation theory. Higher order loop corrections involving $W^\pm$ bosons come in with an expansion parameter $\sim m_H^2/m_W^2$ and perturbation theory breaks down for $m_H \gtrsim m_W$. Lattice computations have shown that the line of first order phase transitions terminates in a second order critical point at $m_H \sim 70$ GeV and for the physical Higgs mass $m_H \sim 125$ GeV there is no phase transition between the broken and symmetric phases, only a smooth crossover [11].

The effective potential in this case is shown in Figure 1.3. To leading order in the loop and high T expansions $V_{\text{eff}}(\phi, T) = \left(-\frac{m_H^2}{2v^2} + a\frac{T^2}{v^2}\right)\frac{\phi^2}{v^2} - b\frac{T}{v}\frac{\phi^3}{v^3} + \frac{m_H^2}{4v^2}\frac{\phi^4}{v^4}$ where $a = \frac{1}{8}y_t^2 + \frac{1+2\cos^2\theta_W}{16\cos^2\theta_W}g^2 \approx 0.2$ (where y_t is the top quark Yukawa coupling) is not to be confused with the cosmological scale factor and $b = \frac{1+2\cos^3\theta_W}{8\pi\sqrt{2}\cos^3\theta_W}g^3 \approx 0.03$. In the plots $m_H = 0.1v \sim 18$ GeV for illustrative purposes. At temperatures above the upper spinodal temperature $T_+ = \frac{2m_H^2}{\sqrt{8am_H^2 - 9b^2v^2}} \approx 0.225v$ the minimum at $\phi = 0$ is the only critical point of the effective potential and the electroweak symmetry is manifest. Between the upper and lower spinodal temperatures, $T_+ > T > T_- = \frac{m_H}{\sqrt{2a}} \approx 0.158v$, a second minimum develops at $\phi \neq 0$ which is separated by a barrier from the minimum at $\phi = 0$. At the critical temperature $T_\star \approx 0.213v$ the new minimum is degenerate with the minimum at $\phi = 0$. At temperatures $T < T_\star$ the broken phase is preferred and for $T < T_-$ the coefficient of ϕ^2 in V_{eff} vanishes and $\phi = 0$ ceases to be a minimum of the effective potential altogether.

The history of the universe in this scenario is as follows. At $T > T_+$ the electroweak symmetry is unbroken. As the universe expands the temperature drops to $T_\star < T < T_+$. In this regime bubbles of broken phase can form by thermal fluctuations but these are unstable and rapidly disappear. At $T = T_\star$ the interior of the bubbles of broken phase are degenerate with the unbroken phase, but there is an energy cost associated with the bubble walls that still renders them unstable. At $T_- < T < T_\star$ the broken phase has an energy advantage $\Delta V = -V_{\rm eff}(\phi_{\rm min}, T)$ and a spherical bubble of broken phase has an energy

$$E = 4\pi R^2 \sigma - \frac{4\pi}{3} R^3 \Delta V, \tag{1.8}$$

where σ is the surface tension of the bubble wall. The bubble will expand if it is larger than the critical size $R_\star = 2\sigma/\Delta V$. The rate of bubble formation is $\Gamma \sim \exp(-E_\star/T)$ where $E_\star = \frac{16\pi}{3}\frac{\sigma^3}{\Delta V^2}$ is the energy of the critical bubble. Bubble nucleation becomes cosmologically significant when this rate exceeds the Hubble rate, which requires that T drop sufficiently below $T_\star$. At this time bubbles begin to form and expand, eventually filling the universe. The volume fraction of broken phase as a function of time and the completion time of the phase transition can be computed from a simple macroscopic model once a few plasma parameters are known, namely the latent heat, surface tension and wall velocity [45, 46]. The wall velocity in turn can be calculated using a hydrodynamic theory given the friction on the bubble wall, at least for sufficiently large bubbles in the thin wall limit [46]. It is the job of a micro-physical model to calculate the wall friction, latent heat and surface tension of a phase transition bubble.

The friction on bubble walls is the result of particles in the unbroken phase scattering on the bubble wall and being driven out of local thermal equilibrium. It is this breaking of local thermal equilibrium which satisfies the Sakharov criterion for baryogenesis in EWBG models. If the scatterings also violate C and CP, e.g. by preferentially scattering left handed particles, then a net asymmetry of left handed particles builds up in the unbroken phase in front of the wall (balanced by an opposite right handed asymmetry which passes through the wall). The left handed particles outside the wall bias sphaleron transitions to produce a net baryon number. It is important that in the broken phase the sphaleron transitions are turned off, otherwise they will wash out the net asymmetry. This requires that the phase transition be sufficiently strong, typically taken to mean $\Delta\phi/T_\star \gtrsim 1$ where $\Delta\phi$ is the difference in ϕ between the phases at $T_\star$, but it is not trivial to determine how strong is really strong enough. Indeed, a proper determination of the washout factor, beyond a few conservative estimates, is one of the main outstanding problems in EWBG [29].

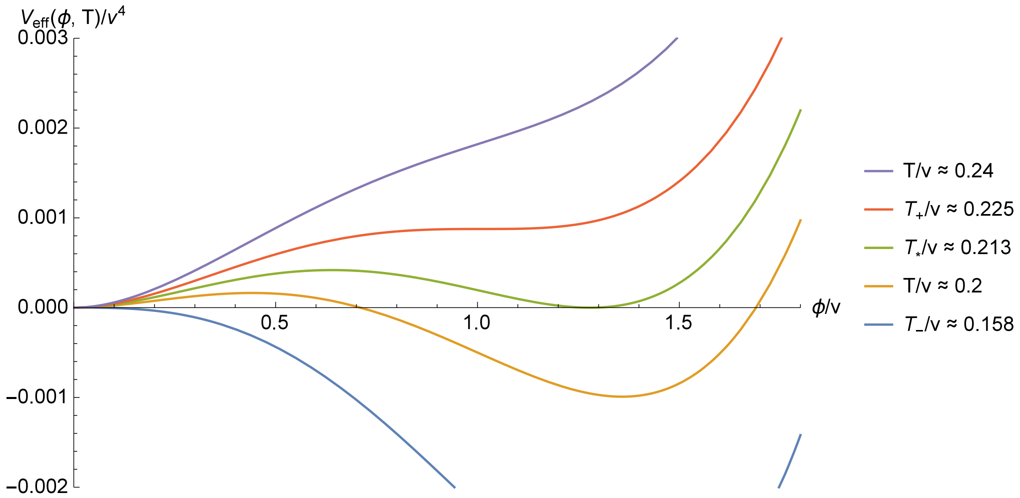
The qualitative reason CP violation leads to baryogenesis is as follows. Suppose that CP is violated in quark-wall scattering such that $q_L + \bar{q}_R$ reflect more readily than $q_R + \bar{q}_L$. Now, temporarily "turn off" sphalerons for illustrative purposes. The $q_R + \bar{q}_L$ build up in front of the bubble wall (recall that Higgs-Yukawa interactions change fermion chirality) while a compensating CP asymmetry passes through into the broken phase. There is no baryon asymmetry at this stage, and if it were not for sphaleron transitions no asymmetry would be generated: though the $q_R + \bar{q}_L$ tend to reflect from the bubble wall they will all eventually pass through, certainly by the time the broken phase fills all of space. Now turn sphalerons back on. Sphalerons only act on the L particles, converting $3n_f\bar{q}_L \to n_f l_L$ and $3n_f q_L \to n_f \bar{l}_L$. The rate of the former process, with $\Delta B = +1$, exceeds the rate of the latter, with $\Delta B = -1$, since there are more $\bar{q}_L$ than q_L in front of the wall. Hence a net baryon number is generated in front of the bubble wall.

Clearly the net baryon asymmetry produced by this mechanism depends on a number of factors: how strong is the CP violation in the reflection/transmission coefficients; how far do reflected particles diffuse into the plasma ahead of the bubble wall and for how long; what is the rate of sphaleron transitions in the plasma ahead of and behind the bubble wall; and how fast are other processes besides sphalerons which tend to wash out CP asymmetries (without producing B or L asymmetries)? The current practice for quantitative calculations is to: (1) calculate bubble wall reflection/transmission coefficients using a non-equilibrium quantum transport theory or semi-classical Boltzmann equation, using bubble wall velocities obtained from a hydrodynamic treatment and ignoring the back-reaction of the generated asymmetries on the bubble wall; (2) perform lattice Monte Carlo calculations of the sphaleron rate, in one of a few simplified models, in equilibrium at various temperatures to calibrate a semi-empirical formula which hopefully applies out of equilibrium to the required accuracy; and (3) combine these elements in a network of continuity equations which are integrated numerically to find the final baryon (and lepton) asymmetry [29, 43].

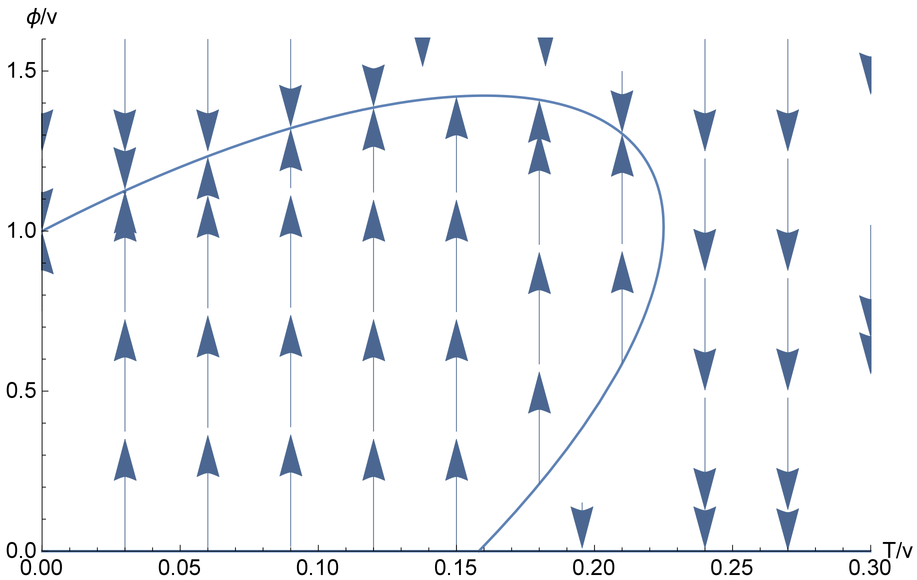
By way of contrast, a second order phase transition is illustrated in Figure 1.4. The key difference from a first order phase transition is that there is never more than one minimum of the effective potential. The upper and lower spinodal temperature are equal to the critical temperature $T_{\pm} = T_{\star}$. As the temperature decreases the universe smoothly evolves from an unbroken to a broken phase and there is never any bubble nucleation. As a result the plasma never departs from local thermal equilibrium and there is no opportunity for EWBG. In the SM the EWPT is second order for $m_H \gtrsim 72$ GeV [29], thus it is necessary to investigate EWBG in the context of SM extensions.

The obvious major problem facing EWBG scenarios is the order of the electroweak phase transition. It is necessary for the transition to proceed through bubble formation as in a first order transition, and also that the scalar field behind the bubble wall be sufficiently large (a measure of the strength of the phase transition) that sphaleron processes are effectively shut off, lest they erase the baryon asymmetry that was so painstakingly created. The most straightforward approach to strengthening the EWPT is adding new scalar fields which are strongly coupled to the Higgs. However, any models with new particles necessarily have undetermined parameters which must be scanned to assess the range of EWBG predictions possible. Typically perturbation theory is used since it is relatively easy to scan parameters, however perturbative results do not necessarily agree with more accurate (but harder to scan) lattice methods. See e.g. [47] for a perturbative discussion of EWBG in the MSSM and contrast this with the lattice Monte Carlo studies using dimensional reduction [48–50] which suggest that the EWPT can be much stronger than the perturbative estimates. Unfortunately these lattice methods have been applied only to a few well studied models such as the MSSM.

It is possible to investigate new physics in a fairly model independent way if a few simplifying assumptions are satisfied. If the new physics, however complicated, exists at a

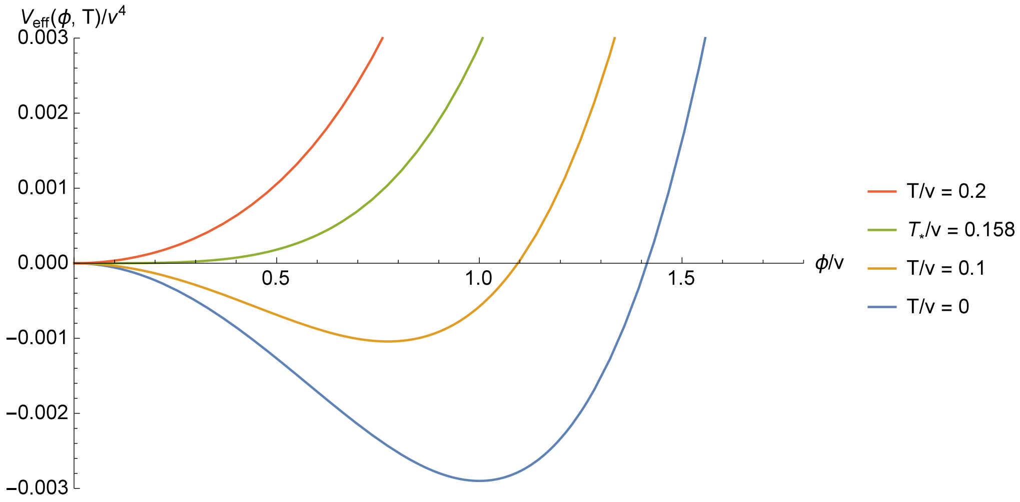


(a)

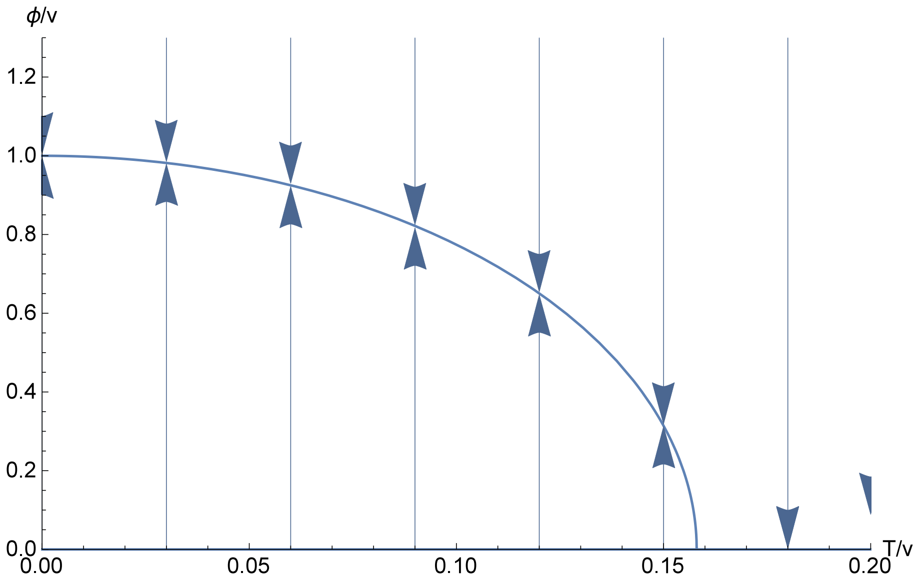


(b)

Figure 1.3: 1.3a: Thermal effective potential of a system with a first order phase transition. Curves are shown for the approximation to the SM discussed in text, with the Higgs mass chosen at the un-physical value $m_H = 0.1v$. 1.3b: Critical points of the effective potential as functions of the temperature with streamlines illustrating their stability.



(a)



(b)

Figure 1.4: Same as Figure 1.3 only for a system with a second order phase transition.

scale M which is heaver than the weak scale v, and there exists a separation scale Λ such that $v \ll \Lambda \lesssim M$ then it is possible to use an effective field theory below the scale Λ with only the SM degrees of freedom. The new physics manifests as additional operators in the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \sum_i \frac{1}{\Lambda} c_i^{(5)} \mathcal{O}_i^{(5)} + \sum_i \frac{1}{\Lambda^2} c_i^{(6)} \mathcal{O}_i^{(6)} + \mathcal{O}\left(\frac{1}{\Lambda^3}\right), \tag{1.9}$$

where $\mathcal{O}_i^{(5,6)}$ are the set of all gauge invariant operators of mass dimension 5 and 6 respectively built only out of SM fields[14]. The $c_i^{(5,6)}$ are dimensionless (*Wilson*) coefficients which are expected to be $\mathcal{O}(1)$ if they come from a generic model of new physics (if the new physics has a symmetry some of the coefficients can vanish, or be loop suppressed). Amplitudes calculated in the effective theory are expected to be accurate to $(E/\Lambda)^3$ corrections, where E is energy scale for the process at hand.

A complete basis of linearly independent operators $\mathcal{O}_i^{(5,6)}$ has been found for the SM degrees of freedom [51]. Altogether there are 65 of them: one dimension 5 term that generates Majorana neutrino masses after electroweak symmetry breaking, and 64 dimension 6 operators: 59 of which conserve B and L, and 5 four-fermion contact interactions which violate[15] B and L (but conserve $B-L$). Of all of these operators, only one affects the Higgs potential: the dimension six term ϕ^6. Thus a conservative, and largely model independent, exploration of BSM modifications to the Higgs potential can be achieved by taking as the Higgs potential

$$V(\phi) = \mu^2\phi^2 + \frac{\lambda}{2}\phi^4 + \frac{1}{\Lambda^2}\phi^6. \tag{1.10}$$

EWBG with this potential has been investigated in [52] which finds that $\Lambda \sim 500-800$ GeV gives a strong first order phase transition for the physical Higgs mass $m_H = 125$ GeV. The ϕ^6 term is very difficult to probe at the LHC, thus the constraints are weak unless the new physics generates other dimension 6 operators at the same scale. As a result, LHC constraints do not yet significantly touch this range of Λ, though with enough luminosity at 14 TeV the LHC could begin to constrain this scenario through measurements of the Higgs cubic self-coupling [47, 52].

Thus it is easy to strengthen the EWPT in BSM models of current phenomenological interest. However, questions can be raised about the computation of the phase transition thermodynamics. A conceptual question of great practical import is the gauge dependence of this theory, which is relevant in the Higgs theory because an expectation value of a gauge-variant quantity like ϕ is physically meaningless. Two methods are available to check the gauge dependence of the result: redo the computation in a general class of gauges such as the R_ξ gauge and check the dependence on the ξ parameter, or develop a formulation using only manifestly gauge invariant quantities. An example of the first method is the jump of the order parameter at the critical temperature, ϕ_c/T_c, calculated by standard techniques in the SM [53]:

$$\frac{\phi_c}{T_c} = \frac{3-\xi^{3/2}}{48\pi\lambda}\left(2g_2^3 + \left(g_1^2+g_2^2\right)^{3/2}\right), \tag{1.11}$$

[14]Do not confuse the operators $\mathcal{O}_i^{(5,6)}$ with the mathematical "big-oh" notation $\mathcal{O}(\cdot)$!

[15]Note that such operators are typically constrained to a very high scale $\Lambda \gtrsim 10^{12}$ GeV as they generate tree-level proton decay, unlike the non-perturbative sphaleron/instanton processes.

where λ is the Higgs quartic coupling and g_1, g_2 are the $\mathrm{U}(1)$, $\mathrm{SU}(2)_L$ gauge couplings respectively (note the normalisation convention for λ is different from the one used here!). Note the spurious gauge dependence of this quantity. Indeed by choosing $\xi = 3^{2/3}$ the order parameter can be made to vanish! Similarly, calculations of the sphaleron rate and washout factors are gauge dependent and so far only lattice results are theoretically solid, but these have only been done in a small number of cases [29]. As a result it is difficult to say whether a given phase transition is indeed strong enough to avoid washing out a baryon asymmetry.

An example of the second method is the lattice Monte Carlo method, where it is necessary to use gauge invariant order parameters from the outset (see [54, 55] for examples showing how gauge invariant observables can be judiciously chosen to extract the desired physics). Unfortunately it is infeasible to implement and interpret lattice computations for every scenario in every model of interest, so it remains valuable to develop appropriately gauge invariant perturbative methods. Also, lattice methods can only be straightforwardly applied to equilibrium problems. Thus real time methods which allow an extension to non-equilibrium situations *and* have manifestly good gauge invariance properties are particularly valuable.

Once a first order phase transition is in hand, the probability of bubble formation can be found in equilibrium by saddle-point evaluation of the Euclidean functional integral. At zero temperature the pioneering theory of Coleman and Callan [56, 57] remains essentially unmodified (c.f. [42, 44, 58] for more recent discussions). They find that the bubble formation probability per unit volume per unit time for a single scalar field ϕ with potential $U(\phi)$ and false vacuum solution ϕ_+ (with the convention $U(\phi_+) = 0$) is, to leading semi-classical order,

$$\frac{\Gamma}{V} = \mathrm{e}^{-B/\hbar} \frac{B^2}{4\pi^2\hbar^2} \left| \frac{\det' \left[-\partial^2 + U''\left(\bar{\phi}\right)\right]}{\det\left[-\partial^2 + U''\left(\phi_+\right)\right]} \right|^{-\frac{1}{2}} \left[1 + \mathcal{O}\left(\hbar\right)\right], \tag{1.12}$$

where B is the *bounce action*

$$B = \int \mathrm{d}^4 x_E \, \left[\frac{1}{2}\left(\partial\bar{\phi}\right)^2 + U\left(\bar{\phi}\right)\right], \tag{1.13}$$

and $\bar{\phi}$ is the solution of the Euclidean (imaginary time) equations of motion with

$$\phi\left(x \to \infty\right) = \phi_+. \tag{1.14}$$

The prime on $\det'$ indicates that the zero modes of the differential operator are excluded from the functional determinant. The functional integrals over the zero modes correspond to integrating over all possible bubble centres and nucleation times, which is accounted for by the $B^2/4\pi^2\hbar^2$ pre-factor. Extension of the theory to finite temperatures is in principle straightforward: one looks for solutions which are periodic in imaginary time with period $\hbar/k_B T$. At high temperatures the problem is effectively reduced to a three dimensional one and $B \to \frac{\hbar}{k_B T} S_3$, where S_3 is the three dimensional action [44, 58].

Generally the evaluation of B is a tractable numerical problem, though analytical results are available for the very widely used *thin wall approximation*, but the evaluation of the functional determinants ranges from very difficult to hopeless. Thus one typically *guesses* the pre-factor based on scaling and dimensional analysis and attempts to validate this guess by comparison to lattice Monte Carlo simulations. So far this program has only been carried out for a small number of benchmark models. In cases of phenomenological interest for EWBG where multiple scalar fields are participating in the phase transition the nucleation rate is in principle a straightforward extension of these standard results, but the computational difficulty increases with the complexity of the scalar sector. Often one finds thin wall results applied without the validity of the approximation being checked. Another important question concerns the gauge dependence of the formalism.

Baryon production is largely dependent on the production of CP violation in fermion scattering off bubble walls and the diffusion of the asymmetric species into the plasma ahead of the bubble wall, which must be subsonic for such diffusion to happen[16]. The faster an asymmetric species diffuses, the longer it remains ahead of the bubble wall, and the longer sphalerons have to convert it into a net baryon asymmetry. Thus a solid understanding of the scattering and transport of plasma particles in the bubble region is essential to a reliable EWBG calculation. Recent progress has achieved this, up to some small open questions, in the context of the MSSM (for a review see [43]). These techniques have not yet seen wide application to BSM models other than the MSSM, although the SM with the dimension six term ϕ^6, a scalar singlet SM extension, 2HDM (Types I and II), MSSM, and NMSSM are each briefly discussed in [43].

In summary, electroweak baryogenesis is a plausible mechanism for explaining one of the greatest mysteries: the matter-antimatter asymmetry of the universe. The size of the asymmetry is well established within seconds of the big bang, and is crucial for the production of light elements in the early universe. If it were not for some mechanism of baryogenesis there would be very little matter in the universe, and essentially no galaxies, stars, heavy elements or life. Yet, remarkably, EWBG provides a testable mechanism for controlling the late time abundance of matter in the universe and is already being constrained by the LHC. EWBG in specific models such as the MSSM is already under a great deal of strain, although ruling it out in a completely model independent test may have to wait for the next collider. At the present time experiments are catching up to theoretical speculations at the electroweak scale, and formulating solid theoretical predictions should be a top priority. In particular, it should be possible to say with confidence whether a first order phase transition exists in any particular extended Higgs sector, and what the critical temperature, nucleation rate, strength of the phase transition, CP violation generation rate and sphaleron rates are in all relevant regimes. These quantities have been well established in a handful of benchmark models but for the most part questionable estimates and conservative rules of thumb are used to establish the viability of EWBG within a particular model.

All of this motivates the investigation of a new approach to non-equilibrium non-abelian

[16] Though see [59] for an alternate EWBG mechanism involving supersonic bubble walls.

gauge theories, incorporating and extending the new insights from recent years to theories of phenomenological interest. The "holy grail" of this investigation would be an approach to Yang-Mills-Higgs-Dirac type theories with chiral fermions which is manifestly gauge invariant, can solve the initial value problem far from equilibrium, with realistic (i.e. non-Gaussian) initial conditions, with a well established and tractable renormalisation procedure, while avoiding infrared singularities, unphysical transients and secularities though consistent and systematically improvable re-summations. It would be desirable to see the spontaneous symmetry breaking of a non-abelian gauge theory or biased sphaleron transitions in real time, and a controlled analytical framework capable of this would provide a valuable check of lattice calculations and perhaps provide useful "boundary markers" in the quest to map out possibilities for beyond standard model physics. It is incredible – maybe wildly optimistic – that such a formulation might be possible. In the meantime one may take some small steps towards this dream by attempting to extend the n-particle irreducible effective action formalism, which has already been demonstrated to be useful in non-perturbative and non-equilibrium quantum field theory [17], to better handle theories with a high degree of symmetry and symmetry changing phase transitions. This hypothesis provides the launching point for this thesis.

1.3 Quantum field theory and functional integrals

The review of electroweak baryogenesis in the preceding section motivates the development of new techniques for non-perturbative quantum field theory based on the n-particle irreducible effective action formalism. However, before diving into that work a brief review of basic quantum field theory is worthwhile. This is provided here. The following review is not comprehensive and cannot replace an actual course in quantum field theory, but it does suffice to establish the notation and basic framework which forms the basis of the rest of this thesis. Further, the original references cited below can provide a jumping off point for further investigation.

A *field* $\phi : \mathcal{M} \to \mathcal{T}$ is simply a mapping from a spacetime manifold $\mathcal{M}$ to a field or *target* space $\mathcal{T}$ assigning to each event in spacetime the value that the field takes at that event. This thesis focuses on the special case of scalar fields, for which $\mathcal{T} = \mathbb{R}$ (later on, when discussing symmetries, $\mathcal{T} = \mathbb{R}^N$ for integer N). This thesis also restricts discussion to flat spacetimes[17], so $\mathcal{M} = \mathbb{R}^d$. A *quantum field* is simply a field which can exist in a quantum superposition of several classical states. Thus quantum field theory (QFT) is often considered a special case of quantum mechanics (QM), taking the limit that the number of degrees of freedom goes to infinity. However, the limit usually cannot be taken in a rigorous way and in fact one proceeds the other way, deriving QM as a special case of QFT in zero spatial dimensions (where $\mathcal{M} = \mathbb{R}$ is just the time-line and $\mathcal{T}$ is a copy of ordinary space for each particle). Furthermore, there are non-trivial QFTs with no particle content whatsoever (so called *conformal field theories* and *topological field theories*). Still,

[17]The generalisation to curved spacetimes is in some respects straightforward and in other respects quite subtle and interesting, though beyond the scope of this thesis. In any event, quantum field theory in curved spacetime is standard lore (see, e.g. [60]).

there is a close relation between QM and QFT: for each formalism available for QM there is a corresponding QFT formalism. Thus, for example, one could study quantum fields in the Schrödinger picture via the *wavefunctional* $\Psi\left[\phi\left(\boldsymbol{x}\right)\right]$ and its evolution equation $i\hbar\partial_t\Psi = \hat{\mathrm{H}}\Psi$. However, in QFT it is often far more convenient to use the functional integral formalism, originally due to Feynman [61], so this section reviews QFT in the functional formalism and some of the connections to the more familiar canonical formalism which will be useful later on.

Start in the canonical formalism. Here, fields are Heisenberg picture operators (actually operator valued distributions) $\hat{\phi}\left(\boldsymbol{x},t\right)$ with accompanying canonical momenta $\hat{\pi}\left(\boldsymbol{x},t\right)$ which obey canonical *equal time* commutation relationships

$$\left[\hat{\phi}\left(\boldsymbol{x},t\right),\hat{\phi}\left(\boldsymbol{y},t\right)\right] = \left[\hat{\pi}\left(\boldsymbol{x},t\right),\hat{\pi}\left(\boldsymbol{y},t\right)\right] = 0, \tag{1.15}$$

$$\left[\hat{\phi}\left(\boldsymbol{x},t\right),\hat{\pi}\left(\boldsymbol{y},t\right)\right] = i\delta^{(3)}\left(\boldsymbol{x}-\boldsymbol{y}\right). \tag{1.16}$$

Commutators at unequal times must be found by evolving the field with its equations of motion. For a free field this can be done exactly. The free field Hamiltonian is quadratic in fields and momenta, exactly analogous to the QM harmonic oscillator:

$$\hat{H}_0 = \int_{\boldsymbol{x}} \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\nabla\hat{\phi}\right)^2 + \frac{1}{2}m^2\hat{\phi}^2. \tag{1.17}$$

m is a parameter with dimensions of inverse length (the name is due to a quantity with dimensions of mass $m\hbar/c$, which is equal to m in natural units because $\hbar = c = 1$). The Hamiltonian equations of motion are then

$$\partial_t\hat{\phi}\left(\boldsymbol{x},t\right) = -i\left[\hat{\phi}\left(\boldsymbol{x},t\right),\hat{H}_0\left(t\right)\right] = \hat{\pi}\left(\boldsymbol{x},t\right), \tag{1.18}$$

$$\partial_t\hat{\pi}\left(\boldsymbol{x},t\right) = -i\left[\hat{\pi}\left(\boldsymbol{x},t\right),\hat{H}_0\left(t\right)\right] = \nabla^2\hat{\phi} - m^2\hat{\phi}, \tag{1.19}$$

which can be rearranged to

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\hat{\phi} = 0 \tag{1.20}$$

which is the *Klein-Gordon* equation for $\hat{\phi}$, and an analogous equation for $\hat{\pi}$. As a linear equation, this can be solved by Fourier decomposition

$$\hat{\phi}\left(\boldsymbol{x},t\right) = \int_{\boldsymbol{k}} N_{\boldsymbol{k}}\left[\hat{a}_{\boldsymbol{k}}\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)} + \hat{a}^{\dagger}_{\boldsymbol{k}}\mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)}\right], \tag{1.21}$$

where

$$\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2} \tag{1.22}$$

and taking into account that, as a real scalar field, $\hat{\phi}^{\dagger} = \hat{\phi}$. Taking the normalisation

$$N_{\boldsymbol{k}} = \sqrt{\frac{\left(2\pi\right)^3}{2\omega_{\boldsymbol{k}}f_{\boldsymbol{k}}}} \tag{1.23}$$

and applying the equal time commutation relations leads then to the commutation relations

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{q}}] = \left[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0, \tag{1.24}$$

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = f_{\boldsymbol{k}} \delta^{(3)} (\boldsymbol{k} - \boldsymbol{q}), \tag{1.25}$$

for the mode operators. It is conventional to take $f_{\boldsymbol{k}} = 1$, which is used in all the following. With this choice $\hat{a}_{\boldsymbol{k}} \left(\hat{a}^\dagger_{\boldsymbol{k}}\right)$ are annihilation(creation) operators for particles with bosonic statistics.

Using the mode expansion the free Hamiltonian becomes

$$\hat{H}_0 = \int_{\boldsymbol{k}} (2\pi)^3 \omega_{\boldsymbol{k}} \left(\hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \frac{1}{2}\delta^{(3)}(\boldsymbol{0})\right), \tag{1.26}$$

where the formal commutator $\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}}\right] = \delta^{(3)}(\boldsymbol{0})$ was used. Using the Fourier representation of the delta function $\delta^{(3)}(\boldsymbol{0}) = (2\pi)^{-3} \int_{\boldsymbol{x}} \mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{0}} = (2\pi)^{-3} \mathrm{V}$ where V is the volume of space, the second term can be written

$$\int_{\boldsymbol{k}} (2\pi)^3 \omega_{\boldsymbol{k}} \frac{1}{2}\delta^{(3)}(\boldsymbol{0}) = \left(\frac{1}{2}\int_{\boldsymbol{k}} \omega_{\boldsymbol{k}}\right) \mathrm{V}, \tag{1.27}$$

which assigns an energy density of $\frac{1}{2}\int_{\boldsymbol{k}} \omega_{\boldsymbol{k}}$ to the vacuum. Subtracting this (divergent) vacuum energy density gives the normal ordered Hamiltonian

$$: \hat{H}_0 := \int \mathrm{d}^3\boldsymbol{k}\ \omega_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} = \int \mathrm{d}^3\boldsymbol{k}\ \omega_{\boldsymbol{k}} \hat{N}_{\boldsymbol{k}}, \tag{1.28}$$

where $\hat{N}_{\boldsymbol{k}}$ is the number operator for the mode $\boldsymbol{k}$. $: \hat{H}_0 :$ describes non-interacting (because the energy is additive) particle-like bosonic excitations of mass m.

Now the unequal time commutator can be found:

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\phi}(y)\right] &= \int_{\boldsymbol{k}} N_{\boldsymbol{k}} \int_{\boldsymbol{q}} N_{\boldsymbol{q}} \left[\hat{a}_{\boldsymbol{k}} \mathrm{e}^{ikx} + \hat{a}^\dagger_{\boldsymbol{k}} \mathrm{e}^{-ikx}, \hat{a}_{\boldsymbol{q}} \mathrm{e}^{iqy} + \hat{a}^\dagger_{\boldsymbol{q}} \mathrm{e}^{-iqy}\right] \\
&= \int_{\boldsymbol{k}} N_{\boldsymbol{k}} \int_{\boldsymbol{q}} N_{\boldsymbol{q}} \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q}) \left(\mathrm{e}^{i(kx - qy)} - \mathrm{e}^{-i(kx-qy)}\right) \\
&= \int \frac{\mathrm{d}^3\boldsymbol{k}}{2\omega_{\boldsymbol{k}}} \left(\mathrm{e}^{ik(x-y)} - \mathrm{e}^{-ik(x-y)}\right),
\end{aligned} \tag{1.29}$$

where the time component of k_μ is $\omega_{\boldsymbol{k}}$ in the exponentials. Note that when x and y are spacelike separated, there is a frame in which the time component of $x^\mu - y^\mu$ vanishes. In this case one can see that the commutator vanishes by setting $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term. To show that this holds in all frames one needs to show that the integration measure is in fact Lorentz invariant, as the exponentials are manifestly invariant. Consider the identity

$$\frac{1}{2\omega_{\boldsymbol{k}}} = \int \mathrm{d}k_0 \delta\left(k_0^2 - \omega_{\boldsymbol{k}}^2\right) \Theta(k_0) = \int \mathrm{d}k_0 \delta\left(k^2 - m^2\right) \Theta(k_0). \tag{1.30}$$

Then

$$\left[\hat{\phi}\left(x\right),\hat{\phi}\left(y\right)\right]=\int_{k}\left(2\pi\right)^{4}\delta\left(k^{2}-m^{2}\right)\Theta\left(k_{0}\right)\left(\mathrm{e}^{ik\left(x-y\right)}-\mathrm{e}^{-ik\left(x-y\right)}\right), \tag{1.31}$$

which is manifestly invariant under the isochronous (i.e. not reversing time) Lorentz transformations. Thus fields commute at spacelike separation. This is the *microcausality* condition and effectively implements the speed of light limit for information transmission. Microcausality holds for the free field theory, but also for the interacting theories of physical interest[18] and is automatic if the theory is constructed from a Lorentz invariant action which is the integral of a local density which is a finite order polynomial of fields and their derivatives [62].

Since spacelike separated fields commute, the set of fields on a spacelike hypersurface Σ forms a complete commuting set of operators and a complete set of eigenvectors can be found. Arbitrary states can be expanded in this basis as

$$\left|\Psi\right\rangle=\int\mathcal{D}\left[\phi\left(x\in\Sigma\right)\right]\Psi\left[\phi\left(x\right)\right]\left|\phi\left(x\right);\Sigma\right\rangle \tag{1.32}$$

where $\Psi\left[\phi\left(x\right)\right]$ is the wavefunctional and the Heisenberg states $\left|\phi\left(x\right);\Sigma\right\rangle$ are eigenvectors of the field operators $\hat{\phi}\left(x\right)$ on Σ with eigenvalues $\phi\left(x\right)$. Note that $\left|\phi\left(x\right);\Sigma\right\rangle$ is generally *not* an eigenvector of a field operator *not* on the hypersurface. The time variable hidden in the notation "$\phi\left(x\right)$" is merely part of the label for the state and implies nothing about its time evolution (indeed Heisenberg states do not evolve). The notation "$;\Sigma$" is used to remind us of this fact. Now consider a transition amplitude between an initial and final state $\Psi_i\to\Psi_f$:

$$\begin{aligned}\left\langle\Psi_{f}\,|\,\Psi_{i}\right\rangle &= \int\mathcal{D}\left[\phi_{i}\left(x\in\Sigma_{i}\right),\phi_{f}\left(x\in\Sigma_{f}\right)\right]\\ &\quad\times\Psi_{f}^{\star}\left[\phi_{f}\left(x\right)\right]\left\langle\phi_{f}\left(x\right);\Sigma_{f}\,|\,\phi_{i}\left(x\right);\Sigma_{i}\right\rangle\Psi_{i}\left[\phi_{i}\left(x\right)\right],\end{aligned} \tag{1.33}$$

where the initial and final hypersurfaces are Σ_i and Σ_f respectively and by assumption Σ_f is later than Σ_i (i.e. any future directed timelike curve passing through Σ_i at proper time τ_i passes through Σ_f at a proper time $\tau_f>\tau_i$). From this equation it is clear that the time evolution problem reduces to finding

$$\left\langle\phi_{f}\left(x\right);\Sigma_{f}\,|\,\phi_{i}\left(x\right);\Sigma_{i}\right\rangle, \tag{1.34}$$

the transition amplitude between field eigenstates on two spacelike hypersurfaces.

This goal is accomplished by the *Feynman path integral* or *functional integral*. The idea is to do the time evolution a little bit at a time. Since Σ_f is later than Σ_i it is possible to break the finite time evolution into a succession of N steps $\Sigma_i\to\Sigma_1\to\Sigma_2\to\cdots\to$

[18] The main exception being electromagnetism in the Coulomb gauge, which explicitly breaks Lorentz invariance. In this case the (gauge variant) commutators do not vanish at spacelike separations but causality problems are avoided because only gauge invariant quantities are physically observable. All gauge invariant quantities can be equally well computed in a Lorentz covariant gauge in which microcausality is manifest at every step. For further details, consult e.g. [62].

$\Sigma_{N-1} \to \Sigma_f$ and take the $N \to \infty$ limit such that each step represents an infinitesimal time evolution. Formally this is done by repeatedly inserting resolutions of the identity

$$\hat{\mathbb{I}} = \int \mathcal{D}\left[\phi\left(x \in \Sigma\right)\right] \left|\phi;\Sigma\right\rangle \left\langle\phi;\Sigma\right|, \tag{1.35}$$

obtaining

$$\begin{aligned} \left\langle \phi_f\left(x\right);\Sigma_f \,\middle|\, \phi_i\left(x\right);\Sigma_i\right\rangle &= \int \prod_{k=1}^{N-1} \mathcal{D}\left[\phi_k\left(x \in \Sigma_k\right)\right] \\ &\quad \times \left\langle \phi_f\left(x\right);\Sigma_f \,\middle|\, \phi_{N-1}\left(x\right);\Sigma_{N-1}\right\rangle \\ &\quad \times \cdots \times \left\langle \phi_{k+1}\left(x\right);\Sigma_{k+1} \,\middle|\, \phi_k\left(x\right);\Sigma_k\right\rangle \\ &\quad \times \cdots \times \left\langle \phi_1\left(x\right);\Sigma_1 \,\middle|\, \phi_i\left(x\right);\Sigma_i\right\rangle. \end{aligned} \tag{1.36}$$

To simplify this further one can assume that the Σs are all flat and parallel (the general case can be recovered from the final result). This means that there is an inertial coordinate system where the time t_k is constant on each Σ_k. One can further assume, without loss of generality, that the hypersurfaces are equally spaced. That is, $t_k = t_i + \epsilon k$ where $\epsilon = \left(t_f - t_i\right)/N$.

Now all one needs is the infinitesimal step

$$\left\langle \phi_{k+1}\left(x\right);t_k+\epsilon \,\middle|\, \phi_k\left(x\right);t_k\right\rangle. \tag{1.37}$$

Intuitively, there is not much time for the fields to evolve, so this should be close to the delta functional $\approx \delta\left[\phi_{k+1}-\phi_k\right]$. To find the $\mathcal{O}\left(\epsilon\right)$ correction it is easiest to convert the states into the Schrödinger picture (indicated using a subscript "S") using the time evolution operator

$$\left\langle \phi_{k+1}\left(x\right);t_k+\epsilon \,\middle|\, \phi_k\left(x\right);t_k\right\rangle = \left\langle \phi_{k+1}\left(\boldsymbol{x}\right);t_k \,\middle|\, \left[\hat{\mathbb{I}} - \frac{i}{\hbar}\epsilon \hat{H}\left(\hat{\phi},\hat{\pi}\right)\right] \,\middle|\, \phi_k\left(\boldsymbol{x}\right);t_k\right\rangle_S, \tag{1.38}$$

where $\hat{H}\left(\hat{\phi},\hat{\pi}\right)$ is the Hamiltonian. Resolving the identity again, this time with $\hat{\pi}$ eigenstates,

$$\begin{aligned} \left\langle \phi_{k+1}\left(x\right);t_k+\epsilon \,\middle|\, \phi_k\left(x\right);t_k\right\rangle &= \int \mathcal{D}\left[\pi_k\left(x\right)\right] \left\langle \phi_{k+1}\left(\boldsymbol{x}\right);t_k \,\middle|\, \pi_k\left(\boldsymbol{x}\right);t_k\right\rangle_S \\ &\quad \times \left\langle \pi_k\left(\boldsymbol{x}\right);t_k \,\middle|\, \left[\hat{\mathbb{I}} - \frac{i}{\hbar}\epsilon \hat{H}\left(\hat{\phi},\hat{\pi}\right)\right] \,\middle|\, \phi_k\left(\boldsymbol{x}\right);t_k\right\rangle_S \end{aligned} \tag{1.39}$$

which becomes

$$\begin{aligned}\langle \phi_{k+1}(x);t_k+\epsilon \mid \phi_k(x);t_k\rangle &= \int \mathcal{D}[\pi_k(x)] \langle \phi_{k+1}(\boldsymbol{x});t_k \mid \pi_k(\boldsymbol{x});t_k\rangle_S \\ &\quad\times \left[1-\frac{i}{\hbar}\epsilon H(\phi_k,\pi_k)\right] \langle \pi_k(\boldsymbol{x});t_k \mid \phi_k(\boldsymbol{x});t_k\rangle_S \qquad (1.40)\end{aligned}$$

Using[19] $\langle \phi \mid \pi \rangle = \exp\left(\frac{i}{\hbar}\int_{\boldsymbol{x}} \pi(\boldsymbol{x},t_k)\,\phi(\boldsymbol{x},t_k)\right)$ this can be written

$$\begin{aligned}\langle \phi_{k+1}(x);t_k+\epsilon \mid \phi_k(x);t_k\rangle &= \int \mathcal{D}[\pi_k(x)] \exp\left(\frac{i}{\hbar}\int_{\boldsymbol{x}} \pi_k(\boldsymbol{x},t_k)\left[\phi_{k+1}(\boldsymbol{x},t_k)-\phi_k(\boldsymbol{x},t_k)\right]\right) \\ &\quad\times \exp\left[-\frac{i}{\hbar}\epsilon H(\phi_k,\pi_k)\right] \\ &= \int \mathcal{D}[\pi_k(x)] \exp\left(\frac{i}{\hbar}\epsilon\int_{\boldsymbol{x}} \left[\pi_k(\boldsymbol{x},t_k)\,\partial_t\phi_k(\boldsymbol{x},t_k)\right.\right. \\ &\quad \left.\left. -\mathcal{H}(\phi_k,\pi_k)\right]\right) \qquad (1.41)\end{aligned}$$

using the finite difference approximation $\phi_{k+1}-\phi_k = \epsilon\partial_t\phi_k$ and that the Hamiltonian $H=\int_{\boldsymbol{x}}\mathcal{H}$ is the integral of a local Hamiltonian density $\mathcal{H}$. Multiplying all time steps gives

$$\langle \phi_f(x);\Sigma_f \mid \phi_i(x);\Sigma_i\rangle = \int_{\phi(x)=\phi_i(x),\ x\in\Sigma_i}^{\phi(x)=\phi_f(x),\ x\in\Sigma_f} \mathcal{D}[\phi,\pi] \exp\left(\frac{i}{\hbar}\int_x \left[\pi(x)\,\partial_t\phi(x)-\mathcal{H}(\phi,\pi)\right]\right) \qquad (1.42)$$

which, in the $N\to\infty$ limit is the Hamiltonian form of the path integral.

If the Hamiltonian is quadratic in the momenta the π integral can be performed by completing the square:

$$\begin{aligned}\int \mathcal{D}[\pi] \exp\left(\frac{i}{\hbar}\int_x \left[\pi(x)\,\partial_t\phi(x)-\frac{1}{2}\pi(x)^2\right]\right) &= \int \mathcal{D}[\pi] \exp\left(\frac{i}{\hbar}\int_x -\frac{1}{2}\left[\pi(x)-\partial_t\phi(x)\right]^2\right) \\ &\quad\times \exp\left(\frac{i}{\hbar}\int_x \frac{1}{2}\left(\partial_t\phi(x)\right)^2\right) \\ &= N\exp\left(\frac{i}{\hbar}\int_x \frac{1}{2}\left(\partial_t\phi(x)\right)^2\right), \qquad (1.43)\end{aligned}$$

where the π integration variable can be shifted $\pi\to\pi+\partial_t\phi$ to give the constant normalisation factor

$$N = \int \mathcal{D}[\pi] \exp\left(\frac{i}{\hbar}\int_x -\frac{1}{2}\pi(x)^2\right), \qquad (1.44)$$

and

$$\langle \phi_f(x);\Sigma_f \mid \phi_i(x);\Sigma_i\rangle = N\int_{\phi(x)=\phi_i(x),\ x\in\Sigma_i}^{\phi(x)=\phi_f(x),\ x\in\Sigma_f} \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}\int_x \mathcal{L}(\phi)\right), \qquad (1.45)$$

[19]Recall from basic QM that $\langle q \mid p\rangle = \exp\left(\frac{i}{\hbar}pq\right)$.

where $\mathcal{L}(\phi)$ is the Lagrangian density[20]

$$\mathcal{L}(\phi) = \pi(x)\,\partial_t\phi(x) - \mathcal{H}(\phi,\pi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \tag{1.46}$$

where $V(\phi)$ is the potential which is equal to $\frac{1}{2}m^2\phi^2$ in the case of a free field theory. The classical action functional is the spacetime integral of the Lagrangian

$$S[\phi] = \int_x \mathcal{L}(\phi)\,. \tag{1.47}$$

(1.45) is a completely Lorentz invariant (indeed, diffeomorphism invariant) formula for transition amplitudes which no longer depends on any particular assumptions about the time slicing used etc. Of course, all of the difficulties are hidden in the notation. N is traditionally absorbed into the definition of the measure $\mathcal{D}[\phi]$ and is unimportant for most physical problems. Some methods of dealing with the overall normalisation are discussed in the next section. Note that matrix elements of arbitrary time ordered products of fields are now easily obtained. Define the functional

$$\mathcal{A}_{fi}[J(x)] = N\int_{\phi(x)=\phi_i(x),\ x\in\Sigma_i}^{\phi(x)=\phi_f(x),\ x\in\Sigma_f} \mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}\left(\int_x \mathcal{L}(\phi) + J(x)\,\phi(x)\right)\right]. \tag{1.48}$$

Then

$$\left\langle \phi_f(x);\Sigma_f \,\middle|\, \mathrm{T}\left[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right] \,\middle|\, \phi_i(x);\Sigma_i\right\rangle = \left(-i\hbar\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\hbar\frac{\delta}{\delta J(x_n)}\right)\mathcal{A}_{fi}[J(x)]\Big|_{J\to 0} \tag{1.49}$$

where it is assumed that all x_k are in the spacetime region bounded by Σ_i and Σ_f. This type of *generating functional* technique is used frequently in the following.

1.4 Time contours, Green functions and non-equilibrium QFT

In order to make practical use of the functional integral it is necessary to have some way of characterising the initial and final states, handling the overall normalisation and evaluating the integral itself. Each of these typically require regularisation to handle divergences and renormalisation to express the input parameters in terms of physical observables. The way all of these are implemented depends on the physical application being considered.

For example, the setting of most particle physics is vacuum field theory. One is usually interested in the scattering of a small number of particles which only interact in a small spacetime region. Long before and after the interaction the particles are effectively free, so the initial and final states can be approximated by free particle states $\sim a^\dagger_{\boldsymbol{k}_1}a^\dagger_{\boldsymbol{k}_2}\cdots a^\dagger_{\boldsymbol{k}_n}|\mathrm{vac}\rangle$ and the appropriate observable is the S-matrix element. The *Lehmann-Symanzik-Zimmermann* (LSZ) reduction formulae connect the S-matrix elements for the asymptotic free particle states to the vacuum-to-vacuum transition matrix elements of a

[20] It is common to refer to $\mathcal{L}$ simply as the Lagrangian, i.e. dropping the "density." Though strictly speaking incorrect, this rarely results in confusion.

time ordered product of interacting fields (a *correlation* or *Green function*). The essential idea of the LSZ formulae is that, while the free field operators create and destroy single particles, interacting fields have some amplitude to create and destroy multi-particle states. Nevertheless, multi-particle wave packets spread differently than single particle wave packets, so that the single particle contributions can be isolated and related to the asymptotic free particle states. The Green functions can be evaluated by using, essentially, the formulae of the previous section. However it is necessary to characterise the initial and final physical vacuum state and handle the overall normalisation. The *Gell-Mann* and *Low* theorem accomplishes both. The idea is that, instead of trying to find the wavefunctional of the vacuum state, one can use *any* physically reasonable initial and final states[21] in (1.45) and deform the time contour in a slight negative imaginary direction, so that one eventually takes the limit $t_i \to -\infty\,(1-i\epsilon)$ and $t_f \to +\infty\,(1-i\epsilon)$. The imaginary time component introduces a dissipative part to the time evolution $\sim \exp\left(-\epsilon t\hat{H}\right)$ which damps all states except for the lowest energy, i.e. vacuum, state. So finally in the limit only the vacuum-to-vacuum matrix element survives[22]. The normalisation factor is taken care of by noting that $\langle \mathrm{vac} \,|\, \mathrm{vac} \rangle = 1$, i.e. an undisturbed vacuum remains vacuum. Thus one can simply divide by this factor, giving

$$\left\langle \mathrm{vac} \,\middle|\, \mathrm{T}\left[\hat{\phi}\left(x_1\right)\cdots\hat{\phi}\left(x_n\right)\right] \,\middle|\, \mathrm{vac} \right\rangle = \frac{\int \mathcal{D}\left[\phi\right] \phi\left(x_1\right)\cdots\phi\left(x_n\right) \exp\left(\frac{i}{\hbar}\int_x \mathcal{L}\left(\phi\right)\right)}{\int \mathcal{D}\left[\phi\right] \exp\left(\frac{i}{\hbar}\int_x \mathcal{L}\left(\phi\right)\right)}, \tag{1.50}$$

where the functional integrals are over all spacetime and any ambiguity in the normalisation of the measure simply cancels between the numerator and denominator of the ratio.

In thermal equilibrium one is not interested in the S-matrix as there are no asymptotic states. Rather one is interested in the partition function

$$Z = \mathrm{Tr}\mathrm{e}^{-\beta\hat{H}}, \tag{1.51}$$

where $\beta = 1/T$ is the inverse temperature. From Z all thermodynamic functions can be derived. The trick to evaluate Z in the *Matsubara formalism* is to note that the operator $\exp\left(-\beta\hat{H}\right)$ is formally an imaginary time evolution operator $\exp\left(-i\hat{H}t/\hbar\right)$ for $t = -i\hbar\beta$. Therefore one *Wick rotates* to imaginary time $t = x^0 = -i\hbar x^4$ where x^4 goes from zero to β. Furthermore, the trace operation implies periodic boundary conditions:

$$\begin{aligned} Z &= \mathrm{Tr}\mathrm{e}^{-\beta\hat{H}} \\ &= \sum_a \left\langle a \,\middle|\, \mathrm{e}^{-\beta\hat{H}} \,\middle|\, a \right\rangle. \end{aligned} \tag{1.52}$$

Writing the matrix element in terms of the functional integral,

[21] The only condition is that they have non-zero overlap with the vacuum state.

[22] This assumes that the vacuum state is non-degenerate and separated by a finite spectral gap from any excited states. In particular, this implies that this formalism for vacuum field theory is strictly speaking invalid for massless particles. Yet, massless particles can be handled in this formalism provided one takes care of infrared (i.e. low energy) divergences and computes inclusive cross sections, i.e. including processes with the emission of extra soft particles in the final state which are invisible in realistic detectors. A detailed treatment of these issues is beyond the scope of this thesis, but can be found in e.g. [36].

$$\begin{aligned}
Z &= \int \mathcal{D}[a] \int_{\phi(t=0)=a}^{\phi(t=-i\hbar\beta)=a} \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}\int_0^{-i\hbar\beta} \mathrm{d}t \int_{\boldsymbol{x}} \mathcal{L}\right) \\
&= \int_{\phi(t=0)=\phi(t=-i\hbar\beta)} \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}\int_0^{-i\hbar\beta} \mathrm{d}t \int_{\boldsymbol{x}} \mathcal{L}\right) \\
&= \int_{\phi(x^4=0)=\phi(x^4=\beta)} \mathcal{D}[\phi] \exp\left(\int_0^{\beta} \mathrm{d}x^4 \int_{\boldsymbol{x}} \mathcal{L}_E\right),
\end{aligned} \tag{1.53}$$

where the *Euclidean* Lagrangian $\mathcal{L}_E$ is obtained by making the replacement $t \to -i\hbar x^4$ in $\mathcal{L}$. The overall normalisation is either irrelevant (because the observable is a ratio, e.g. $\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{O}\mathrm{e}^{-\beta\hat{H}}\right)/Z$) or can be fixed from the requirement that thermodynamic functions (e.g. pressure $P = \partial_V (T \ln Z)$) vanish at zero temperature.

These aspects can be illustrated by the case of a free field theory, for which

$$\int_{x_E} \mathcal{L}_E = -\frac{1}{2}\int_{x_E} \left((\partial_{x^4}\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right) = -\frac{1}{2}\int_{x_E} \phi D \phi \tag{1.54}$$

where

$$D = -\partial_{x^4}^2 - \nabla^2 + m^2 = -\nabla_E^2 + m^2 \tag{1.55}$$

and ∇_E is the obvious Euclidean space gradient operator. In going from the first form to the second in (1.54), an integration by parts was performed and the surface terms were dropped. This is justified by the use of periodic boundary conditions below. The discussion from here to the expression for the energy density ρ closely follows [11]. The functional integral is easiest to evaluate in the eigenbasis of D, i.e. periodic functions of the form $\exp i\left(\boldsymbol{p}\cdot\boldsymbol{x} + \omega_n x^4\right)$ where the Matsubara frequencies $\omega_n = 2\pi n/\beta$ with $n \in \mathbb{Z}$ implement the periodic boundary condition. It is also convenient to quantise in a spatial box of volume $\mathrm{V} = L^3$ with periodic boundary conditions, making $\boldsymbol{p}$ discrete with lattice spacing $2\pi/L$. The convenient basis functions are

$$\phi_{n\boldsymbol{p}} = \sqrt{\frac{\beta}{\mathrm{V}}}\mathrm{e}^{i\left(\boldsymbol{p}\cdot\boldsymbol{x} + \omega_n x^4\right)}, \tag{1.56}$$

which are normalised so that

$$\langle \phi_{m\boldsymbol{q}} \,|\, \phi_{n\boldsymbol{p}} \rangle = \beta^2 \delta_{mn} \delta_{\boldsymbol{pq}}, \tag{1.57}$$

where, on functions, $\langle f \,|\, g \rangle \equiv \int_{x_E} f^\star g$. With these normalisations the Fourier amplitudes $A_{n\boldsymbol{p}}$ in

$$\phi = \sum_{n\boldsymbol{p}} A_{n\boldsymbol{p}} \phi_{n\boldsymbol{p}} \tag{1.58}$$

are dimensionless. Then

$$\int_{x_E} \mathcal{L}_E = -\frac{1}{2}\sum_{n\boldsymbol{p}} |A_{n\boldsymbol{p}}|^2 \beta^2 \left(\omega_n^2 + \boldsymbol{p}^2 + m^2\right). \tag{1.59}$$

The functional integrand does not depend on the phases of $A_{n\boldsymbol{p}}$, so these contribute to the normalisation. The integrals over the magnitudes are Gaussian, giving

$$Z = N' \prod_{n\boldsymbol{p}} \left[\beta^2 \left(\omega_n^2 + \boldsymbol{p}^2 + m^2\right)\right]^{-1/2} \tag{1.60}$$

where all the normalisation factors are absorbed in N'. Then

$$\begin{aligned}
\ln Z &= \ln N' - \frac{1}{2}\sum_{n\boldsymbol{p}} \ln\left[\beta^2\left(\omega_n^2 + \boldsymbol{p}^2 + m^2\right)\right] \\
&= \ln N' - \frac{1}{2}\sum_{n\boldsymbol{p}} \left\{ \ln\left[(2\pi n)^2 + 1\right] + \int_1^{\beta^2\left(\boldsymbol{p}^2+m^2\right)} \frac{\mathrm{d}\theta^2}{\theta^2 + (2\pi n)^2} \right\} \\
&= \ln N' - \frac{1}{2}\sum_{\boldsymbol{p}} \left\{ \sum_n \ln\left[(2\pi n)^2 + 1\right] + \int_1^{\beta\omega_{\boldsymbol{p}}} \mathrm{d}\theta \left(1 + \frac{2}{\mathrm{e}^\theta - 1}\right) \right\} \\
&= \ln N' - \frac{1}{2}\sum_{\boldsymbol{p}} \left\{ \sum_n \ln\left[(2\pi n)^2 + 1\right] + \beta\omega_{\boldsymbol{p}} - 1 + \int_1^{\beta\omega_{\boldsymbol{p}}} \mathrm{d}\theta \frac{2}{\mathrm{e}^\theta - 1} \right\} \\
&= \ln N' - \sum_{\boldsymbol{p}} \left\{ \frac{1}{2}\sum_n \ln\left[(2\pi n)^2 + 1\right] + \frac{1}{2} + \ln\frac{1}{\mathrm{e} - 1} \right\} \\
&\quad - \sum_{\boldsymbol{p}} \left[\frac{1}{2}\beta\omega_{\boldsymbol{p}} + \ln\left(1 - \mathrm{e}^{-\beta\omega_{\boldsymbol{p}}}\right)\right].
\end{aligned} \tag{1.61}$$

The identity $\sum_n 1/\left(n^2 + (\theta/2\pi)^2\right) = \left(2\pi^2/\theta\right)\left(1 + 2/\left(\mathrm{e}^\theta - 1\right)\right)$ was used to get the third line. The rest of the steps are straightforward evaluations.

Demanding that the entropy $S = \partial\left(T\ln Z\right)/\partial T \to 0$ as $T \to 0$ fixes the normalisation

$$N' = \exp\sum_{\boldsymbol{p}} \left\{ \frac{1}{2}\sum_n \ln\left[(2\pi n)^2 + 1\right] + \frac{1}{2} + \ln\frac{1}{\mathrm{e} - 1} \right\}, \tag{1.62}$$

and finally (on taking the infinite volume limit $\sum_{\boldsymbol{p}} \to \mathrm{V}\int_{\boldsymbol{p}}$)

$$\ln Z = -\mathrm{V}\int_{\boldsymbol{p}} \left[\frac{1}{2}\beta\omega_{\boldsymbol{p}} + \ln\left(1 - \mathrm{e}^{-\beta\omega_{\boldsymbol{p}}}\right)\right]. \tag{1.63}$$

The energy density is

$$\rho = \frac{E}{\mathrm{V}} = -\frac{\partial_\beta \ln Z}{\mathrm{V}} = \int_{\boldsymbol{p}} \left(\frac{1}{2}\omega_{\boldsymbol{p}} + \frac{\omega_{\boldsymbol{p}}}{\mathrm{e}^{\beta\omega_{\boldsymbol{p}}} - 1}\right) \tag{1.64}$$

which is the familiar Bose-Einstein contribution plus a vacuum energy contribution which can be subtracted. Similarly, the pressure is

$$P = T\partial_{\mathrm{V}} \ln Z = -\int_{\boldsymbol{p}} \left[\frac{1}{2}\omega_{\boldsymbol{p}} + T\ln\left(1 - \mathrm{e}^{-\beta\omega_{\boldsymbol{p}}}\right)\right]. \tag{1.65}$$

Note that there is a vacuum contribution to the pressure as well. The fact that this contribution is equal to the negative of the vacuum energy is in fact a consequence of

Lorentz invariance[23] and this provides a nice check of the formalism.

The formalisms sketched above for vacuum and finite temperature field theory can be naturally generalised to the contour or non-equilibrium Green function method which also allows the treatment of non-equilibrium field theory. The contour method has been used extensively in various fields of condensed matter and particle physics, from molecular transport and ultra-cold atomic gases to nano-electronics, nuclear physics and cosmology. The research literature using these techniques is vast, and there are good recent books introducing them in condensed matter [63] and high energy/cosmology [64] contexts. The contour method (as defined by [63]) contains the ordinary vacuum quantum field theory (as discussed above and in [18, 36] and many other references), the finite temperature equilibrium quantum field theory in the *Matsubara* or *imaginary time* formalism (discussed above and in e.g. [11]), and the non-equilibrium Green function theory developed by Schwinger and Keldysh (independently; see [65, 66] in addition to [63]) as special cases.

The basic idea of the method is quite simple. Consider a quantum system in some initial state $|\Psi_0\rangle$ at an initial time t_0. Then the expectation value $O(t)$ of some operator $\hat{O}(t)$ at time t is

$$O(t) = \left\langle \Psi(t) \middle| \hat{O}(t) \middle| \Psi(t) \right\rangle = \left\langle \Psi_0 \middle| \hat{U}(t_0, t)\, \hat{O}(t)\, \hat{U}(t, t_0) \middle| \Psi_0 \right\rangle, \tag{1.66}$$

where $\hat{U}(t, t_0)$ is the time evolution operator. As usual $\hat{U}(t_2, t_1)$ for $t_2 > t_1$ can be expressed in terms of a time ordered exponential

$$\hat{U}(t_2, t_1) = \mathrm{T}\left\{\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{d}\tau\ \hat{H}(\tau)\right)\right\}, \tag{1.67}$$

but, for $t_2 < t_1$, an *anti-time ordered* exponential is required:

$$\hat{U}(t_2, t_1) = \tilde{\mathrm{T}}\left\{\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{d}\tau\ \hat{H}(\tau)\right)\right\}, \tag{1.68}$$

where $\tilde{\mathrm{T}}$ orders operators in the opposite sense to T, i.e. earlier times to the left.

Substituting these in the equation for $O(t)$ gives

$$O(t) = \left\langle \Psi_0 \middle| \tilde{\mathrm{T}}\left\{\exp\left(-\frac{i}{\hbar}\int_{t}^{t_0} \mathrm{d}\tau\ \hat{H}(\tau)\right)\right\} \hat{O}(t)\, \mathrm{T}\left\{\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau\ \hat{H}(\tau)\right)\right\} \middle| \Psi_0 \right\rangle. \tag{1.69}$$

This can be simplified by interpreting the time evolution as along a *two branched* contour C which runs from t_0 to t then doubles back to t_0. Introducing a contour time variable z as an affine parameter on the contour[24] and a *contour time ordering* T_C, $O(t)$ can be written as

$$O(t) = \left\langle \Psi_0 \middle| \mathrm{T}_C\left\{\exp\left(-\frac{i}{\hbar}\int_{C} \mathrm{d}z\ \hat{H}(z)\right)\hat{O}(t)\right\} \middle| \Psi_0 \right\rangle. \tag{1.70}$$

[23] The energy-momentum tensor of a Lorentz invariant state must be proportional to the metric $T_{\mu\nu} \propto \eta_{\mu\nu}$ since there are no other spacetime tensors available to construct it from. Thus the pressure (i.e. the diagonal space-space components) is equal and opposite to the energy density (the time-time component).

[24] An explicit mapping of $z \in (t_0, 2t - t_0)$ to ordinary time $\tau \in (t_0, t)$ is $\tau = z$ for $t_0 \leq z \leq t$ and $\tau = 2t - z$ for $t < z < 2t - t_0$, but this is rarely needed in practice.

Further, both branches of C can be extended to an arbitrary time $T > t$, removing the t dependence of the contour. Since there are no operators in the time interval (t, T) the segment of the evolution from t to T and back is given by $\hat{U}(t, T)\,\hat{U}(T, t) = \hat{\mathbb{I}}$ and cancels outs, so no modification of the above expression is necessary. There is an ambiguity where $\hat{O}(t)$ should go on the contour, i.e. how to generalise $\hat{O}(t) \to \hat{O}(z)$, but for convenience $\hat{O}(z)$ can be taken to be the same operator on both branches.

The generalisation of this result to multiple operators and a general initial density matrix $\hat{\rho}_0 = \sum_n p_n \left|n\right\rangle \left\langle n\right|$, where p_n is the probability that the system is in state $\left|n\right\rangle$, is obvious:

$$\left\langle O_1(z_1)\, O_2(z_2) \cdots \right\rangle = \mathrm{Tr}\left\{\hat{\rho}_0 \mathrm{T}_C \left[\exp\left(-\frac{i}{\hbar}\int_C \mathrm{d}z\ \hat{H}(z)\right) \hat{O}_1(z_1)\, \hat{O}_2(z_2) \cdots\right]\right\}. \quad (1.71)$$

Proceeding still further, a general density matrix can be written

$$\hat{\rho}_0 = Z^{-1} \exp\left(-\beta \hat{H}_M\right), \quad (1.72)$$

for *some* Hermitian operator $\hat{H}_M$, where Z is chosen to ensure $\mathrm{Tr}\hat{\rho}_0 = 1$ (in fact Z can be absorbed into a shift of $\hat{H}_M$ if so desired). If the Hamiltonian of the system is time independent and $\hat{H}_M = \hat{H}$ then this describes a thermal equilibrium state at temperature $1/\beta$. However, there is no loss of generality in writing $\hat{\rho}_0$ in this way with a general $\hat{H}_M$, and out of equilibrium the value of β is entirely arbitrary, equivalent to an arbitrary normalisation convention for $\hat{H}_M$. Now note that $\exp\left(-\beta\hat{H}_M\right) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t_0 - i\beta\hbar} \mathrm{d}\tau\ \hat{H}_M\right) = \hat{U}\left(t_0 - i\beta\hbar, t_0; \hat{H}_M\right)$, i.e. up to an overall normalisation factor the initial density matrix can be written as the time evolution operator for an *imaginary* time interval with "Hamiltonian" $\hat{H}_M$ (assuming a τ-independent $\hat{H}_M$). Extending the contour C with a segment on the imaginary axis from t_0 to $t_0 - i\beta\hbar$ and *defining* $\hat{H}(z) = \hat{H}_M$ for z on this segment then, taking care with the normalisation,

$$\begin{aligned} \left\langle \mathrm{T}_C \left[O_1(z_1)\, O_2(z_2) \cdots\right]\right\rangle &= \frac{\mathrm{Tr}\left\{\mathrm{T}_C\left[\exp\left(-\frac{i}{\hbar}\int_C \mathrm{d}z\ \hat{H}(z)\right)\hat{O}_1(z_1)\,\hat{O}_2(z_2)\cdots\right]\right\}}{\mathrm{Tr}\left\{\mathrm{T}_C\left[\exp\left(-\frac{i}{\hbar}\int_C \mathrm{d}z\ \hat{H}(z)\right)\right]\right\}} \\ &= \frac{\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}\int_C \mathcal{L}\right) \hat{O}_1(z_1)\,\hat{O}_2(z_2)\cdots}{\int \mathcal{D}[\phi]\exp\left(\frac{i}{\hbar}\int_C \mathcal{L}\right)}, \end{aligned} \quad (1.73)$$

where the second line specialises to field theory and extends the path integral and Lagrangian to all branches of the contour. The imaginary branch of the contour is called the *Matsubara* branch, as this is the same contour used in the imaginary time field theory formalism for thermal equilibrium problems. The similarity of this result to the usual vacuum field theory formula should be obvious, however *no* assumptions have been made about the initial/final state or spectrum of the theory. This greatly extends the applicability of the formalism compared to the usual vacuum field theory. Despite this, much of the technical machinery of the vacuum theory, including path integrals, Feynman diagrams, effective actions and renormalisation, can be carried over with suitable modifications for

Table 1.3: Keldysh components for any function $k(z,z')$ taking two contour time arguments. (Note: $t_0 = 0$ here.)

Keldysh component	Symbol	Contour arguments
Singular	$k^{\delta}(z)$	$k(z,z') = \delta_C(z,z')\, k^{\delta}(z) +$ regular functions
Time ordered	$k^{T}(t,t')$	z, z' on forward contour
Anti-time ordered	$k^{\tilde{T}}(t,t')$	z, z' on backward contour
Lesser	$k^{<}(t,t')$	z on forward, z' on backward contour
Greater	$k^{>}(t,t')$	z on backward, z' on forward contour
Left hook	$k^{\ulcorner}(\tau,t')$	$z = -i\tau$, z' on forward/backward contour
Right hook	$k^{\urcorner}(t,\tau)$	z on forward/backward contour, $z' = -i\tau$
Matsubara	$k^{M}(\tau,\tau')$	$z = -i\tau$, $z' = -i\tau'$
Retarded	$k^{R}(t,t')$	$k^{R}(t,t') = k^{\delta}(t-t')\,\delta(t-t') + \Theta(t-t')\,[k^{>}(t,t') - k^{<}(t,t')]$
Advanced	$k^{A}(t,t')$	$k^{A}(t,t') = k^{\delta}(t-t')\,\delta(t-t') - \Theta(t'-t)\,[k^{>}(t,t') - k^{<}(t,t')]$

contour time ordering as opposed to regular time ordering. Difficulties come from the extra degrees of freedom (fields on the backward branch of the contour), the explicit appearance of memory effects, and choosing a $\hat{H}_M$ that allows for consistent and practically solvable renormalisation conditions.

The main objects of interest are the correlation functions of field operators, i.e. objects like

$$\Delta(x,y) = -i\left\langle \mathrm{T}_C\left[\hat{\phi}(x)\,\hat{\phi}(y)\right]\right\rangle, \tag{1.74}$$

in a scalar field theory. Now all the time coordinates run on the time contour C. There are nine possibilities for the time arguments of the Green function: z_1 and z_2 can be on either of the forward, backward or Matsubara branches of the contour independently. This leads to the definition of *Keldysh component* Green functions, with real time arguments, for each case. There is also the possibility for a singular part, proportional to a contour delta function $\delta_C(z_1,z_2)$. These components and two "derived" Green functions (the advanced and retarded functions) are listed in Table 1.3.

The origin of some of the names can be seen by visualising the function arguments on the time contour. For example, the lesser function is given by

$$\Delta^{<}(x,y) = -i\left\langle \hat{\phi}(y)\,\hat{\phi}(x)\right\rangle, \tag{1.75}$$

i.e. the second argument appears to the left of the first in the matrix element, regardless of the relation between x^0 and y^0. So the lesser function orders the fields *as if* $x^0 < y^0$, even if it is not. The hook functions are similarly visual. Consider the left hook function $\Delta^{\ulcorner}(x,y)$: the left argument is on the Matsubara (vertical) branch and the right argument is on a real time (horizontal) branch (it does not matter which one since the contour ordering is the same in both cases). The contour delta function is defined so that $\int_C \mathrm{d}z'\; \delta_C(z,z')\, f(z') =$

$f(z)$ as usual. The only non-trivial reduction to an ordinary delta function is when (taking $t_0 = 0$ for simplicity) $z = -i\tau$ lies on the Matsubara branch:

$$\int_C \mathrm{d}z' \ \delta_C\left(-i\tau, z'\right) f\left(z'\right) = f\left(-i\tau\right). \tag{1.76}$$

The only contribution can come from z' on the Matsubara branch, so substitute $z' = -i\tau'$ giving

$$\int_0^{-i\beta\hbar} \mathrm{d}\left(-i\tau'\right) \delta_C\left(-i\tau, -i\tau'\right) f\left(-i\tau'\right) = f\left(-i\tau\right), \tag{1.77}$$

which gives $\delta_C\left(-i\tau, -i\tau'\right) = i\delta\left(\tau - \tau'\right)$.

When working with the Keldysh components it is convenient to have identities for the reduction of frequently occurring convolutions and products of contour time functions to their components. These identities are called *Langreth rules* and are summarised in Table 1.4. These identities are proved in [63] chapter 5. The most common use of these identities is in the reduction of equations of motion for contour ordered Green functions to equations for real times functions. For example, if a scalar field Green function obeys a Dyson equation of the form

$$\Box_x \Delta\left(x, y\right) = i\delta_C\left(x, y\right) + \int_z \Sigma\left(x, z\right) \Delta\left(z, y\right), \tag{1.78}$$

where $\Box_x = \partial_\mu^{(x)} \partial^{(x)\mu}$ and the self energy function Σ is also defined on the time contour, the Langreth rules show that $\Delta^<$ obeys (suppressing arguments)

$$\Box_x \Delta^< = \Sigma^< \cdot \Delta^A + \Sigma^R \cdot \Delta^< + \Sigma^{\urcorner} \star \Delta^{\ulcorner}, \tag{1.79}$$

an equation suitable for numerical solution.

The Green functions contain the physical information of the theory. This can be seen explicitly in the simplifying case of free fields where $\Sigma = 0$ and the Dyson equation can be solved simply by Fourier transform. Writing

$$\Delta^{(a)}\left(x, y\right) = \int_p \mathrm{e}^{-ip(x-y)} \Delta^{(a)}\left(p\right), \tag{1.80}$$

where (a) denotes the Keldysh component, one has [67]

$$\Delta^{\mathrm{T}}\left(p\right) = \frac{1}{p^2 - m^2 + i\epsilon} - 2\pi i f\left(p\right) \delta\left(p^2 - m^2\right), \tag{1.81}$$

$$\Delta^<\left(p\right) = \Delta^>\left(-p\right) = -2\pi i\left[\Theta\left(p^0\right) + f\left(p\right)\right] \delta\left(p^2 - m^2\right), \tag{1.82}$$

$$\Delta^{\tilde{\mathrm{T}}}\left(p\right) = -\frac{1}{p^2 - m^2 - i\epsilon} - 2\pi i f\left(p\right) \delta\left(p^2 - m^2\right), \tag{1.83}$$

where $\Delta^<\left(p\right) = \Delta^>\left(-p\right)$ follows by Hermiticity of the field $\hat{\phi}$, and the fact that the same function $f\left(p\right)$ is found on all of the right hand sides is due to the fact that the commutator $\left[\hat{\phi}\left(x\right), \hat{\phi}\left(y\right)\right]$ is a c-number, so its expectation value $-i\left\langle\left[\hat{\phi}\left(x\right), \hat{\phi}\left(y\right)\right]\right\rangle = \Delta^>\left(x, y\right) - \Delta^<\left(x, y\right)$ is independent of the state. By considering the anticommutator

Table 1.4: Langreth rules for convolutions and products (with $f \cdot g \equiv \int_0^\infty \mathrm{d}t\ f(t)\, g(t)$ and $f \star g \equiv -i \int_0^\beta \mathrm{d}\tau\ f(\tau)\, g(\tau)$)

Green function	$c(z,z') = \int_C \mathrm{d}\bar{z}\ a(z,\bar{z})\, b(\bar{z},z')$	$c(z,z') = a(z,z')\, b(z',z)$ $(a^\delta = b^\delta = 0)$
Greater	$c^> = a^> \cdot b^A + a^R \cdot b^> + a^\rceil \star b^\ulcorner$	$c^> = a^> b^<$
Lesser	$c^< = a^< \cdot b^A + a^R \cdot b^< + a^\rceil \star b^\ulcorner$	$c^< = a^< b^>$
Retarded	$c^R = a^R \cdot b^R$	$c^R = \begin{cases} a^R b^< + a^< b^A \\ a^R b^> + a^> b^A \end{cases}$
Advanced	$c^A = a^A \cdot b^A$	$c^A = \begin{cases} a^A b^< + a^< b^R \\ a^A b^> + a^> b^R \end{cases}$
Time ordered	$c^T = a^T \cdot b^T - a^< \cdot b^> + a^\rceil \star b^\ulcorner$	$c^T = \begin{cases} a^< b^T + a^R b^< \\ a^T b^> + a^> b^A \end{cases}$
Anti-time ordered	$c^{\tilde{T}} = a^> \cdot b^< - a^{\tilde{T}} \cdot b^{\tilde{T}} + a^\rceil \star b^\ulcorner$	$c^{\tilde{T}} = \begin{cases} a^{\tilde{T}} b^< - a^< b^A \\ a^> b^{\tilde{T}} - a^R b^> \end{cases}$
Right hook	$c^\rceil = a^R \cdot b^\rceil + a^\rceil \star b^M$	$c^\rceil = a^\rceil b^\ulcorner$
Left hook	$c^\ulcorner = a^\ulcorner \cdot b^A + a^M \star b^\ulcorner$	$c^\ulcorner = a^\rceil b^\ulcorner$
Matsubara	$c^M = a^M \star b^M$	$c^M = a^M b^M$

$-i\left\{\hat{\phi}(x), \hat{\phi}(y)\right\} = \Delta^>(x,y) + \Delta^<(x,y)$ one finds that $f(p)$ must be real and even. Another constraint is that $\int_{xy} h^\star(x)\, i\Delta^<(x,y)\, h(y) \geq 0$ for all real functions $h(x)$, which implies that $f(p) \geq 0$. Finally, an equilibrium computation shows that there $f(p)$ is just the Bose-Einstein occupation number of the mode p. Thus, $f(p)$ naturally has the interpretation of a statistical distribution function which can be extended to arbitrary states. Thus knowledge of the Green function is equivalent to knowledge of the statistical distribution function.

Other observables of the theory are simply related to the Green functions since they are polynomials in the fields and their derivatives. For example, in a theory with N scalar fields ϕ_a, $a = 1, \cdots, N$, with an $\mathrm{O}(N)$ symmetry, the conserved $\mathrm{O}(N)$ currents are

$$j_\mu^A = i\left\langle \phi_a T_{ab}^A \overleftrightarrow{\partial_\mu} \phi_b \right\rangle, \tag{1.84}$$

where $T_{ab}^A = -T_{ba}^A$ is a generator of rotations which has been written in a basis of the $A = 1, \cdots, N(N-1)/2$ independent generators and $f \overleftrightarrow{\partial_\mu} g \equiv f(\partial_\mu g) - (\partial_\mu f)\, g$. This can be written in terms of the Green function as

$$j_\mu^A(x) = iT_{ab}^A \left(\partial_\mu^{(y)} - \partial_\mu^{(x)}\right) \langle \phi_a(x)\, \phi_b(y) \rangle \Big|_{y\to x} = T_{ab}^A \left(\partial_\mu^{(x)} - \partial_\mu^{(y)}\right) \Delta_{ab}^>(x,y) \Big|_{y\to x}. \tag{1.85}$$

Similarly, if the theory has a quartic coupling the energy-momentum tensor is

$$T_{\mu\nu} = \left\langle \partial_\mu \phi_a \partial_\nu \phi_a - \eta_{\mu\nu} \left(\frac{1}{2} \partial_\rho \phi_a \partial^\rho \phi_a - \frac{1}{2} m^2 \phi_a \phi_a - \frac{\lambda}{4!} (\phi_a \phi_a)^2 \right) \right\rangle, \tag{1.86}$$

which can be written in terms of Δ and $G^{(4)} = \langle \mathrm{T}_C [\phi_a \phi_b \phi_c \phi_d] \rangle$.

1.5 Summary

This chapter has tried to motivate the investigation which forms the remainder of this thesis. This started with a general discussion of the gaps which exist between theory and experiment in situations involving strongly interacting particles and many particles out of equilibrium. Electroweak baryogenesis was then given as a concrete example and discussed in some detail, with an emphasis on those areas which have been hindered by an inability to compute. Then, laying the groundwork for the remainder of the thesis, basic quantum field theory was reviewed focusing on the functional integral formalism. It has been shown that a common formula for the generating functional can be used for scattering, thermal equilibrium and generic non-equilibrium problems by specifying a choice of time contour. The fundamental objects of field theory are the Green functions which can be straightforwardly connected to physical observables such as densities, currents and so forth.

The remainder of this thesis is structured as follows. Chapter 2 introduces the n-particle irreducible effective action (nPIEA) techniques with an emphasis on their application to non-perturbative quantum field theory. This includes a review of the standard theory surrounding the methods and concludes with a novel investigation into the analytic properties of the solution of the 2PIEA in a toy model which can be compared to the exact solution and contrasted with other non-perturbative techniques. Chapter 3 begins the consideration of symmetries in the thesis by introducing the $\mathrm{O}(N)$ scalar field theory which forms the basis of the remainder of the work. The Ward identities for the theory (which encode the symmetries of the theory in effective action language) are derived and it is shown that the expected identities no longer hold for truncations of nPIEA. This result is known in the literature but is derived here in a way which is, to the author's knowledge, new. The 2PIEA is then solved in the Hartree-Fock approximation demonstrating this result explicitly. A contrast is then drawn between these results and those of the external propagator method and large N method. Chapter 4 introduces the symmetry improved nPIEA (SI-nPIEA) as a means of overcoming the difficulty posed by violations of symmetries in truncated nPIEA. The novel work in this chapter is an extension of the symmetry improvement known in the literature from $\mathrm{O}(2), \mathrm{O}(4) \to \mathrm{O}(N)$ and from the $\mathcal{O}\left(\hbar^2\right)$ SI-2PIEA to the $\mathcal{O}\left(\hbar^3\right)$ SI-3PIEA. Additionally, several ambiguities in the original symmetry improvement procedure are identified and addressed. The renormalisation theory of these actions is discussed and solutions are presented in the Hartree-Fock approximation, as well as a few checks of the basic field theoretical consistency of the new actions. Chapter 5 introduces a novel technique called soft symmetry improvement (SSI) aimed at overcoming the difficulties of the SI-nPIEA. The SSI-2PIEA is derived, renormalised and solved in the Hartree-Fock approximation. It is shown that the method is sensitive to the way the infinite volume / low temperature limit is taken and there are three limiting cases: a trivial reduction to the unimproved 2PIEA, a reduction to the SI-2PIEA, and a novel limit which possess some unusual features which are discussed. Chapter 6 studies for the first time the non-equilibrium aspects of the SI-2PIEA by formulating the linear response theory for a system governed by the SI-2PIEA being driven away from equilibrium by a weak external perturbation. It is shown that the SI-2PIEA, and by a simple extension all SI-nPIEA, are inconsistent with the linear response approximation. Finally, chapter 7 is the discussion chapter drawing everything together and pinpointing directions for future work. Some derivations and results concerning the renormalisation theory which are inessential to the main flow of the text are collected in the appendices A through C.

Chapter 2

Non-perturbative Quantum Field Theory with n-Particle Irreducible Effective Action Techniques

2.1 Synopsis

This chapter reviews the n-particle irreducible effective action (nPIEA) techniques for quantum field theory. For $n > 1$ the nPIEAs are non-perturbative, effecting resummations of 2- through n-th order correlation functions to infinite order in perturbation theory. The equations of motion derived from nPIEAs are closed, unlike the otherwise very similar Schwinger-Dyson equations in quantum field theory or Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) equations of kinetic theory. Because of this and the fact the the methods can be derived from first principles and are manifestly self-consistent, nPIEAs have been advocated for applications in non-equlibrium field theory ranging from cold atom research to cosmology (see, e.g., [16, 17]). For $n \geq 3$, nPIEAs can provide the basis for a self-consistent first principles derivation of kinetic theory (see, e.g. [16, 68, 69]).

In section 2.2 the venerable (but perturbative) 1PIEA is introduced following standard textbook treatments. Then in section 2.3 the non-perturbative 2PIEA is reviewed, following the reviews by Berges [16, 17] and the extensive book by Calzetta and Hu [64]. Here the full expression for the 2PIEA of a quartically coupled scalar field theory is derived to three loop order. The general pattern of higher nPIEAs is then discussed in section 2.4, where the 3PIEA is explicitly derived for future reference. Finally, section 2.5 contains a novel discussion of the analytic properties of effective actions using the 2PIEA of a toy model where exact results are available, so that the 2PIEA results can be compared to perturbation theory and other resummation methods. This discussion is based on a paper published by the author [3].

2.2 The 1PI effective action

Generating functionals and effective action methods form the basis for the rest of this work. This section reviews the one particle irreducible effective action. Originally developed by Goldstone, Salam and Weinberg [20] and, independently, Jona-Lasinio [70], the 1PIEA is widely used, particularly in discussions of theories with spontaneous symmetry breaking or gauge symmetries. The treatment starts with the generating functional $Z[J]$, closely related to the partition function described in section 1.4. $Z[J]$ generates all correlation functions of the field theory and can be related to a Feynman diagram expansion. From $Z[J]$ one derives $W[J]$ which contains only the "connected" correlation functions, i.e. those whose diagrams are made of one connected component. The 1PIEA $\Gamma[\varphi]$ is then found by Legendre transforming $W[J]$ and is the generating function of "proper" correlation functions, meaning those which cannot be divided by cutting one line of the corresponding Feynman diagram. The skeleton expansion of the connected generating functional provides an elegant and systematic means of relating connected correlation functions with the proper correlation functions. Hence $\Gamma[\varphi]$ contains all of the physical information of a field theory in relatively compact form. Further, terms in the derivative expansion of $\Gamma[\varphi]$ have a direct physical meaning. Most importantly, the term with no derivatives (the *effective potential*) is equal to the minimum of the energy, over all states $|\Psi\rangle$, subject to the constraint that the mean field is equal to $\langle\phi\rangle = \varphi$.

The previous chapter outlined how using various choices of time contour in the functional integral allow one to compute vacuum, finite temperature and non-equilibrium properties of a field theory. The differences between these formalisms are largely irrelevant to most of the following considerations. The important thing is that all observables of the theory can be computed from a formula formally identical to (1.50), only where the range of the integration variables, boundary conditions and time contour are interpreted appropriately according to the particular physical situation. To that end it proves useful to generalise the generating functional (1.48) to a quantity usually called the partition function:

$$Z[J] = \int \mathcal{D}[\phi] \exp\left[\frac{i}{\hbar}\left(S[\phi] + \int_x J(x)\,\phi(x)\right)\right], \tag{2.1}$$

where

$$S[\phi] = \int_x \mathcal{L}(\phi), \tag{2.2}$$

is the classical action. The only difference between (2.1) and (1.48) is that the time contour and boundary conditions of the functional integral are hidden in the notation and can be re-interpreted as needed. Then, the sought after generalisation of (1.50) is

$$G^{(n)}(x_1, \cdots, x_n) \equiv \left\langle \mathrm{T}_C\left[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right]\right\rangle = \frac{1}{Z[J]}\left(-i\hbar\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\hbar\frac{\delta}{\delta J(x_n)}\right)Z[J]\bigg|_{J=0}, \tag{2.3}$$

which defines the functions $G^{(n)}$, $n = 1, 2, 3, \cdots$, which are the contour ordered Green functions of the theory. (There is no harm in defining also the trivial cases $G^{(0)} \equiv \langle 1\rangle = 1$ and $G^{(n)} = 0$ for $n < 0$, which makes some formulae simpler.)

Applying the trivial identity

$$0 = \int \mathcal{D}[\phi] \frac{\delta}{\delta\phi(x_1)} F[\phi], \tag{2.4}$$

where $F[\phi]$ is an arbitrary functional, to the case

$$F[\phi] = \phi(x_2)\cdots\phi(x_n)\exp\left[\frac{i}{\hbar}\left(\int_x \mathcal{L}(\phi) + J(x)\phi(x)\right)\right] \tag{2.5}$$

and working out the derivative explicitly gives the *Schwinger-Dyson equation*

$$\begin{aligned}&\left\langle \mathrm{T}_C\left[\phi(x_2)\cdots\phi(x_n)\left(\frac{\delta S[\phi]}{\delta\phi(x_1)} + J(x_1)\right)\right]\right\rangle \\ &\qquad = i\hbar\sum_{j=2}^{n}\delta_C(x_1, x_j)\, G^{(n-2)}(\phi(x_2), \cdots\phi(x_{j-1}), \phi(x_{j+1}), \cdots, \phi(x_n)),\end{aligned} \tag{2.6}$$

which says that the classical equation of motion $\frac{\delta S[\phi]}{\delta\phi(x)} + J(x) = 0$ is obeyed inside correlation functions up to the existence of delta function *contact terms* on the right hand side. Substituting an explicit form for $S[\phi]$ gives a hierarchy of equations, one for each n, connecting correlation functions for different values of n. This is exactly analogous to the BBGKY hierarchy in statistical mechanics (see, e.g., [71, 72]). Solving this hierarchy of equations in practice requires truncation at some finite order, which necessitates invoking a closure ansatz approximating some higher order correlation functions in terms of lower order ones. This ansatz can be physically motivated but always involves some degree of arbitrariness. In contrast, while the higher nPIEA discussed later provide a superficially very similar framework (a hierarchy of equations of motion connecting correlation functions of different order), the equations of motion resulting from nPIEA are automatically self-consistent and closed, without requiring any arbitrary ansatz or approximation.

Usually the Lagrangian can be written

$$\mathcal{L}(\phi) = \frac{1}{2}\phi\Delta_0^{-1}\phi - V_{\text{int}}(\phi), \tag{2.7}$$

where the first term is the quadratic part of the Lagrangian, Δ_0^{-1} is a differential operator and $V_{\text{int}}(\phi)$ is a polynomial containing cubic and higher terms in ϕ. In this case there is a perturbation expansion for Z which can be derived by expanding the exponential in $V_{\text{int}}(\phi)$:

$$\begin{aligned} Z[J] &= \int \mathcal{D}[\phi]\exp\left[-\frac{i}{\hbar}\int_x V_{\text{int}}(\phi)\right]\exp\left[\frac{i}{\hbar}\left(\int_x \frac{1}{2}\phi\Delta_0^{-1}\phi + J(x)\phi(x)\right)\right] \\ &= \int \mathcal{D}[\phi]\exp\left[-\frac{i}{\hbar}\int_x V_{\text{int}}\left(-i\hbar\frac{\delta}{\delta J}\right)\right]\exp\left[\frac{i}{\hbar}\left(\int_x \frac{1}{2}\phi\Delta_0^{-1}\phi + J(x)\phi(x)\right)\right]\end{aligned} \tag{2.8}$$

The $V_{\text{int}}\left(-i\hbar\frac{\delta}{\delta J}\right)$ term can now be extracted from the integral[1], giving

[1] It is this step that is responsible for the asymptotic nature of perturbation theory, c.f. section 2.5.

$$
\begin{aligned}
Z[J] &= \exp\left[-\frac{i}{\hbar}\int_x V_{\text{int}}\left(-i\hbar\frac{\delta}{\delta J}\right)\right]\int \mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}\left(\int_x \frac{1}{2}\phi\Delta_0^{-1}\phi + J(x)\,\phi(x)\right)\right] \\
&= \exp\left[-\frac{i}{\hbar}\int_x V_{\text{int}}\left(-i\hbar\frac{\delta}{\delta J}\right)\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}J\Delta_0 J\right]Z[0], \qquad (2.9)
\end{aligned}
$$

where the last line is obtained by completing the square in the exponent of the Gaussian integral. Thus

$$
\begin{aligned}
G^{(n)}(x_1,\cdots,x_n) = &\left(-i\hbar\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\hbar\frac{\delta}{\delta J(x_n)}\right) \\
&\times \exp\left[-\frac{i}{\hbar}\int_x V_{\text{int}}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)\right]\exp\left[-\frac{i}{\hbar}\int_{xy}\frac{1}{2}J(x)\,\Delta_0(x,y)\,J(y)\right]\Bigg|_{J=0}. \qquad (2.10)
\end{aligned}
$$

Expanding the exponentials one finds the usual Wick expansion of the Green functions which is embodied in the *Feynman rules*:

1. To find a contribution to $G^{(n)}(x_1,\cdots,x_n)$ draw n points ("external legs"), associated to x_1 through x_n, and

2. Draw a k point vertex for every interaction $\frac{1}{k!}g_k\phi^k \subset V_{\text{int}}$ appearing in the expansion of the exponential. Each such vertex is associated with a factor $\left(-\frac{i}{\hbar}\right)g_k$ and an integral $\int_x$ over position of the vertex, and

3. Connect external legs and vertices with lines ("propagators") in all possible ways such that: all external legs are attached to one "line end," each k point vertex is attached to k line ends (nothing forbids attaching both ends of a line to the same vertex) and no line ends except on a vertex or external leg. Each line is associated with a factor $i\hbar\Delta_0(x,y)$ where x and y are the position of the two ends.

4. Also, associate to the diagram an overall symmetry factor $1/|\Gamma|$, where $|\Gamma|$ is the order of the automorphism group of the diagram under interchanges of lines and vertices.

Using these rules it is possible to compute $G^{(n)}$ systematically, order by order, in a perturbation expansion in the coupling constants in V_{int}. From $G^{(n)}$, one can also recover $Z[J]$:

$$
Z[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_{x_1\cdots x_n} J(x_1)\cdots J(x_n)\,G^{(n)}(x_1,\cdots,x_n). \qquad (2.11)
$$

The diagram expansion of $G^{(n)}$contains many disconnected diagrams, i.e. diagrams with more than one connected component. This leads to an undesirable duplication of computation (since a disconnected contribution to $G^{(n)}$ is also a contribution to various $G^{(k)}$ with $k < n$) and non-extensivity of $G^{(n)}$ since a disconnected contribution factorises $\sim G^{(k)}(x_1,\cdots,x_k)\,G^{(n-k)}(x_{n-k},\cdots,x_n)$ and so is independent of the relative separation between points in the two clusters $x_1,\cdots,x_k$ and $x_{n-k},\cdots,x_n$. Clearly the connected components are the basic objects which one would prefer to compute. Fortunately, though the combinatorics will not be proved here[2], the connected diagrams obey the simple relationship given schematically as

$$\text{all diagrams} = \exp\left(\text{connected diagrams}\right). \tag{2.12}$$

This strongly motivates the introduction of the *connected generating functional*

$$W\left[J\right] = -i\hbar \ln Z\left[J\right]. \tag{2.13}$$

The diagram expansion for $W\left[J\right]$ has the same Feynman rules as above, except that one only includes connected diagrams and, traditionally, the extra overall factor $-i\hbar$. To lowest order

$$W\left[J\right] = W\left[0\right] + \int_x J\left(x\right)\left\langle \phi\left(x\right)\right\rangle - \frac{1}{2}\int_{xy} J\left(x\right)\Delta_0\left(x,y\right)J\left(y\right) + \mathcal{O}\left(V_{\text{int}}\right). \tag{2.14}$$

Note that $\left\langle \phi\left(x\right)\right\rangle \sim \mathcal{O}\left(V_{\text{int}}\right)$ if there is no linear term in $\mathcal{L}\left(\phi\right)$, which will be assumed below. As usual, for most physical purposes the overall normalisation $W\left[0\right]$ is unimportant. The connected correlation functions are

$$G_C^{(n)}\left(x_1,\cdots,x_n\right) = \left(-i\hbar\frac{\delta}{\delta J\left(x_1\right)}\right)\cdots\left(-i\hbar\frac{\delta}{\delta J\left(x_n\right)}\right)W\left[J\right]\Big|_{J=0}, \tag{2.15}$$

so that, at leading order in interactions,

$$\begin{aligned} G_C^{(2)}\left(x,y\right) &= \hbar^2\Delta_0\left(x,y\right) + \mathcal{O}\left(V_{\text{int}}\right), & (2.16)\\ G_C^{(1)}\left(x\right) &= G_C^{(n\geq 3)}\left(x_1,\cdots,x_n\right) = 0 + \mathcal{O}\left(V_{\text{int}}\right). & (2.17)\end{aligned}$$

From these $W\left[J\right]$ can be recovered if need be:

$$W\left[J\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_{x_1\cdots x_n} J\left(x_1\right)\cdots J\left(x_n\right)G_C^{(n)}\left(x_1,\cdots,x_n\right). \tag{2.18}$$

It is also convenient to define

$$\Delta\left(x,y\right) = \hbar^{-2}G_C^{(2)}\left(x,y\right), \tag{2.19}$$

i.e. simply stripping off the factor of $\hbar^2$. It is useful for later to note that, in terms of the $G^{(n)}$,

$$G_C^{(2)}\left(x,y\right) = -i\hbar\left[G^{(2)}\left(x,y\right) - G^{(1)}\left(x\right)G^{(1)}\left(y\right)\right], \tag{2.20}$$

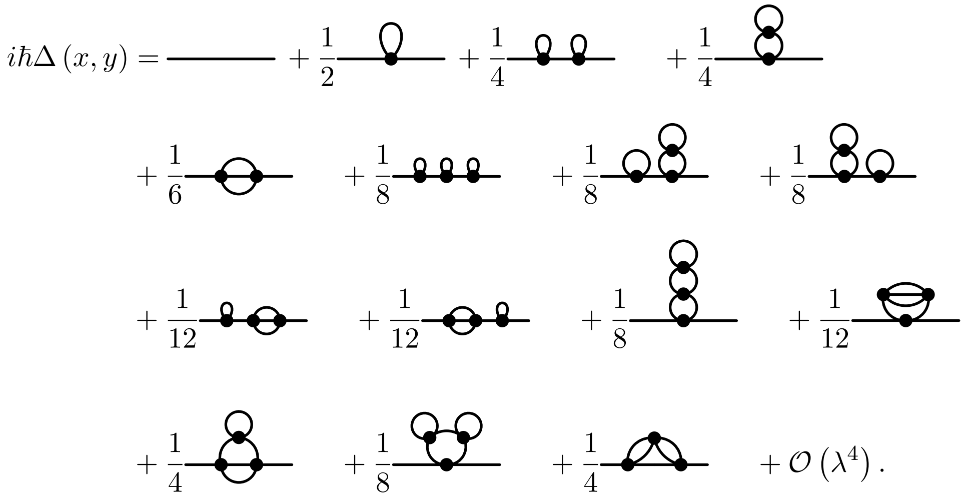
thus

$$G^{(2)}\left(x,y\right) = i\hbar\Delta\left(x,y\right) + \left\langle \phi\left(x\right)\right\rangle\left\langle \phi\left(y\right)\right\rangle. \tag{2.21}$$

There is a great deal of duplication of computation, even in the connected correlation functions. This can be seen by the example of the two point function of a theory with

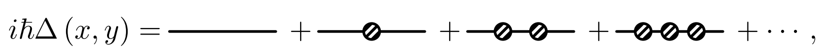
[2]It can be found in many quantum field theory books, see e.g. [73].

$V_{\text{int}}(\phi) = \frac{1}{4!}\lambda\phi^4$ expanded to third order in the interaction:



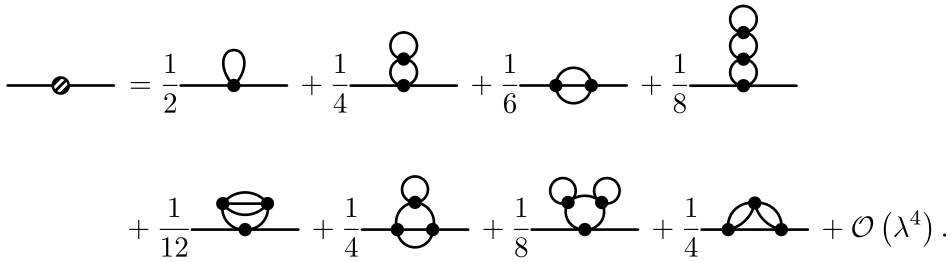
(2.22)

There are obviously a number of repeated structures in this expression, which can be seen to follow the pattern of a geometric series:



(2.23)

where the shaded "blobs" represent the sum of all two point *one particle irreducible* (1PI) Feynman diagrams, that is, the set of diagrams which do not fall apart upon the cutting of any one line. The first few terms of the self-energy are seen to be



(2.24)

The geometric series can be easily summed. Let $-\frac{i}{\hbar}\Sigma$ denote the 1PI "blob." Then (2.23) can be written

$$\begin{aligned} i\hbar\Delta &= i\hbar\Delta_0 + (i\hbar\Delta_0)\left(-\frac{i}{\hbar}\Sigma\right)(i\hbar\Delta_0) + (i\hbar\Delta_0)\left(-\frac{i}{\hbar}\Sigma\right)(i\hbar\Delta_0)\left(-\frac{i}{\hbar}\Sigma\right)(i\hbar\Delta_0) + \cdots \\ &= i\hbar\Delta_0 + i\hbar\Delta_0\Sigma\Delta. \end{aligned} \tag{2.25}$$

Acting on the right with Δ^{-1} and on the left with Δ_0^{-1} one finds

$$\Delta^{-1} = \Delta_0^{-1} - \Sigma, \tag{2.26}$$

which is known as the *Dyson equation* for Δ. Σ is called the *self-energy*. A similar *skeleton* expansion occurs for the higher order correlation functions $G^{(n\geq3)}$ which can also be written in terms of 1PI subgraphs.

There is a generating function for the one particle irreducible diagrams: the 1PI effective action $\Gamma[\varphi]$ which is the Legendre transform of $W[J]$,

$$\Gamma[\varphi] = W[J] - J\frac{\delta W}{\delta J}, \tag{2.27}$$

where J on the right hand side is eliminated in terms of the mean field $\varphi = \langle \phi \rangle$ by solving[3]

$$\frac{\delta W}{\delta J} = \varphi. \tag{2.28}$$

Then the equation of motion followed by $\Gamma[\varphi]$ is

$$\frac{\delta \Gamma}{\delta \varphi} = -J. \tag{2.29}$$

A useful result is

$$\int_y \frac{\delta^2 \Gamma}{\delta\varphi(x)\,\delta\varphi(y)} \frac{\delta^2 W}{\delta J(y)\,\delta J(z)} = \int_y -\frac{\delta J(y)}{\delta\varphi(x)} \frac{\delta\varphi(z)}{\delta J(y)} = -\delta(x-z), \tag{2.30}$$

so that

$$\frac{\delta^2 \Gamma}{\delta\varphi(x)\,\delta\varphi(y)} = \Delta^{-1}(x,y). \tag{2.31}$$

Taking another derivative gives

$$\begin{aligned}
0 &= \frac{\delta}{\delta\varphi(w)} \int_y \frac{\delta^2 \Gamma}{\delta\varphi(x)\,\delta\varphi(y)} \frac{\delta^2 W}{\delta J(y)\,\delta J(z)} \\
&= \int_y \frac{\delta^3 \Gamma}{\delta\varphi(w)\,\delta\varphi(x)\,\delta\varphi(y)} \frac{\delta^2 W}{\delta J(y)\,\delta J(z)} + \int_{yv} \frac{\delta^2 \Gamma}{\delta\varphi(x)\,\delta\varphi(y)} \frac{\delta J(v)}{\delta\varphi(w)} \frac{\delta^3 W}{\delta J(v)\,\delta J(y)\,\delta J(z)} \\
&= \int_y \frac{\delta^3 \Gamma}{\delta\varphi(w)\,\delta\varphi(x)\,\delta\varphi(y)} \frac{\delta^2 W}{\delta J(y)\,\delta J(z)} \\
&\quad - \int_{yv} \frac{\delta^2 \Gamma}{\delta\varphi(x)\,\delta\varphi(y)} \frac{\delta \Gamma}{\delta\varphi(w)\,\delta\varphi(v)} \frac{\delta^3 W}{\delta J(v)\,\delta J(y)\,\delta J(z)},
\end{aligned} \tag{2.32}$$

from which

$$\frac{\delta^3 \Gamma}{\delta\varphi(w)\,\delta\varphi(x)\,\delta\varphi(u)} = -\int_{yzv} \Delta^{-1}(w,v)\,\Delta^{-1}(x,y)\,\Delta^{-1}(u,z) \frac{\delta^3 W}{\delta J(v)\,\delta J(y)\,\delta J(z)}. \tag{2.33}$$

[3]There are physically relevant cases where this equation cannot be inverted. The most important example in this class is spontaneous symmetry breaking. There the saddle point method of evaluating Γ derived below gives a non-convex potential which also has an imaginary part. The non-convex portion of the potential is unphysical, corresponding to a metastable state, and the imaginary part of the potential is related to the lifetime of that state. The quantity actually found by the saddle point method is an effective action which expands about a single classical configuration. This is equal to Γ in the case that it is convex. However, for a non-convex effective potential the two quantities differ and the true form of Γ in these cases is given by a Maxwell construction much like that used for the free energy in thermodynamics (see, e.g. [18, 74] and references therein).

Similar results can be derived for higher order correlation functions by taking more derivatives. One eventually finds that W is given by a sum of *tree* diagrams (i.e., no loops) where the vertices are given by the derivatives of Γ. This is the skeleton expansion of W, and from it one can see that Γ consists of 1PI diagrams only. A detailed proof is not needed here and can be found in, e.g., chapter 5 of [18].

$\Gamma[\varphi]$ is most often evaluated in a semi-classical or saddle point approximation. Here the path integral is expanded around the field configuration $\phi_{\text{cl.}}$ of stationary phase, which is the one that satisfies the classical equation of motion

$$\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\text{cl.}}[J]} + J(x) = 0. \tag{2.34}$$

Writing $\phi = \phi_{\text{cl.}} + \hbar^{1/2}\tilde{\phi}$,

$$S[\phi] = S[\phi_{\text{cl.}}] + \hbar^{1/2}\tilde{\phi}\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\text{cl.}}[J]} + \hbar\int\left(\frac{1}{2}\tilde{\phi}\Delta_0^{-1}[\phi_{\text{cl.}}]\tilde{\phi} - V\left(\phi_{\text{cl.}},\tilde{\phi}\right)\right), \tag{2.35}$$

which defines $\Delta_0^{-1}[\phi_{\text{cl.}}]$ as (up to a factor $\hbar$) the part of $S[\phi]$ which is quadratic in $\tilde{\phi}$ and $V\left(\phi_{\text{cl.}},\tilde{\phi}\right)$ as (up to a factor $\hbar$) the part of V_{int} which is greater than quadratic in $\tilde{\phi}$. For example, in a theory with cubic and quartic couplings $V_{\text{int}}(\phi) = \frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4$, $V\left(\phi_{\text{cl.}},\tilde{\phi}\right)$ is

$$V\left(\phi_{\text{cl.}},\tilde{\phi}\right) = \frac{1}{3!}\hbar^{1/2}(g+\lambda\phi_{\text{cl.}})\tilde{\phi}^3 + \frac{1}{4!}\hbar\lambda\tilde{\phi}^4. \tag{2.36}$$

If one substitutes $S\left[\phi_{\text{cl.}} + \hbar^{1/2}\tilde{\phi}\right]$ in $Z[J]$ the linear term in $\tilde{\phi}$ cancels the $J\tilde{\phi}$ term due to the equation of motion for $\phi_{\text{cl.}}$ and one has

$$\begin{aligned} Z[J] &= \exp\left[\frac{i}{\hbar}(S[\phi_{\text{cl.}}] + J\phi_{\text{cl.}})\right]\int\mathcal{D}[\phi]\exp\left[i\int\left(\frac{1}{2}\tilde{\phi}\Delta_0^{-1}[\phi_{\text{cl.}}]\tilde{\phi} - V\left(\phi_{\text{cl.}},\tilde{\phi}\right)\right)\right] \\ &= \exp\left[\frac{i}{\hbar}(S[\phi_{\text{cl.}}] + J\phi_{\text{cl.}})\right]\int\mathcal{D}[\phi](1+\mathcal{O}(\hbar))\exp\left[i\int\frac{1}{2}\tilde{\phi}\Delta_0^{-1}[\phi_{\text{cl.}}]\tilde{\phi}\right] \\ &= \exp\left[\frac{i}{\hbar}(S[\phi_{\text{cl.}}] + J\phi_{\text{cl.}})\right]\left(N\left(\det\Delta_0^{-1}[\phi_{\text{cl.}}]\right)^{-1/2} + \mathcal{O}(\hbar)\right), \end{aligned} \tag{2.37}$$

Note that while $V\left(\phi_{\text{cl.}},\tilde{\phi}\right) = \mathcal{O}\left(\hbar^{1/2}\right)$, $Z[J]$ necessarily involves an even number of cubic vertices so the leading correction is $\mathcal{O}(\hbar)$. The constant N is determined by $Z[0] = 1$. Also note that $\Delta_0^{-1}[\phi_{\text{cl.}}]$ depends on $\phi_{\text{cl.}}$. Thus[4]

$$W[J] = S[\phi_{\text{cl.}}] + J\phi_{\text{cl.}} + \frac{i\hbar}{2}\left(\operatorname{Tr}\ln\Delta_0^{-1}[\phi_{\text{cl.}}] - \operatorname{Tr}\ln\Delta_0^{-1}[0]\right) + \mathcal{O}\left(\hbar^2\right). \tag{2.38}$$

To perform the Legendre transform one needs to relate $\phi_{\text{cl.}}$ to φ:

[4] Recall the identity $\ln\det M = \operatorname{Tr}\ln M$.

$$\varphi = \frac{\delta W}{\delta J} = \frac{\delta \phi_{\text{cl.}}}{\delta J}\frac{\delta S}{\delta \phi_{\text{cl.}}} + \phi_{\text{cl.}} + J\frac{\delta \phi_{\text{cl.}}}{\delta J} + \frac{\delta \phi_{\text{cl.}}}{\delta J}\frac{\delta}{\delta \phi_{\text{cl.}}}\frac{i\hbar}{2}\text{Tr}\ln\Delta_0^{-1}\left[\phi_{\text{cl.}}\right] + \mathcal{O}\left(\hbar^2\right) = \phi_{\text{cl.}} + \mathcal{O}\left(\hbar\right). \tag{2.39}$$

Using $\phi_{\text{cl.}} = \varphi + \mathcal{O}(\hbar)$ in S gives

$$S\left[\phi_{\text{cl.}}\right] + J\phi_{\text{cl.}} = S\left[\varphi\right] + J\varphi + \mathcal{O}\left(\hbar^2\right), \tag{2.40}$$

because the classical equation of motion for $\phi_{\text{cl.}}$ implies the cancellation of the $\mathcal{O}(\hbar)$ term, so that

$$\Gamma\left[\varphi\right] = S\left[\varphi\right] + \frac{i\hbar}{2}\left(\text{Tr}\ln\Delta_0^{-1}\left[\varphi\right] - \text{Tr}\ln\Delta_0^{-1}\left[0\right]\right) + \mathcal{O}\left(\hbar^2\right). \tag{2.41}$$

The $\Delta_0^{-1}[0]$ term is just a shift by an overall constant which is often neglected, leading one to write

$$\Gamma\left[\varphi\right] = S\left[\varphi\right] + \frac{i\hbar}{2}\text{Tr}\ln\Delta_0^{-1}\left[\varphi\right] + \mathcal{O}\left(\hbar^2\right). \tag{2.42}$$

There is a nice topological property of Feynman diagrams which makes the organisation of the semi-classical expansion at higher orders easy. A diagram contributing to Γ has an overall factor of $\hbar$ coming from the definition of $W[J]$, another $\hbar$ for each $i\hbar\Delta_0$ propagator appearing in the diagram, and a factor of $\hbar^{-1}$ for each vertex. Thus the total power of $\hbar$ for a diagram is $I - V + 1$ where I is the number of lines and V the number of vertices. However, the Euler characteristic $L - I + V = 1$, where L is the number of faces or "loops" in the graph, is a topological invariant for all planar graphs. Thus, for planar graphs the contribution to Γ is $\mathcal{O}\left(\hbar^L\right)$. The definition of L can be extended from planar graphs to all graphs by using the same formula. Thus, in the semi-classical expansion, powers of $\hbar$ count the number of loops in a graph. This agrees with (2.42): the $\mathcal{O}\left(\hbar^0\right)$ term is just the classical action and the $\mathcal{O}\left(\hbar^1\right)$ term is the sum of all one loop graphs, as can be seen explicitly by expanding the logarithm in a Taylor series in V_{int}. The terms not shown above are the 1PI diagrams with two or more loops.

2.3 The 2PI effective action

Two-particle irreducible effective actions (2PIEA) and approximation schemes based on them are useful when it is necessary to go beyond the perturbative field theory embodied by the 1PIEA, with applications to thermal and non-equilibrium plasmas/fluids, strongly coupled quantum field theories and systems dominated by many-body collective effects. The technique was originally developed by Lee and Yang [75], Luttinger and Ward [76], Baym [77] and others in the context of many-body theory, then extended by Cornwall, Jackiw and Tomboulis [78] to relativistic field theory in terms of functional integrals. Since then a broad literature has developed surrounding 2PI effective actions and their generalisations (see [16] for a good introductory review).

The essential idea behind the 2PIEA is to define a functional $\Gamma[\varphi, \Delta]$ of not only the mean field φ, but also the (connected) two point correlation function Δ. By allowing Δ to vary it is possible to eliminate the diagrams yielding propagator corrections. This is achieved by a double Legendre transform procedure, first defining a generalised partition functional

$$Z[J,K] = \int \mathcal{D}[\phi] \exp\left[\frac{i}{\hbar}\left(S[\phi] + J\phi + \frac{1}{2}\phi K \phi\right)\right], \tag{2.43}$$

where spacetime integrations are hidden for brevity, i.e. $\phi K \phi \equiv \int_{xy} \phi(x) K(x,y) \phi(y)$ etc. Then $W[J,K] = -i\hbar \ln Z[J,K]$ as before and

$$\Gamma[\varphi, \Delta] = W[J,K] - J\frac{\delta W[J,K]}{\delta J} - K\frac{\delta W[J,K]}{\delta K}, \tag{2.44}$$

where J and K are eliminated by solving

$$\frac{\delta W}{\delta J(x)} = \varphi(x), \tag{2.45}$$

$$\frac{\delta W}{\delta K(x,y)} = \frac{1}{2}\left(i\hbar\Delta(x,y) + \varphi(x)\varphi(y)\right). \tag{2.46}$$

The equations of motion obeyed by the 2PIEA are then

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi} = -J - K\varphi, \tag{2.47}$$

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta} = -\frac{1}{2}i\hbar K. \tag{2.48}$$

The easiest way to compute the double Legendre transform is stepwise, using (2.42) with $S \to S + \frac{1}{2}\phi K \phi$ and $\Delta_0^{-1} \to \Delta_0^{-1} + K$ as an intermediate step, then performing the transform with respect to K. This leads to

$$\Gamma[\varphi,\Delta] = S[\varphi] + \frac{i\hbar}{2}\mathrm{Tr}\ln\Delta^{-1} + \frac{i\hbar}{2}\mathrm{Tr}\left(\Delta_0^{-1}\Delta - 1\right) + \Gamma_2[\varphi,\Delta], \tag{2.49}$$

where $\Gamma_2[\varphi,\Delta] \sim \mathcal{O}\left(\hbar^2\right)$ on using

$$K = \Delta^{-1} - \Delta_0^{-1} + \mathcal{O}(\hbar). \tag{2.50}$$

The equation of motion for the propagator becomes

$$\Delta^{-1} = \Delta_0^{-1} - \Sigma, \tag{2.51}$$

where

$$\Sigma = \frac{2i}{\hbar}\frac{\delta\Gamma_2[\varphi,\Delta]}{\delta\Delta}, \tag{2.52}$$

which one recognises as the Dyson equation and self-energy respectively. Since Σ consists of 1PI graphs and the effect of $\delta/\delta\Delta$ is to remove a propagator from any graph in Γ_2 in all possible ways, Γ_2 must consist of *two particle irreducible* (2PI) graphs, i.e. those which do not fall apart under any cutting of two lines. Note that Δ depends on $\hbar$ through (2.51), which must be accounted for when comparing $\Gamma[\varphi,\Delta[\varphi]]$ to the 1PIEA order by order in $\hbar$.

For reference it is useful to give the 2PIEA for a quartic scalar field theory, generalised to a set of fields ϕ_a, $a = 1, \cdots, N$, up to three loop order. No symmetry is yet imposed, though the study of $\mathrm{O}(N)$ symmetric theories forms the bulk of this thesis. The classical action is given by

$$S[\phi] = \frac{1}{2}\phi_a\left(-\Box\delta_{ab} - m^2_{ab}\right)\phi_b - \frac{1}{3!}g_{abc}\phi_a\phi_b\phi_c - \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d, \tag{2.53}$$

where the linear term is dropped without loss of generality[5]. The couplings m^2_{ab}, g_{abc} and λ_{abcd} can always be taken to be totally symmetric under permutations of indices since the ϕ_as commute. There are $N(N+1)/2$, $N(N+1)(N+2)/3!$ and $N(N+1)(N+2)(N+3)/4!$ linearly independent quadratic, cubic and quartic couplings respectively, though $N(N-1)/2$ rotations of the fields into each other can be used to remove some of these, e.g. by diagonalising m^2_{ab}. In the case of $\mathrm{O}(N)$ symmetry there are only two constants with $m^2_{ab} = m^2\delta_{ab}$, $g_{abc} = 0$ and $\lambda_{abcd} \propto \lambda(\delta_{ab}\delta_{cd} + \text{permutations})$. One has the bare propagator

$$\begin{aligned}\Delta^{-1}_{0ab}[\varphi](x,y) &= \left.\frac{\delta^2 S}{\delta\phi_a(x)\,\delta\phi_b(y)}\right|_{\phi=\varphi} \\ &= -\left(\Box_x\delta_{ab} + m^2_{ab} + g_{abc}\varphi_c(x) + \frac{1}{2}\lambda_{abcd}\varphi_c(x)\,\varphi_d(x)\right)\delta(x-y),\end{aligned} \tag{2.54}$$

and the effective interaction Lagrangian (i.e. the cubic and quartic part of S after shifting the field à la $\phi = \varphi + \hbar^{1/2}\tilde{\phi}$)

$$-\hbar V\left(\varphi,\tilde{\phi}\right) = -\frac{1}{3!}\hbar^{3/2}\left(g_{abc} + \lambda_{abcd}\varphi_d\right)\tilde{\phi}_a\tilde{\phi}_b\tilde{\phi}_c - \frac{1}{4!}\hbar^2\lambda_{abcd}\tilde{\phi}_a\tilde{\phi}_b\tilde{\phi}_c\tilde{\phi}_d. \tag{2.55}$$

It is convenient to introduce the shorthands

$$V_{0abc} \equiv \left.\frac{\delta^3 S}{\delta\phi_a\delta\phi_b\delta\phi_c}\right|_{\phi=\varphi} = -\left(g_{abc} + \lambda_{abcd}\varphi_d\right), \tag{2.56}$$

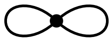
$$W_{abcd} \equiv \left.\frac{\delta^4 S}{\delta\phi_a\delta\phi_b\delta\phi_c\delta\phi_d}\right|_{\phi=\varphi} = -\lambda_{abcd}. \tag{2.57}$$

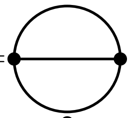
(This matches the notation of [3].) $\Gamma_2[\varphi,\Delta]$ is the sum of 2PI Feynman diagrams with interactions given by the above and propagators given by the variational propagator $i\hbar\Delta_{ab}$. Up to three loop order this is

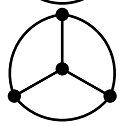
$$\Gamma_2 = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \mathcal{O}\left(\hbar^4\right), \tag{2.58}$$

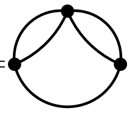
[5]If a linear term is present, it can be removed by shifting ϕ_a. This results in a constant term which can also be dropped without consequence.

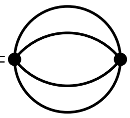
where

$\Phi_1 =$  $= -\dfrac{\hbar^2}{8} W_{abcd}\Delta_{ab}\Delta_{cd},$ (2.59)

$\Phi_2 =$  $= \dfrac{\hbar^2}{12} V_{0abc}V_{0def}\Delta_{ad}\Delta_{be}\Delta_{cf},$ (2.60)

$\Phi_3 =$  $= \dfrac{i\hbar^3}{4!} V_{0abc}V_{0def}V_{0ghi}V_{0jkl}\Delta_{ad}\Delta_{bg}\Delta_{cj}\Delta_{eh}\Delta_{fk}\Delta_{il},$ (2.61)

$\Phi_4 =$  $= -\dfrac{i\hbar^3}{8} W_{abcd}V_{0efg}V_{0hij}\Delta_{ae}\Delta_{bf}\Delta_{ch}\Delta_{di}\Delta_{gj},$ (2.62)

$\Phi_5 =$  $= \dfrac{i\hbar^3}{48} W_{abcd}W_{efgh}\Delta_{ae}\Delta_{bf}\Delta_{cg}\Delta_{dh}.$ (2.63)

These diagrams are called “EIGHT,” “EGG,” “MERCEDES,” “HAIR,” and “BBALL,” respectively, in the nomenclature of [79, 80]. Keeping only Φ_1 is called the *Hartree-Fock* (HF) approximation, which has special properties due to the simple form of the self-energy, while keeping both Φ_1 and Φ_2 is called the *sunset* approximation. Note that at four loop order there are eleven new diagrams (including the first non-planar diagram), so the number of diagrams increases rapidly with order. However, the growth is much slower than for 1PI or connected diagrams and the 2PIEA gives a relatively compact formulation of the theory.

It is also convenient to give explicitly the equations of motion in the sunset approximation for later reference. They are:

$$\begin{aligned}
-\left(\Box\delta_{ab} + m_{ab}^2\right)\varphi_b &= \frac{1}{2}g_{abc}\varphi_b\varphi_c + \frac{1}{3!}\lambda_{abcd}\varphi_b\varphi_c\varphi_d \\
&\quad + \frac{i\hbar}{2}\left(g_{abc} + \lambda_{abcd}\varphi_d\right)\Delta_{cb} + \frac{\hbar^2}{6}\lambda_{abcd}V_{0efg}\Delta_{be}\Delta_{cf}\Delta_{dg} \\
&\quad - J_a - K_{ab}\varphi_b + \mathcal{O}\left(\hbar^3\right), \qquad (2.64) \\
\Delta_{ab}^{-1} &= \Delta_{0ab}^{-1} + \frac{i\hbar}{2}W_{abcd}\Delta_{cd} - \frac{i\hbar}{2}V_{0acd}V_{0bef}\Delta_{ce}\Delta_{df} \\
&\quad + K_{ab} + \mathcal{O}\left(\hbar^2\right). \qquad (2.65)
\end{aligned}$$

Notice the following phenomenon: a loopwise truncation of the 2PIEA is not a loopwise truncation of the equations of motion. That is, starting from the action to order $\mathcal{O}\left(\hbar^2\right)$, one obtains the equation of motion for φ to $\mathcal{O}\left(\hbar^2\right)$ but the equation of motion for Δ to only $\mathcal{O}\left(\hbar\right)$. This is because taking a derivative with respect to Δ removes a propagator and therefore opens up a loop. This behaviour holds also in higher nPIEAs, and in Chapter 4 some of the consequences of this for physics will be discussed.

2.4 Higher nPI effective actions

The construction for 2PIEA via a two-fold Legendre transform can be immediately generalised to arbitrary n-fold Legendre transforms giving the nPIEA. Apparently de Dominicis and Martin [81] were the first to realise this and study $n \geq 3$. They were mentioned, but not used, in the pioneering 2PIEA paper [78] and developed further in [82]. More early work on higher effective actions was carried on by Vasiliev [83], whose book was unfortunately not available in English for more than twenty years, though there were some reviews suggesting that the English literature was at least aware of these developments (e.g. [84, 85]). The recent resurgence of interest in higher nPIEAs has largely been driven by their advantages for non-equilibrium problems and can likely be credited to the reviews by Berges [16, 17] and advances in computer power. Explicit expressions are given here for the three loop 3PIEA. Higher order nPIEA can be found in the literature (e.g. [16, 79, 80, 86–88]).

The definition of the nPIEA is conceptually simple enough, having seen now the 1PIEA and 2PIEA cases[6]. One defines a generalised partition function

$$\begin{aligned} Z^{(n)}\left[J, K^{(2)}, \cdots, K^{(n)}\right] = \int \mathcal{D}[\phi] \exp\Bigg[\frac{i}{\hbar}\Big(S[\phi] + J_x\phi_x \\ + \frac{1}{2}\phi_x K^{(2)}_{xy}\phi_y + \cdots + \frac{1}{n!}K^{(n)}_{x_1\cdots x_n}\phi_{x_1}\cdots\phi_{x_n}\Big)\Bigg], \end{aligned} \tag{2.66}$$

and generating function $W^{(n)} = -i\hbar \ln Z^{(n)}$, and perform the multiple Legendre transform

$$\Gamma^{(n)}\left[\varphi, \Delta, V, \cdots, V^{(n)}\right] = W^{(n)} - J\frac{\delta W^{(n)}}{\delta J} - K^{(2)}\frac{\delta W^{(n)}}{\delta K^{(2)}} - \cdots - K^{(n)}\frac{\delta W^{(n)}}{\delta K^{(n)}}, \tag{2.67}$$

where the sources J, $K^{(2)}$ through $K^{(n)}$ are eliminated in terms of φ, Δ and the proper (1PI) three- through n-point vertex functions V through $V^{(n)}$. The relation of the Vs to the correlation functions can be worked out using the skeleton expansion so that, e.g.,

$$\begin{aligned} \hbar^2\Delta_{ad}\Delta_{be}\Delta_{cf}V_{def} &= \langle \mathrm{T}[\phi_a\phi_b\phi_c]\rangle - \langle \mathrm{T}[\phi_a\phi_b]\rangle\langle\phi_c\rangle \\ &\quad - \langle \mathrm{T}[\phi_c\phi_a]\rangle\langle\phi_b\rangle - \langle \mathrm{T}[\phi_b\phi_c]\rangle\langle\phi_a\rangle \\ &\quad + 2\langle\phi_a\rangle\langle\phi_b\rangle\langle\phi_c\rangle, \end{aligned} \tag{2.68}$$

which can be connected to $W^{(n)}$ using

$$\begin{aligned} 3!\frac{\delta W^{(n)}}{\delta K^{(3)}_{abc}} &= \langle \mathrm{T}[\phi_a\phi_b\phi_c]\rangle \\ &= \hbar^2\Delta_{ad}\Delta_{be}\Delta_{cf}V_{def} + i\hbar\Delta_{ab}\varphi_c \\ &\quad + i\hbar\Delta_{ca}\varphi_b + i\hbar\Delta_{bc}\varphi_a + \varphi_a\varphi_b\varphi_c, \end{aligned} \tag{2.69}$$

[6] Note however that the nPIEA is defined by the Legendre transform, *not* by any irreducibility property of the Feynman graphs, though for low enough loop orders the graphs are irreducible as the name implies. At high enough loop order for $n > 2$ the name becomes misleading. For example, the five-loop 5PIEA contains graphs that are not five-particle irreducible [80]!)

and similar results for the higher order Vs. The nPI equation of motion is then, in the absence of sources,

$$\frac{\delta\Gamma^{(n)}}{\delta\varphi}=\frac{\delta\Gamma^{(n)}}{\delta\Delta}=\frac{\delta\Gamma^{(n)}}{\delta V}=\cdots\frac{\delta\Gamma^{(n)}}{\delta V^{(n)}}=0. \tag{2.70}$$

As before, the 3PIEA can be computed using the previous result for the 2PIEA (2.49), with $S\to S+\frac{1}{3!}K^{(3)}_{abc}\phi_a\phi_b\phi_c$ and $\Delta^{-1}_{0ab}\to\Delta^{-1}_{0ab}+K^{(3)}_{abc}\varphi_c$which results in a shifted vertex $\tilde{V}=V_0+K^{(3)}$ in Γ_2, then performing the Legendre transform with respect to $K^{(3)}$. One gets, on cancelling the disconnected contributions,

$$\begin{aligned}\Gamma^{(3)}\left[\varphi,\Delta,V\right]=S\left[\varphi\right]+\frac{i\hbar}{2}\mathrm{Tr}\ln\Delta^{-1}+\frac{i\hbar}{2}\mathrm{Tr}\left(\Delta_0^{-1}\Delta-1\right)+\Gamma_2\left[\varphi,\Delta,\tilde{V}\right]\\-K^{(3)}_{abc}\frac{1}{3!}\hbar^2\Delta_{ad}\Delta_{be}\Delta_{cf}V_{def},\end{aligned} \tag{2.71}$$

which can be further simplified using the fact that, by definition of the Legendre transform, the right hand side is stationary under variations of $K^{(3)}$ at fixed V. This gives to order $\mathcal{O}\left(\hbar^4\right)$:

$$\begin{aligned}\frac{\delta\Gamma_2}{\delta K^{(3)}_{abc}}&=\frac{\delta\left(\Phi_2+\Phi_3+\Phi_4\right)}{\delta\tilde{V}_{abc}}\\&=\frac{\hbar^2}{3!}\tilde{V}_{def}\Delta_{ad}\Delta_{be}\Delta_{cf}+\frac{i\hbar^3}{3!}\tilde{V}_{def}\tilde{V}_{ghi}\tilde{V}_{jkl}\Delta_{ad}\Delta_{bg}\Delta_{cj}\Delta_{eh}\Delta_{fk}\Delta_{il}\\&-\frac{i\hbar^3}{4}W_{hijd}\tilde{V}_{efg}\left(\frac{1}{3}\Delta_{ah}\Delta_{bi}\Delta_{ej}\Delta_{df}\Delta_{cg}+\text{cyclic permutations }\;(a\to b\to c\to a)\right)\\&=\frac{1}{3!}\hbar^2\Delta_{ad}\Delta_{be}\Delta_{cf}V_{def},\end{aligned} \tag{2.72}$$

where the second line comes from the definitions of the diagrams and the third line is from the source term in $\Gamma^{(3)}\left[\varphi,\Delta,V\right]$. Note that the permutations of the $W\tilde{V}\Delta^5$ term enter because $K^{(3)}$ is symmetric in its indices. The $1/3$ compensates for the would-be over-counting. This shows that $\tilde{V}=V+\mathcal{O}\left(\hbar\right)$, so that, up to higher order terms which are not written, the $\tilde{V}$s in the $\mathcal{O}\left(\hbar^3\right)$ terms can be replaced by Vs. Then one has

$$\begin{aligned}\tilde{V}_{xyz}&=V_{0xyz}+K^{(3)}_{xyz}\\&=V_{xyz}-i\hbar V_{xef}V_{yhi}V_{zkl}\Delta_{eh}\Delta_{fk}\Delta_{il}\\&+\frac{i\hbar}{2}\left(W_{xyjd}V_{efz}\Delta_{ej}\Delta_{df}+\text{cyclic permutations }\;(x\to y\to z\to x)\right)\\&+\mathcal{O}\left(\hbar^2\right).\end{aligned} \tag{2.73}$$

Again, the stationarity of the Legendre transform under variations of $K^{(3)}$ at fixed V implies that the $\mathcal{O}\left(\hbar\right)$ correction to $K^{(3)}$ only contributes to Γ at $\mathcal{O}\left(\hbar^4\right)$ (one order higher than the naive $\mathcal{O}\left(\hbar^3\right)$). Thus, to three loop order one only needs that $K^{(3)}=V-V_0+\mathcal{O}\left(\hbar\right)$. Substituting this into the expression for Γ gives, finally

$$\Gamma^{(3)}\left[\varphi,\Delta,V\right]=S\left[\varphi\right]+\frac{i\hbar}{2}\mathrm{Tr}\ln\Delta^{-1}+\frac{i\hbar}{2}\mathrm{Tr}\left(\Delta_0^{-1}\Delta-1\right)+\Gamma_3\left[\varphi,\Delta,V\right], \tag{2.74}$$

where

$$\Gamma_3 = \Phi_1 - \Phi_2 + \frac{\hbar^2}{3!} V_{0abc} \Delta_{ad} \Delta_{be} \Delta_{cf} V_{def} + \Phi_3 + \Phi_4 + \Phi_5 + \mathcal{O}\left(\hbar^4\right), \tag{2.75}$$

where the three point vertices in the Φs are now the variational function V. From (2.67) the 3PI equations of motion are

$$\begin{aligned}
\frac{\delta\Gamma^{(3)}}{\delta\varphi_a} &= -J_a - K^{(2)}_{ab}\varphi_b - \frac{1}{2}K^{(3)}_{abc}\left(i\hbar\Delta_{bc} + \varphi_b\varphi_c\right), && (2.76)\\
\frac{\delta\Gamma^{(3)}}{\delta\Delta_{ab}} &= -\frac{i\hbar}{2}K^{(2)}_{ab} - \frac{i\hbar}{2}K^{(3)}_{abc}\varphi_c \\
&\quad -\frac{\hbar^2}{3!2}\left(K^{(3)}_{acd}\Delta_{ce}\Delta_{df}V_{bef} + K^{(3)}_{bcd}\Delta_{ce}\Delta_{df}V_{aef}\right) && (2.77)\\
\frac{\delta\Gamma^{(3)}}{\delta V_{abc}} &= -\frac{1}{3!}\hbar^2\Delta_{ad}\Delta_{be}\Delta_{cf}K^{(3)}_{def}, && (2.78)
\end{aligned}$$

while from (2.74) the left hand sides are

$$\begin{aligned}
\frac{\delta\Gamma^{(3)}}{\delta\varphi_a} &= \frac{\delta S}{\delta\varphi_a} - \frac{i\hbar}{2}\left(g_{abc} + \lambda_{abcd}\varphi_d\right)\Delta_{cb} - \frac{\hbar^2}{3!}\lambda_{abcd}\Delta_{be}\Delta_{cf}\Delta_{dg}V_{efg} + \mathcal{O}\left(\hbar^3\right), && (2.79)\\
\frac{\delta\Gamma^{(3)}}{\delta\Delta_{ab}} &= -\frac{i\hbar}{2}\left(\Delta^{-1}_{ab} + \mathrm{Tr}\Delta^{-1}_{0ab} - \Sigma_{ab}\right) + \mathcal{O}\left(\hbar^4\right), && (2.80)\\
\frac{\delta\Gamma^{(3)}}{\delta V_{abc}} &= \frac{\hbar^2}{3!}\left(V_{0def} - V_{def}\right)\Delta_{ad}\Delta_{be}\Delta_{cf} + \frac{i\hbar^3}{3!}V_{def}V_{ghi}V_{jkl}\Delta_{ad}\Delta_{bg}\Delta_{cj}\Delta_{eh}\Delta_{fk}\Delta_{il} \\
&\quad -\frac{i\hbar^3}{4}W_{hijd}V_{efg}\left(\frac{1}{3}\Delta_{ah}\Delta_{bi}\Delta_{ej}\Delta_{df}\Delta_{cg} + \text{cyclic permutations } (a \to b \to c \to a)\right) \\
&\quad +\mathcal{O}\left(\hbar^4\right), && (2.81)
\end{aligned}$$

where now

$$\Sigma_{ab} = \frac{2i}{\hbar}\frac{\delta\Gamma_3}{\delta\Delta_{ab}}. \tag{2.82}$$

Note that, as in the 2PI case, there are $\mathcal{O}\left(\hbar^2\right)$ corrections to the φ and Δ equations of motion, but only $\mathcal{O}\left(\hbar\right)$ corrections to V. Thus a loopwise truncation of $\Gamma^{(3)}$ is not a loopwise truncation of the corresponding equations of motion.

Note that if one truncates $\Gamma^{(3)}$ to two loop order and sets $K^{(3)} = 0$, then the V equation of motion becomes simply $V = V_0$. Substituting this back in $\Gamma^{(3)}$ gives the 2PIEA to two loop order. That is

$$\Gamma^{(3)}\left[\varphi, \Delta, V\right] = \Gamma^{(2)}\left[\varphi, \Delta\right], \text{ at two loops.} \tag{2.83}$$

This is a specific example of a general equivalence hierarchy

$$\begin{aligned}
&\Gamma^{(1)}[\varphi] = \Gamma^{(2)}[\varphi,\Delta] = \cdots && \text{at one loop,}\\
&\Gamma^{(1)}[\varphi] \neq \Gamma^{(2)}[\varphi,\Delta] = \Gamma^{(3)}[\varphi,\Delta,V] = \cdots && \text{at two loops,}\\
&\Gamma^{(1)}[\varphi] \neq \Gamma^{(2)}[\varphi,\Delta] \neq \Gamma^{(3)}[\varphi,\Delta,V] = \Gamma^{(4)}\left[\varphi,\Delta,V,V^{(4)}\right] = \cdots && \text{at three loops,}\\
&\quad\vdots && \vdots
\end{aligned}$$

where extra arguments are evaluated at the solution of their source-free equations of motion when comparisons are made. This implies that if one is willing to work at L loop order, there is no profit in using an nPIEA with $n > L$. However, the L-loop LPIEA is *self-consistently complete* in the sense that all Feynman diagram topologies which are possible to resum using $\leq L$ loop diagrams as skeletons are being resummed fully self-consistently. Not all diagrams at $> L$ loops are included, however. For example, taking the 2PI equations of motion at two loop order and solving them by iteration, one finds that not all three loop diagrams are present. This is because at three loops there are primitive vertex correction diagrams which cannot be represented by iterated propagator corrections. Thus there is profit in increasing n as long as $n < L$. Finally, it will be shown in Chapter 4 that symmetry improvement breaks the equivalence hierarchy, e.g. the symmetry improved 3PIEA *does not* reduce to the symmetry improved 2PIEA at the two loop level.

2.5 Analytic properties of the 2PI effective action and resummation schemes

2.5.1 2PIEA as a resummation scheme

One of the advantages of nPIEA ($n > 1$) is the summation of perturbation theory effected by the self-consistent equations of motion. This can be illustrated simply with the example of the 2PIEA in the Hartree-Fock approximation. The propagator equation of motion reads

$$\Delta^{-1}_{ab}(x,y) = -\left(\Box_x\delta_{ab} + m^2_{ab} + g_{abc}\varphi_c(x) + \frac{1}{2}\lambda_{abcd}\varphi_c(x)\varphi_d(x) + \frac{i\hbar}{2}\lambda_{abcd}\Delta_{cd}(x,x)\right)\delta(x-y). \tag{2.84}$$

Consider for simplicity the case where φ_a is independent of position and time. This is the case, for example, in thermal equilibrium or vacuum since by assumption spacetime is also homogeneous. Then this equation can be solved in the Fourier domain. Introducing

$$\Delta^{-1}_{ab}(p) = \int_{x-y} \mathrm{e}^{ip(x-y)}\Delta^{-1}_{ab}(x,y), \tag{2.85}$$

the equation of motion is

$$\Delta^{-1}_{ab}(p) = -\left(-p^2\delta_{ab} + m^2_{ab} + g_{abc}\varphi_c + \frac{1}{2}\lambda_{abcd}\varphi_c\varphi_d + \frac{i\hbar}{2}\lambda_{abcd}\int_k \Delta_{cd}(k)\right). \tag{2.86}$$

The important point is that $\int_k \Delta_{cd}(k)$ is actually independent of p. Thus one can write $\Delta_{ab}^{-1}(p) = p^2\delta_{ab} - M_{ab}^2$ where the effective mass matrix is given by the *gap equation*

$$M_{ab}^2 = m_{ab}^2 + g_{abc}\varphi_c + \frac{1}{2}\lambda_{abcd}\varphi_c\varphi_d + \frac{i\hbar}{2}\lambda_{abcd}\int_k \left[k^2\delta_{cd} - M_{cd}^2\right]^{-1}. \tag{2.87}$$

There is a basis of fields in which the matrix $M_{ab}^2 = M_a^2\delta_{ab}$ (no sum) is diagonalised. In that basis

$$M_a^2 = \mu_a^2 + \frac{i\hbar}{2}\sum_c \lambda_{ac}\int_k \frac{1}{k^2 - M_c^2}, \tag{2.88}$$

where μ_a^2 are the eigenvalues of $m_{ab}^2 + g_{abc}\varphi_c + \frac{1}{2}\lambda_{abcd}\varphi_c\varphi_d$ and λ_{ac} are, for each c, the eigenvalues of the matrix λ_{abcc} (no sum).

The integral is divergent in $d \geq 2$ dimensions and must be regularised. Using dimensional regularisation in $d = 4 - 2\epsilon$ dimensions,

$$\begin{aligned}\int_k \frac{i}{k^2 - M_c^2} = &-\frac{M_c^2}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma + 1 + \ln(4\pi) + \mathcal{O}(\epsilon)\right] \\ &+ \frac{M_c^2}{16\pi^2}\ln\left(\frac{M_c^2}{\mu^2}\right) + \int_{\boldsymbol{k}} \frac{1}{\sqrt{\boldsymbol{k}^2 + M_c^2}}\frac{1}{\mathrm{e}^{\beta\sqrt{\boldsymbol{k}^2+M_c^2}} - 1},\end{aligned} \tag{2.89}$$

where μ is the renormalisation point and $\gamma \approx 0.577$ is the Euler-Mascheroni constant. The three terms are the $\overline{\mathrm{MS}}$ divergent contribution, the finite vacuum contribution and the finite temperature Bose-Einstein contribution respectively. The renormalisation procedure (not carried out in full here; see section 3.4 and appendix C for details), effectively results in the removal of the infinite part. The ambiguity in the removal of a finite part is reflected in the freedom to choose μ. For simplicity set the temperature to zero (the finite temperature contribution is not important for the point that follows). Then,

$$M_a^2 = \mu_a^2 + \frac{\hbar}{2}\sum_c \lambda_{ac}\frac{M_c^2}{16\pi^2}\ln\left(\frac{M_c^2}{\mu^2}\right). \tag{2.90}$$

Now contrast this with the perturbative, or 1PI, evaluation of the same quantity. In the 1PI self-energy only the bare Δ_0 propagators appear, so the analogue of (2.84) is identical except for the replacement $\Delta \to \Delta_0$ on the right hand side. This results eventually in the replacement $M_c^2 \to \mu_c^2$ on the right hand side above, giving

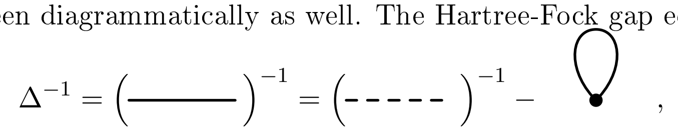
$$M_a^2(\mathrm{1PI}) = \mu_a^2 + \frac{\hbar}{2}\sum_c \lambda_{ac}\frac{\mu_c^2}{16\pi^2}\ln\left(\frac{\mu_c^2}{\mu^2}\right), \tag{2.91}$$

which is what is obtained by iterating the 2PI equation once and ignoring terms of order $\mathcal{O}(\hbar^2)$. The 2PI solution has higher order contributions, starting with

$$M_a^2 = M_a^2(\mathrm{1PI}) + \frac{\hbar^2}{1024\pi^4}\sum_{cd}\left\{\lambda_{ac}\lambda_{cd}\left[1 + \ln\left(\frac{\mu_c^2}{\mu^2}\right)\right]\mu_d^2\ln\left(\frac{\mu_d^2}{\mu^2}\right)\right\} + \mathcal{O}(\hbar^3). \tag{2.92}$$

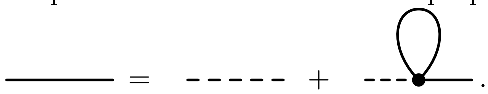
This is a large correction if, parametrically, $\lambda \ln\left(\mu_a^2/\mu^2\right) \sim 16\pi^2$. This can occur in theories with very strong couplings or a hierarchy of masses so that some μ_a^2/μ^2 is inevitably large. An important case is $\mu_a^2 = 0$. In this case the 1PI gap equation gives zero, but the 2PI gap equation gives a non-perturbative solution $M_a^2 \sim \mu^2 \exp\left(32\pi^2/\hbar\lambda\right)$ which may or may not be physical.

This can be seen diagrammatically as well. The Hartree-Fock gap equation is



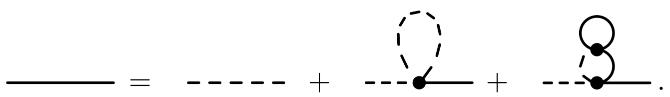
(2.93)

where the full propagator represents $i\Delta$ and the dashed propagator $i\Delta_0$. This can be rearranged into



(2.94)

Iterating this equation gives



(2.95)

The 1PI Dyson equation consists of the first two terms on the right hand side only. The additional term gives, upon repeated iteration, the sum of all *daisy* (or *ring*) diagrams as well as all *superdaisy* (i.e., daisies within daisies in a fractal pattern) diagrams. The highly non-trivial fact about 2PI approximation schemes is that all of these diagrams appear with the correct combinatoric factors. (This no longer holds for all diagrams in nPIEA with $n \geq 3$.) From the point of view of someone engineering *ad hoc* equations of motion to effect a resummation, this is a highly desirable property which would be difficult to prove, especially when diagrams with more complicated topologies are included. Furthermore, since perturbation theories are generically asymptotic expansions, resummation in general is a dangerous procedure. Mathematically speaking an asymptotic series can be summed to *any* value whatsoever. Additional non-perturbative properties are required of any resummation scheme that claims to faithfully represent the original theory. For these reasons, while nPIEA ($n > 1$) *do* effect resummations of perturbation theory, speaking of resummations reflects a limited understanding of why nPIEA actually work. This section examines this issue in the context of the 2PIEA for a toy model which is exactly solvable so that a deeper understanding of the analytic features that support nPIEAs in general and 2PIEAs in particular can be found. These results are then contrasted with those for some other common resummation schemes used in the literature, particularly the Padé and Bore-Padé methods. Much of this section is based on a publication by the author [3].

Unfortunately robust comparisons are difficult because the large order behaviour of perturbation theory is known, at best, in a sketchy form for most field theories of interest. The use of a genuinely trivial model "field theory" in zero spacetime dimensions (i.e. probability theory) allows for exact results to be obtained and removes all complications due to renormalisation, etc. This model nevertheless accurately represents the typical combinatoric structure of large order perturbation theory, at least in those cases where the behaviour is known in more interesting field theories. A "spectral function" representation of the Green function (similar to the one first introduced by Bender and Wu [89, 90]) is used to capture the non-analyticity of the solutions in the various methods.

The existence of the spectral representation is connected to the branch cut of physical amplitudes on the negative coupling (λ) axis. This branch cut is due to the non-existence of the theory at negative couplings: the path integral diverges due to a potential unbounded from below. In a higher dimensional field theory this has a simple physical interpretation: the vacuum is unstable and, after tunnelling through a barrier, the system rolls down the potential [56]. For weak coupling the semi-classical approximation is valid and the tunnelling is exponentially suppressed, giving an imaginary contribution $\sim \exp(-1/\lambda)$ to the vacuum persistence amplitude which is inherited by the Green function. This exponential behaviour can also be seen in the spectral function.

Dyson [13] argued that a very similar phenomenon occurs in quantum electrodynamics (QED). In QED physical observables are calculated in a perturbation series of the form $F\left(e^2\right) = a_0 + a_2 e^2 + a_4 e^4 + \cdots$ where $e \sim 0.3$ is the charge of the electron. Now if one imagines a world where $e^2 < 0$, i.e. like charges attract, it is easy to see that the ordinary vacuum is unstable to the production of many electron-positron pairs which separate into clouds of like-charged particles. At weak coupling there is a large tunnelling barrier to overcome because one must pay for the rest mass of the pairs and separate them far enough for the wrong-sign Coulomb potential to compensate. Thus there is a finite but exponentially suppressed rate of vacuum decay. A Taylor series expansion in e^2 cannot capture this non-analyticity so the perturbation series must be divergent.

Similarly, the perturbation series in λ diverges for the toy model considered here. Padé approximants can represent physical quantities more accurately than Taylor series because they can develop isolated poles in the complex λ plane, however they struggle to capture the strong coupling behaviour at any fixed order in the approximation. Padé approximants are better able to capture the non-analyticities of the Borel transform, however, and the widely used combination Borel-Padé approximants give a better global approximation. This occurs because the Padé approximated Borel transform has poles in the Borel plane, which lead to branch cuts when the Laplace transform is taken to return to physical variables. Similarly, the self-consistent 2PI approximations develop branch point non-analyticities and approximate the exact Green function rather well in the entire complex λ plane already at the leading non-trivial truncation. However, the branch cuts in the 2PI case arise because the 2PI Green function obeys self-consistent equations of motion, and is connected to the existence of unphysical solution branches.

A question that naturally arises is: how do these methods compare? Both 2PI and Borel-Padé methods have the ability to accurately represent non-analyticities of the exact theory and so out-perform other methods. However, a natural conjecture is that self-consistently derived equations of motion "know more" about the analytic structure of the underlying theory than do the generic Borel-Padé approximants. This section tests the hypothesis that the 2PI methods should be more accurate than Borel-Padé and, indeed, one finds this to be the case, at least in certain regimes.

The theory discussed here, although admittedly a toy model, also has physical relevance. Beneke and Moch found this toy model as the theory governing the zero mode of scalar fields in Euclidean de Sitter space [91]. They performed an analysis very similar to this one, finding that a non-perturbative treatment is necessary and compared 2PI and (Borel-)Padé resummed approximations. However, they present this analysis very briefly as part of a larger discussion of scalar fields in de Sitter space. Further, their comparison of the 2PI and resummed techniques, while correct, is not very detailed. The analysis presented here is a detailed discussion of the interplay between 2PI effective actions and various resummation techniques. Use of the spectral function to quantify the non-analyticities present in the Green function in aid of this comparison is, to the author's knowledge, a new aspect.

2.5.2 A toy model and its exact solution

The toy model is a Euclidean quartic theory in zero dimensions, i.e. a probability theory for a single real variable q given by the partition function

$$Z\left[K\right] = N \int_{-\infty}^{\infty} \mathrm{d}q \exp\left(-\frac{1}{2}m^2 q^2 - \frac{1}{4!}\lambda q^4 - \frac{1}{2}Kq^2\right) \tag{2.96}$$

with a source K for the two point function. N is a normalisation factor chosen so that $Z\left[0\right] = 1$. This theory has been discussed before in the context of exact and non-perturbative methods in field theory (see, e.g., [92, 93] and references therein) because, despite the absence of spacetime, the theory possesses a perturbative expansion in terms of Feynman diagrams with the same combinatorial structure as more realistic theories. It was also discussed in [91] as an effective field theory for scalar field zero modes in Euclidean de Sitter space.

Attention is restricted to the $m^2 > 0$ theory since, though the $m^2 < 0$ theory exists and may be interesting for other purposes, it possesses no sensible weak coupling limit and in zero dimensions does not give a broken symmetry phase anyway. In the absence of symmetry breaking there is no need for a source term for q. The integral diverges for $\mathrm{Re}\lambda < 0$, but can be defined for all complex λ by analytic continuation. Then $Z\left[K\right]$ possesses a branch cut along the negative λ axis. Physically, this signals the instability of the negative λ vacuum due to tunnelling away from the local minimum at $q \sim 0$ to $q \sim \pm\sqrt{-6m^2/\lambda}$ followed by rolling down the inverted quartic potential which is unbounded from below. The branch point at $\lambda = 0$ means that the weak coupling perturbation series has zero radius of convergence, agreeing with the general analysis of Dyson [13].

Introducing the conveniently rescaled variables $k = K/m^2$ and $\rho = 3m^4/4\lambda$, the integral can be performed giving

$$Z\left[K\right] = \frac{2N}{m}\sqrt{\rho\left(1+k\right)} \exp\left(\rho\left(1+k\right)^2\right) \mathrm{K}_{1/4}\left(\rho\left(1+k\right)^2\right), \tag{2.97}$$

where $K_{1/4}(\cdots)$ is a modified Bessel function of the second kind. This expression is valid so long as

$$\mathrm{Re}\left(\sqrt{\rho}\,(1+k)\right) > 0, \tag{2.98}$$

which extends the definition to the entirety of the cut λ-plane. The normalisation factor is

$$N = \frac{m}{2}\frac{\exp(-\rho)}{\sqrt{\rho}\mathrm{K}_{1/4}(\rho)}, \tag{2.99}$$

so finally

$$Z[K] = \sqrt{1+k}\exp\left(\rho\left[(1+k)^2 - 1\right]\right)\frac{\mathrm{K}_{1/4}\left(\rho(1+k)^2\right)}{\mathrm{K}_{1/4}(\rho)}. \tag{2.100}$$

Also define the generating function

$$W[K] = -\ln Z[K] \tag{2.101}$$

and note that connected correlations are found by taking derivatives of W. For example,

$$\frac{\partial}{\partial K}W[K] = \frac{1}{2}\left\langle q^2\right\rangle_K \equiv \frac{1}{2}\bar{G}, \tag{2.102}$$

$$\frac{\partial^2}{\partial K^2}W[K] = -\frac{1}{4}\left(\left\langle q^4\right\rangle - \left\langle q^2\right\rangle^2\right)_K \equiv \frac{1}{4}\left(V^{(4)}\bar{G}^4 - 2\bar{G}^2\right), \tag{2.103}$$

where the subscript K indicates the average is taken at a fixed value of K. $\bar{G}$ and $V^{(4)}$ are the proper two and four point functions respectively. To lowest order in perturbation theory and with $K=0$, $\bar{G} = 1/m^2 + \mathcal{O}(\lambda)$ and $V^{(4)} = \lambda + \mathcal{O}\left(\lambda^2\right)$.

The exact value of $W[K]$ is easily obtained directly from the definition, giving

$$W[K] = -\frac{1}{2}\ln(1+k) - \rho\left[(1+k)^2 - 1\right] - \ln\mathrm{K}_{1/4}\left(\rho(1+k)^2\right) + \ln\mathrm{K}_{1/4}(\rho). \tag{2.104}$$

By direct differentiation the exact two point correlation function is

$$m^2\left\langle q^2\right\rangle_K = 4\rho(1+k)\left[\frac{\mathrm{K}_{3/4}\left(\rho(1+k)^2\right)}{\mathrm{K}_{1/4}\left(\rho(1+k)^2\right)} - 1\right], \tag{2.105}$$

or for the original ($K=0$) theory,

$$m^2\bar{G} = 4\rho\left(\frac{\mathrm{K}_{3/4}(\rho)}{\mathrm{K}_{1/4}(\rho)} - 1\right). \tag{2.106}$$

Like Z, $\bar{G}$ possesses a branch cut discontinuity from $\lambda = 0$ to $\lambda = -\infty$. At $\lambda = 0$ one obtains the usual free (Gaussian) theory result $\bar{G} = G_0 = m^{-2}$. In the strong coupling limit, $\lambda \to \infty$, $\bar{G} \sim \left[2\sqrt{6}\Gamma\left(\frac{3}{4}\right)/\Gamma\left(\frac{1}{4}\right)\right]\lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right)$. $\bar{G}$ is shown in Figure 2.1, from which the branch cut is obvious. This can also be seen in more detail in the complex λ plane as shown in Figure 2.2. One sees that not only does $\bar{G}$ possess a branch cut, it is analytic in the cut plane and is in fact a Herglotz-Nevanlinna function (i.e. $\bar{G}(\lambda)^\star = \bar{G}(\lambda^\star)$ where $\star$ is complex conjugation). This gives rise to an integral representation in terms of

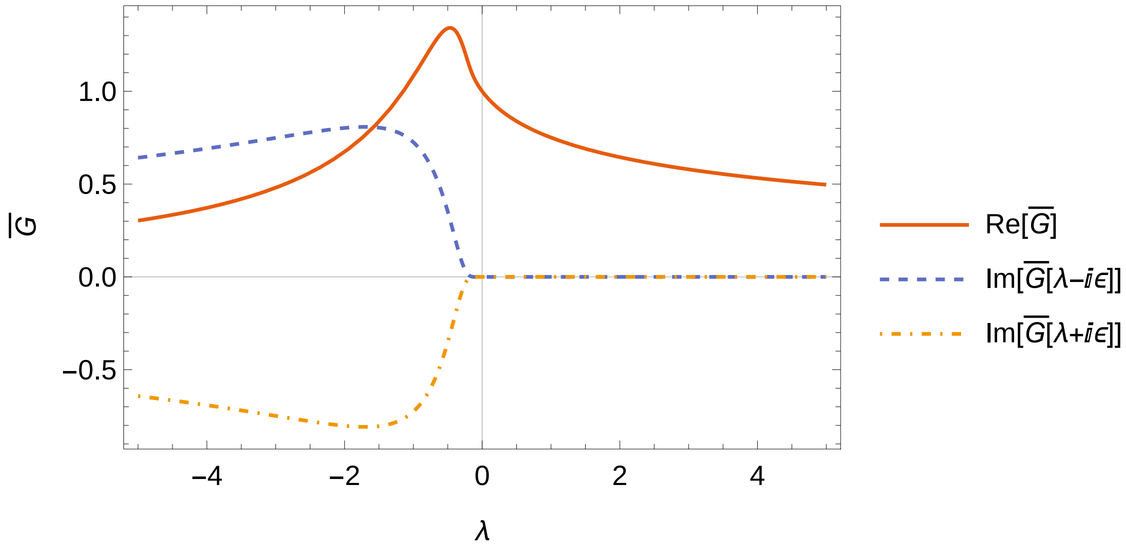


Figure 2.1: $\bar{G}$ from (2.106) as a function of λ for $m = 1$. The sign of the imaginary part depends on whether one approaches the $\lambda < 0$ cut from above or below. Note the imaginary part is exponentially suppressed near the origin because the vacuum decay process is non-perturbative.

a spectral function which can be used in order to quantify the branch cut.

Start with the integral representation for $\bar{G}$ using the Cauchy formula

$$\bar{G}(\lambda) = \frac{1}{2\pi i}\oint_C \frac{\bar{G}(\lambda')}{\lambda' - \lambda}\mathrm{d}\lambda', \tag{2.107}$$

where the contour C circles λ in the counter-clockwise direction and avoids the cut. Deforming the contour to run on the circle at infinity and around the cut and using $\bar{G}(\lambda) \to 0$ as $|\lambda| \to \infty$, the integral can be written in terms of a spectral function

$$\sigma(s) = \frac{\bar{G}(-s - i\epsilon) - \bar{G}(-s + i\epsilon)}{2\pi i}\Theta(s) = \frac{\mathrm{Im}\left[\bar{G}(-s - i\epsilon)\right]}{\pi}\Theta(s), \tag{2.108}$$

such that

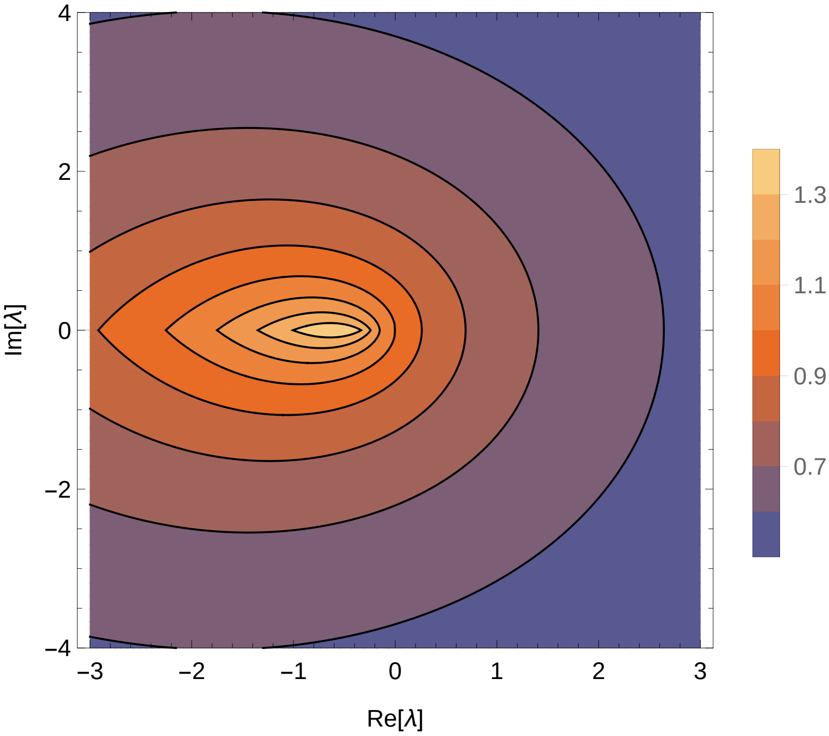
$$\bar{G}(\lambda) = \int_0^\infty \mathrm{d}s \frac{\sigma(s)}{s + \lambda}. \tag{2.109}$$

Then,

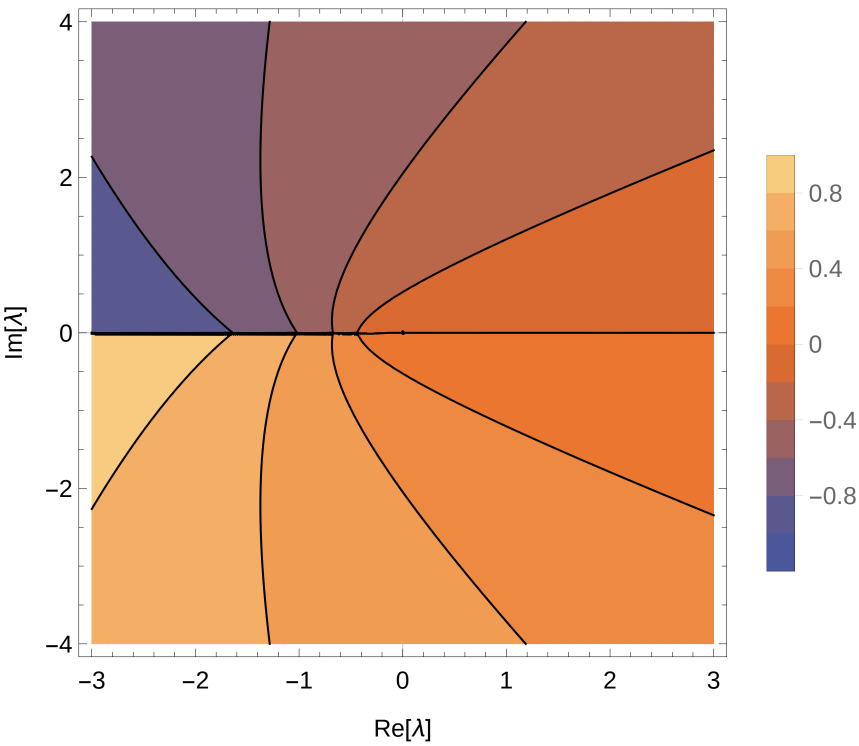
$$\sigma(\lambda) = -\frac{4\sqrt{2}}{m^2\pi^2}\frac{1}{\mathrm{Im}\left[I_{-\frac{1}{4}}(-\rho)^2 - I_{\frac{1}{4}}(-\rho)^2\right]}\Theta(\lambda), \tag{2.110}$$

where $I_{\pm\frac{1}{4}}(\rho)$ are modified Bessel functions of the first kind[7]. $\sigma(\lambda)$ is shown in Figure 2.3.

[7]Note that the physical interpretation of this spectral function is unrelated to the usual one in field theory since, for one thing, there is no such thing as energy in zero dimensions. One can consider $\sigma(s)$ a purely formal device that gives information about the analytic structure of $\bar{G}$.



(a)



(b)

Figure 2.2: Modulus 2.2a and phase 2.2b of $\bar{G}$ in (2.106) for $m^2 = 1$ in the complex λ plane.

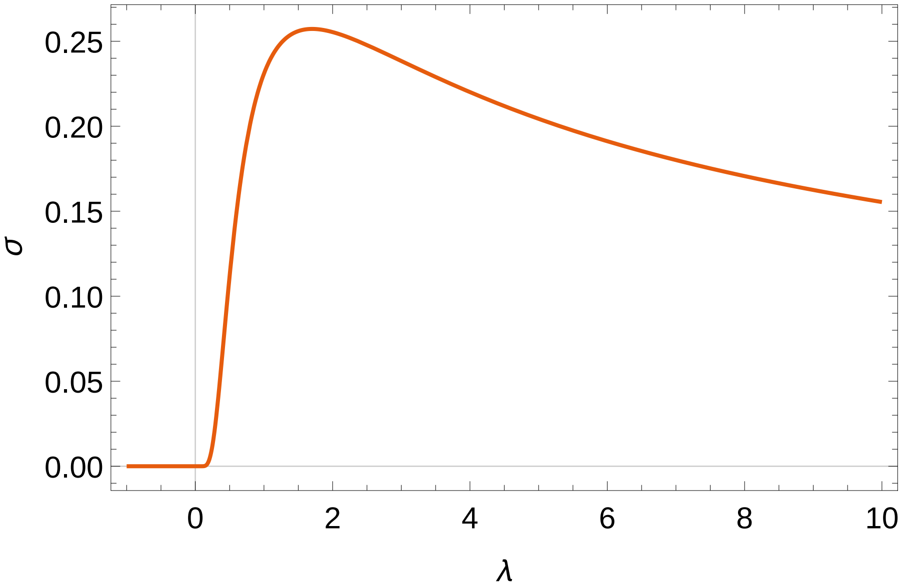


Figure 2.3: Spectral function $\sigma(\lambda)$ of (2.110) for $m = 1$.

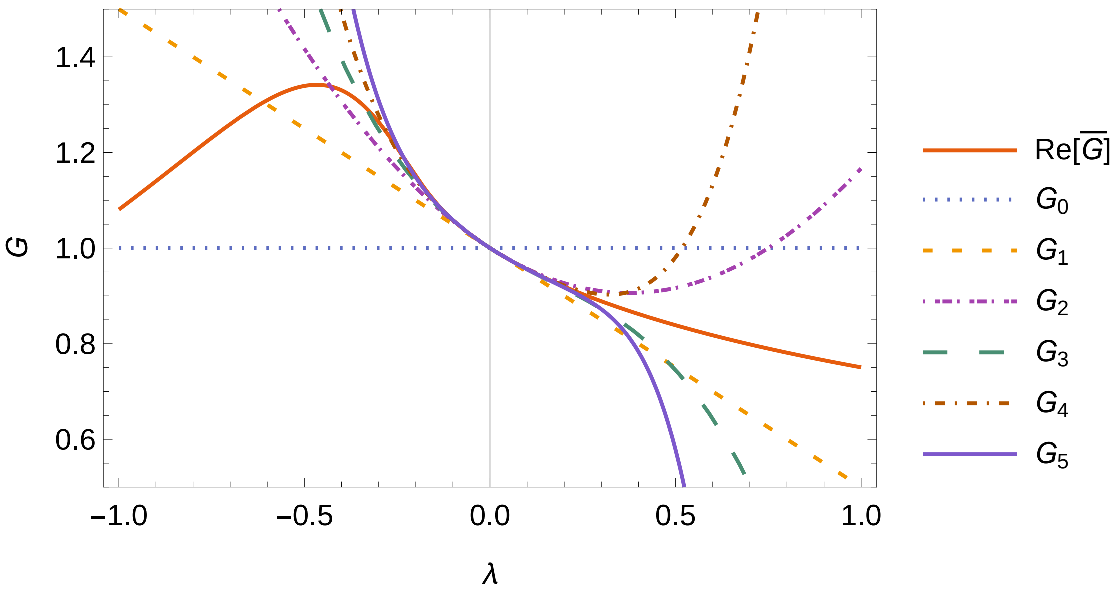


Figure 2.4: Perturbative approximations to $\bar{G}$ up to $\mathcal{O}\left(\lambda^5\right)$ for $m = 1$, compared to the exact solution.

Expanding $\bar{G}$ in a Taylor series in λ gives

$$\begin{aligned}\bar{G} &= \frac{1}{m^2} + \frac{4}{m^2}\frac{\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(-\frac{1}{4!}\lambda\right)^n 2^{2n}\Gamma\left(2n+\frac{1}{2}\right)\frac{1}{m^{4n}}}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{4!}\lambda\right)^n 2^{2n}\Gamma\left(2n+\frac{1}{2}\right)\frac{1}{m^{4n}}} \\ &= \frac{1}{m^2} - \frac{\lambda}{2m^6} + \frac{2\lambda^2}{3m^{10}} - \frac{11\lambda^3}{8m^{14}} + \frac{34\lambda^4}{9m^{18}} + \cdots . \end{aligned} \tag{2.111}$$

The perturbative approximations G_n are the $\mathcal{O}(\lambda^n)$ truncations of this series. The first few are shown compared to the exact $\bar{G}$ in Figure 2.4. These approximations apparently converge poorly to the exact solution, and in fact the series diverges.

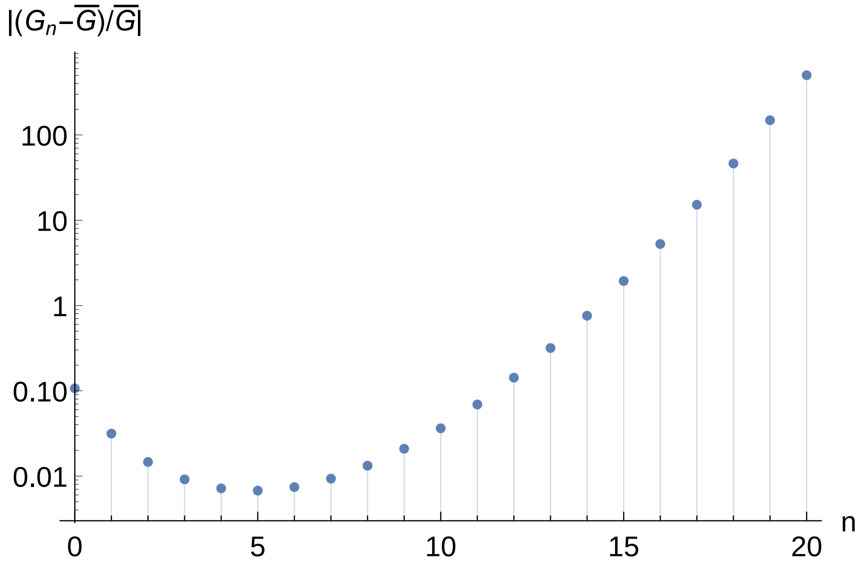


Figure 2.5: Relative error $\left|\left(G_n - \bar{G}\right)/\bar{G}\right|$of the n-th perturbative approximation to $\bar{G}$ for $m = 1$ and $\lambda = 1/4$, showing the decreasing then increasing behaviour typical of asymptotic series

The series for $\bar{G}$ can also be described in terms of Feynman diagrams by the following rules:

1. Draw all connected graphs with two external lines constructed from lines and four point vertices.

2. Associate to each line a factor $G_0 = 1/m^2$.

3. Associate to each vertex a factor $-\lambda$.

4. Divide by an overall symmetry factor being the order of the symmetry group of the diagram under permutations of lines and vertices.

The diagrams of this series are exactly those seen before in the $\lambda\phi^4$ theory in (2.22). The series (2.111) has the form $\bar{G} = m^{-2} \sum_{n=0}^{\infty} c_n \left(\lambda/m^4\right)^n$ where the coefficients asymptotically obey $c_{n+1} \sim -\frac{2}{3} n c_n$ as $n \to \infty$, thus the radius of convergence of the series is zero. This is consistent with the fact that one is perturbing around a branch point of the exact solution: no approximation of $\bar{G}$ in terms of analytic functions can converge at $\lambda = 0$ because $Z[K]$ is itself undefined for $\mathrm{Re}\lambda < 0$. The terms of the series start to increase when $c_{n+1}\left(\lambda/m^4\right) \sim c_n$, i.e. $n \sim 3m^4/2\lambda$, meaning the series is useful for $\lambda \ll m^4$ but fails immediately for a moderately strong coupling $\lambda \approx m^4$. This is typical asymptotic series behaviour as shown in Figure 2.5. Extrapolating perturbation theory to strong coupling $\lambda \gg m^4$ is simply impossible, although the exact solution is well behaved there. (In fact $\bar{G}$ can be expanded as $\bar{G} = m^{-2} \sum_{n=1}^{\infty} \tilde{c}_n \left(\lambda/m^4\right)^{-n/2}$, displaying explicitly the branch point at $\lambda = \infty$.)

2.5.3 Borel summation

The series expansion for $\bar{G}$ diverges for all $\lambda \neq 0$. This is typical of perturbation series and usually signals some singularity of the exact solution for unphysical values of λ. Indeed, the toy model does not exist for $\mathrm{Re}\lambda < 0$ and the exact solution possesses a branch cut on the negative λ axis, a feature which cannot be reproduced in any order of perturbation theory. However, the perturbation series is asymptotic and does contain true information about the exact solution, even if λ is large enough that the series is not useful practically. Because of the ubiquity of this phenomenon, mathematicians have invented a number of series summation techniques which assign a finite value to certain types of divergent series and which obey certain consistency properties (e.g. the value assigned to a convergent series is just its naïve sum). This section discusses Borel summation, which is capable of summing factorially divergent series like (2.111).

Suppose that $\sum_{n=0}^{\infty} a_n$ is a divergent series but that the *Borel transform* of the series, defined as

$$\phi(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}, \tag{2.112}$$

converges for sufficiently small x. Then, if the integral

$$B(x) = \int_0^{\infty} \mathrm{e}^{-t} \phi(xt)\, \mathrm{d}t \tag{2.113}$$

exists the *Borel sum* [14, 94, 95] of the divergent series is defined as $\mathcal{B}\left[\sum_{n=0}^{\infty} a_n\right] \equiv B(1)$. This definition is justified by substituting the series for $\phi(xt)$ into the integral and evaluating term-wise and noting that $B(x) \sim \sum_{n=0}^{\infty} a_n x^n$. The main drawback of Borel summation is that one must know the precise form of a_n for all n to compute $\phi(x)$, which is rarely the case in field theory. For this reason Borel summation cannot be usefully applied directly in practice. However, one may use Padé approximants as discussed in the next section to recast the Borel transform in a useful way. In the case of the toy model, though, one can verify that the perturbation series is Borel summable and the Borel sum is equal to the exact solution (the computation is unenlightening and best done by a computer algebra package).

Note that key to Borel-summability of the perturbation series is the alternating sign $(-1)^n$ of the n-th order term. To see this consider the two series

$$S_1 = \sum_{n=0}^{\infty} (-\lambda)^n n!, \tag{2.114}$$

$$S_2 = \sum_{n=0}^{\infty} \lambda^n n!, \tag{2.115}$$

which differ only by the alternating sign. The Borel transforms $\phi_{1,2}(x)$ are

$$\phi_1(x) = \sum_{n=0}^{\infty} (-\lambda x)^n = \frac{1}{1+\lambda x}, \tag{2.116}$$

$$\phi_2(x) = \sum_{n=0}^{\infty} (\lambda x)^n = \frac{1}{1-\lambda x}, \tag{2.117}$$

and the Borel sums are

$$\mathcal{B}[S_1] = B_1(1) = \int_0^{\infty} \frac{\mathrm{e}^{-t}}{1+\lambda t} \mathrm{d}t, \tag{2.118}$$

$$\mathcal{B}[S_2] = B_2(1) = \int_0^{\infty} \frac{\mathrm{e}^{-t}}{1-\lambda t} \mathrm{d}t. \tag{2.119}$$

In the first case the integral exists and $\mathcal{B}[S_1] = \lambda^{-1}\mathrm{e}^{1/\lambda}\Gamma\left(0, \frac{1}{\lambda}\right)$ where $\Gamma(a,b) = \int_b^{\infty} t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$ is the incomplete gamma function. However, the second integral hits a pole at $t = 1/\lambda$. There is no natural prescription for avoiding the pole, which leads to an ambiguity in the sum of $\pm\pi i\lambda^{-1}\mathrm{e}^{-1/\lambda}$. This is a non-perturbative ambiguity called a *renormalon* [96]. In every known case where this arises in field theory the renormalon is connected to a non-perturbative finite action solution of the field equations, i.e. an instanton or soliton, and a correct evaluation of the path integral which sums over *all* saddle points (not just perturbative ones) removes the ambiguity. Key to the practical application of Borel summation is the location and classification of all renormalons in a given theory [93]. Sophisticated techniques have been developed to deal with this situation which are beyond the scope of this thesis [95, 97, 98].

2.5.4 Padé approximation and Borel-Padé resummation

Borel summation is limited in its usefulness because one often only knows a few low order terms of perturbation theory (as well as the possibility of renormalons). Padé approximation is a technique which often improves perturbation series and is far more useful in practice, and can be combined with Borel summation. Padé approximants are rational polynomials which generally converge rapidly to the function being fitted, are useful for numerical computation, and help to estimate the location of singularities of the function in the complex plane. Many software packages include standard routines for evaluating Padé approximants, for instance the PadeApproximant function in Mathematica [99]. This section applies Padé approximants to both the Green function and its Borel transform, finding that the latter approach is clearly the better one.

The (N, M)-Padé approximant of a function $\sum_{n=0}^{\infty} a_n x^n$ is given by [14]

$$P_M^N(x) = \frac{\sum_{n=0}^{N} A_n x^n}{\sum_{n=0}^{M} B_n x^n}, \tag{2.120}$$

where, without loss of generality, one takes $B_0 = 1$. The remaining $N + M + 1$ coefficients are chosen so that the Taylor series of $P_M^N(x)$ matches the perturbation series up to $\mathcal{O}\left(x^{N+M}\right)$. Due to the denominator, Padé approximants develop poles in the complex

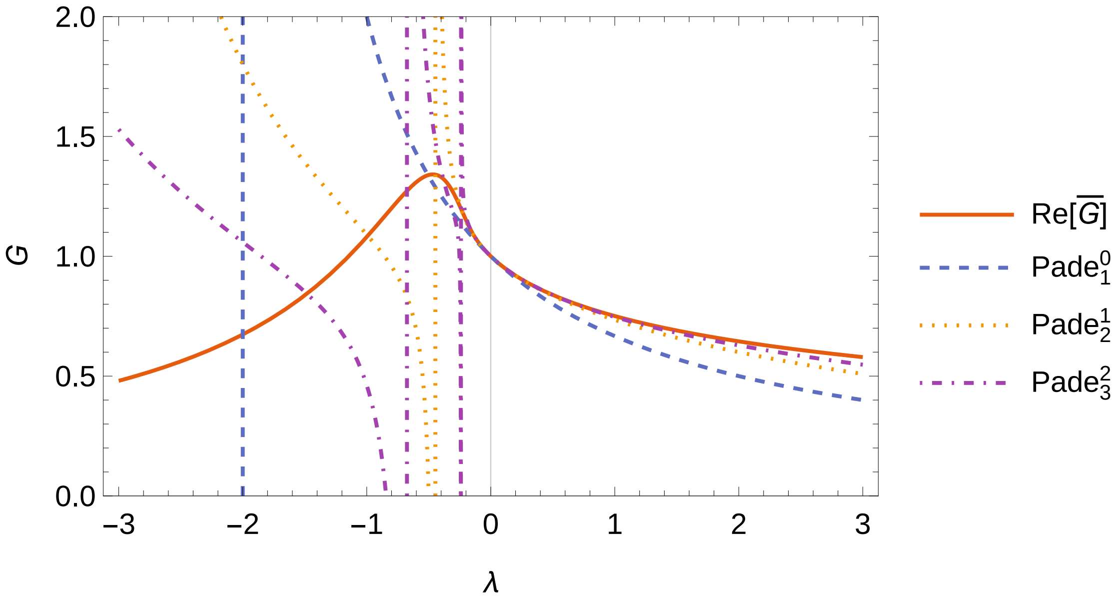


Figure 2.6: First three Padé approximants to $\bar{G}$ with $m = 1$. Note the simple poles developed for $\lambda < 0$.

x-plane, allowing the close approximation of more singular functions than Taylor series are capable of. Examples of the use of Padé approximants in field theory can be found in [94, 100] and references therein.

To find the approximants to $m^2\bar{G}$, with $x = \lambda/m^4$, one must restrict attention to the approximants where $M = N + 1$. This guarantees that $P^N_M \to 0$ as $\lambda \to \infty$. The usual diagonal approximants ($N = M$) would give an unphysical constant $P^N_N \to A_N/B_N \neq 0$ as $\lambda \to \infty$. Note that it is impossible to match the true $1/\sqrt{\lambda}$ behaviour of $\bar{G}$ as $\lambda \to \infty$ using Padé approximants centred on the origin. The best that is possible in this limit is $\sim \lambda^{-1}$. (As it happens, using $x \propto \sqrt{\lambda}$ does not allow one to resolve this issue: the same approximants are found only with x^2 everywhere in place of x. This is because the $1/\sqrt{\lambda}$ behaviour is due to the branch point at infinity, which is infinitely far from the origin where the Padé approximants are matched to perturbation theory. Low order Padé approximants can extract information about the branch point near the origin, but evidently not the one at infinity.) The first five approximants for $m^2\bar{G}$ are shown in Table 2.1 and the first three are plotted in Figure 2.6 with comparison to the exact $\bar{G}$. Note that the existence of the integral representation (2.109) for $\bar{G}$ implies that $\bar{G}$ is a *Stieltjes* function, meaning one can prove convergence properties for the Padé approximants as $N, M \to \infty$, though this analysis is not presented here (see [14] for examples in other contexts).

The $(N, N+1)$-Padé approximant can also be written as

$$P^N_{N+1} = \sum_{i=0}^{N} \frac{r_i}{\lambda - p_i}, \tag{2.121}$$

where r_i and p_i are the i-th residue and pole respectively. Note that since all of the coefficients in the denominators of Table 2.1 are positive and real, all of the poles must

Table 2.1: First few Padé approximants for $m^2\bar{G}$.

N	P^N_{N+1}
0	$\dfrac{1}{1+\frac{\lambda}{2m^4}}$
1	$\dfrac{1+\frac{2\lambda}{m^4}}{1+\frac{5\lambda}{2m^4}+\frac{7\lambda^2}{12m^8}}$
2	$\dfrac{1+\frac{16\lambda}{3m^4}+\frac{59\lambda^2}{12m^8}}{1+\frac{35\lambda}{6m^4}+\frac{43\lambda^2}{6m^8}+\frac{77\lambda^3}{72m^{12}}}$
3	$\dfrac{1+\frac{10\lambda}{m^4}+\frac{155\lambda^2}{6m^8}+\frac{15\lambda^3}{m^{12}}}{1+\frac{21\lambda}{2m^4}+\frac{365\lambda^2}{12m^8}+\frac{295\lambda^3}{12m^{12}}+\frac{385\lambda^4}{144m^{16}}}$
4	$\dfrac{1+\frac{16\lambda}{m^4}+\frac{315\lambda^2}{4m^8}+\frac{1190\lambda^3}{9m^{12}}+\frac{7945\lambda^4}{144m^{16}}}{1+\frac{33\lambda}{2m^4}+\frac{259\lambda^2}{3m^8}+\frac{11935\lambda^3}{72m^{12}}+\frac{14315\lambda^4}{144m^{16}}+\frac{7315\lambda^5}{864m^{20}}}$

either be on the negative real axis or they must be complex and come in complex conjugate pairs. Numerical experiments suggest that all the poles lie on the negative λ axis, though the author is not aware of a proof of this for all N. Assuming this is generally true, Padé approximants give a representation of $\bar{G}$ which approximates the continuous spectral function by a sum of delta functions

$$\sigma\left(s\right) \approx \sigma_N\left(s\right) \equiv \sum_{i=0}^{N} r_i \delta\left(s+p_i\right). \tag{2.122}$$

As $N \to \infty$ the poles become denser and fill the negative λ axis, eventually merging into a continuous branch cut. Similarly the spectral function turns into a dense sum of delta functions which, when considered acting on any sufficiently smooth test function, smooths into a continuous function. The first few σ_N are shown next to the exact spectral function in Figure 2.7 for comparison.

Now consider Padé approximation of the Borel transform of $\bar{G}$. This is aided by noting the following connections between the Borel transform ϕ and $\bar{G}$ and σ:

$$v^{-1}\sigma\left(\frac{\lambda}{v}\right) \xleftarrow[x\to v]{\mathcal{L}^{-1}} \phi\left(x\right) \xrightarrow[x\to s]{\mathcal{L}} s^{-1}\bar{G}\left(\frac{\lambda}{s}\right). \tag{2.123}$$

That is, the Green function is related to the Laplace transform of the Borel transform, while the spectral function is related to the inverse Laplace transform of the Borel transform. These relations can be shown using the definitions of ϕ and σ and the integral representation of the (inverse) Laplace transform. This allows one to extract the spectral function directly from the Borel transform. Note that each pole of ϕ yields by the inverse Laplace transform a term of the form $\lambda^{-1}\exp\left(-k/\lambda\right)$ in σ, where k is controlled by the location of the pole. The general Borel-Padé approximation for σ is a superposition of terms of this form.

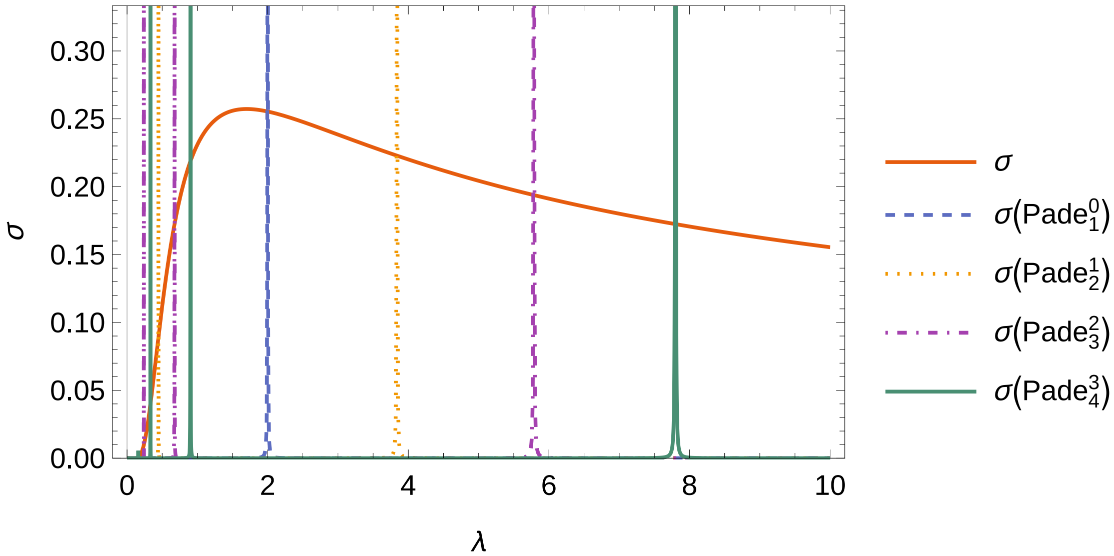


Figure 2.7: Padé approximations to the spectral function $\sigma(\lambda)$ (for $m = 1$) consist of an increasingly dense set of delta functions. (Note the delta functions have been smoothed for visual purposes only.)

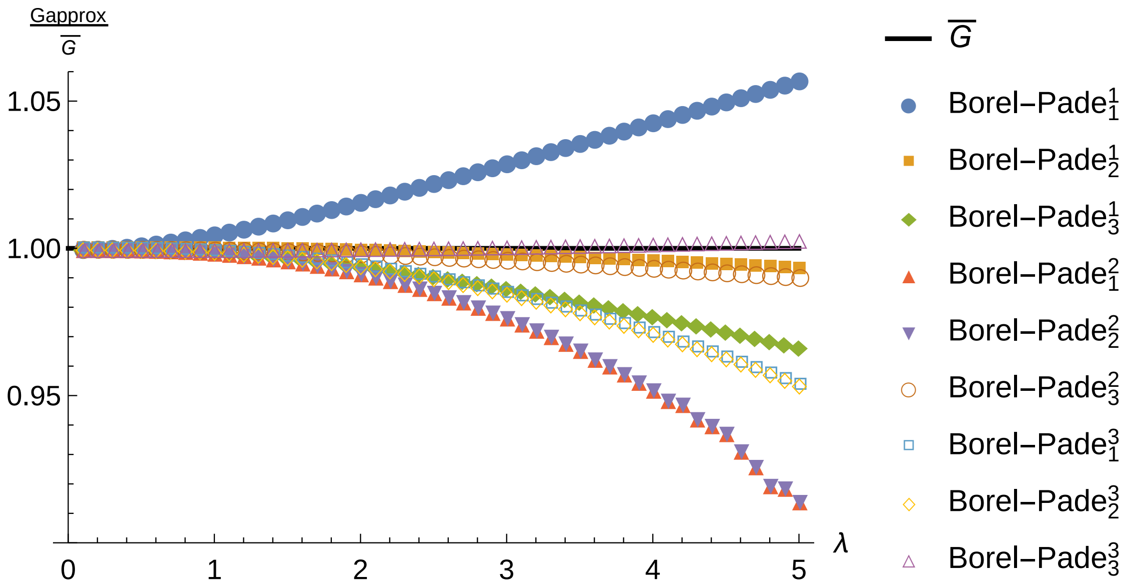


Figure 2.8: Borel-Padé approximations to the Green function $\bar{G}(\lambda)$ (ratio of approximant / exact) for $m = 1$. Approximants are calculated by numerical integration of (2.113).

The first few approximants to $\bar{G}$ and σ are shown in Figures 2.8 and 2.9 respectively. The low order Borel-Padé Green functions are reasonably accurate (within a few percent for $\lambda \leq 5m^4$) but the spectral functions are not approximated particularly well. Certain approximants to σ oscillate erratically and even become negative for certain values of λ. Except for the fact that the best approximant is the highest order one plotted, there is no clear sense in which the Borel-Padé approximations appear to converge to σ. However, even this bad approximation is at least a continuous function, as opposed to a sum of delta functions.

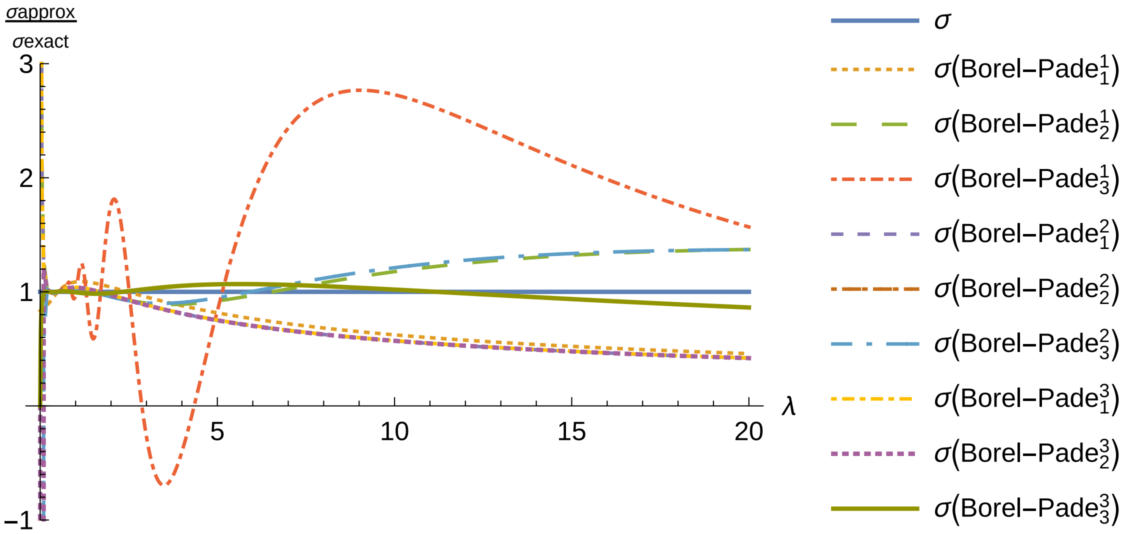


Figure 2.9: Borel-Padé approximations to the spectral function $\sigma(\lambda)$ (ratio of approximant / exact) for $m = 1$. Note that the (2,2) approximant is off scale.

2.5.5 2PI results

The 2PI effective action in the toy model is

$$\Gamma[G] = W[K] - K\partial_K W[K], \tag{2.124}$$

where K is solved for in terms of G. $\Gamma[G]$ obeys the equation of motion

$$\partial_G \Gamma[G] = -\frac{1}{2}K. \tag{2.125}$$

Explicitly,

$$\Gamma[G] = \frac{1}{2}\ln\left(G^{-1}\right) + \frac{1}{2}m^2 G + \gamma_{2\mathrm{PI}} + \mathrm{const.}, \tag{2.126}$$

where $-\gamma_{2\mathrm{PI}}$ is (minus due to Euclidean conventions) the sum of two particle irreducible vacuum graphs, i.e. those graphs which do not fall apart when any two lines are cut, where the lines are given by G and vertices by $-\lambda$. The equation of motion in the absence of sources is

$$G^{-1} = m^2 + 2\partial_G \gamma_{2\mathrm{PI}}, \tag{2.127}$$

which is Dyson's equation where the second term on the right hand side, $-\Sigma = 2\partial_G\gamma_{2\mathrm{PI}}$, represents the exact one-particle irreducible self-energy of the propagator G.

By power counting (or counting line ends in the corresponding diagrams), $\gamma_{2\mathrm{PI}} = \sum_{n=0}^{\infty}\gamma_n\left(\lambda G^2\right)^n$. It is possible to derive the γ_n by considering the symmetry factors of the two particle irreducible Feynman diagrams, but they can also be obtained using knowledge of the exact solution $G = \bar{G}$. Substituting $\bar{G}$ into the equation of motion and the expansion for $\gamma_{2\mathrm{PI}}$, expanding about $\lambda = 0$ and matching powers of λ, one finds

$$\begin{aligned}\gamma_{2\mathrm{PI}} = & \frac{G^2\lambda}{8} - \frac{G^4\lambda^2}{48} + \frac{G^6\lambda^3}{48} - \frac{5G^8\lambda^4}{128} + \frac{101G^{10}\lambda^5}{960} - \frac{93G^{12}\lambda^6}{256} \\ & + \frac{8143G^{14}\lambda^7}{5376} - \frac{271217G^{16}\lambda^8}{36864} + \frac{374755G^{18}\lambda^9}{9216} - \frac{5151939G^{20}\lambda^{10}}{20480} \\ & + \frac{697775057G^{22}\lambda^{11}}{405504} - \frac{3802117511G^{24}\lambda^{12}}{294912} + \frac{201268707239G^{26}\lambda^{13}}{1916928} \\ & - \frac{11440081763125G^{28}\lambda^{14}}{12386304} + \frac{5148422676667G^{30}\lambda^{15}}{589824} - \frac{1665014342007385G^{32}\lambda^{16}}{18874368} \\ & + \frac{4231429245358235G^{34}\lambda^{17}}{4456448} - \frac{92113806767869739 5G^{36}\lambda^{18}}{84934656} + \mathcal{O}\left(\lambda^{19}G^{38}\right). \quad (2.128)\end{aligned}$$

The author does not know of any closed form expression for either the coefficients of this series or its sum (implicit analytic expressions can be derived; however, these require the inversion of $G = \bar{G}(\lambda)$ for $\lambda(G)$, which is not known in closed form). There is a recursion formula for the coefficients [101] which, however, will not be needed here. After the first few terms the coefficients are well approximated by $\gamma_{i+1} \sim -\frac{2}{3}i\gamma_i$, the same as for the perturbation series. This has the hallmark of an asymptotic series and, like perturbation theory, the 2PI series does not converge.

The first non-trivial contribution to the equation of motion gives

$$G_{(1)}^{-1} = m^2 + \frac{\lambda}{2}G_{(1)}, \tag{2.129}$$

where the subscript (1) indicates terms of order $\mathcal{O}\left(\lambda^1\right)$ have been kept. This has two solutions

$$G_{(1)} = \frac{-m^2 \pm \sqrt{m^4 + 2\lambda}}{\lambda}. \tag{2.130}$$

One of these solutions is unphysical and the $+$ sign must be chosen. As $\lambda \to 0$,

$$G_{(1)} \to \frac{1}{m^2} - \frac{\lambda}{2m^6} + \frac{\lambda^2}{2m^{10}} + \mathcal{O}\left(\lambda^3\right), \tag{2.131}$$

which matches perturbation theory up to $\mathcal{O}\left(\lambda^2\right)$ terms as expected. However, unlike perturbation theory, the strong coupling limit $\lambda \to \infty$ exists and gives

$$G_{(1)} \to \sqrt{\frac{2}{\lambda}} - \frac{m^2}{\lambda} + \frac{m^4}{(2\lambda)^{3/2}} + \mathcal{O}\left(\frac{1}{\lambda^{5/2}}\right). \tag{2.132}$$

This series has the correct form in powers of $\lambda^{-1/2}$, though the leading coefficient is incorrect by $\approx 15\%$, and the sub-leading coefficients are incorrect by $\approx 40\%$, 70% and 100% etc. This level of accuracy is remarkable considering the simple nature of the approximation and the fact that $\gamma_{2\mathrm{PI}}$ was truncated at leading order in λ! $G_{(1)}$ is a much more uniform approximation to $\bar{G}$ than the perturbative approximation $m^{-2} - \lambda/2m^6$.

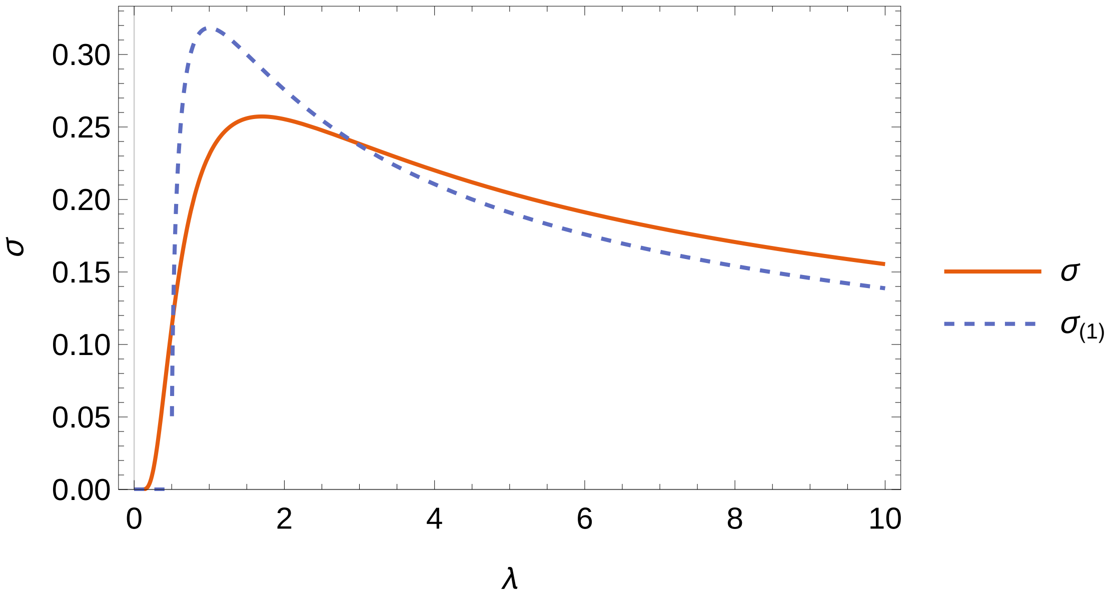


Figure 2.10: Comparison of exact spectral function $\sigma(\lambda)$ (2.110) with the two loop 2PI approximation $\sigma_{(1)}(\lambda)$ (2.134) for $m = 1$. Already, the simplest non-trivial 2PI truncation gives a much better approximation than perturbation theory or Padé approximants to arbitrary order.

This uniformity is possible because of the branch cut $G_{(1)}$ possesses on the negative λ axis. The discontinuity across the cut

$$G_{(1)}(\lambda + i\epsilon) - G_{(1)}(\lambda - i\epsilon) = 2i\frac{\sqrt{-m^4 - 2\lambda}}{\lambda}\Theta\left(-\lambda - \frac{m^4}{2}\right), \tag{2.133}$$

gives the spectral function

$$\sigma_{(1)}(\lambda) = \frac{\sqrt{-m^4 + 2\lambda}}{\pi\lambda}\Theta\left(\lambda - \frac{m^4}{2}\right), \tag{2.134}$$

which is a far better approximation to the exact spectral function than obtained from any of the other techniques, as can be seen from Figure 2.10.

All of the first order approximations are shown in Figure 2.11. Note that for $\lambda/m^4 \geq 0$ the best approximation is the 2PI, followed by the Borel-Padé, then by Padé, then perturbation theory last of all. The situation for negative λ is complicated. The best approximation overall is the 2PI, though it has an unphysical cusp where the branch cut starts ($\lambda/m^4 = -1/2$ in Figure 2.11). It appears the 2PI approximation trades sensitivity to the exponentially small portion of σ in exchange for a better global approximation. The Padé approximation is good at small negative λ but hits a pole at $\lambda/m^4 = -2$ and there loses all validity. The Borel-Padé approximation is smooth and more accurate than the 2PI near the peak of $\bar{G}$, but eventually becomes invalid, even negative, at sufficiently large negative λ ($\lambda/m^4 \lesssim -5.37$).

At n-th order, (2.127) is a degree $2n$ polynomial in G which has $2n$ roots, only one of which is physical. For $n = 2$ there are analytical expressions for the roots, though they are very bulky, and for $n \geq 3$ (2.127) must be solved numerically. Picking out the correct

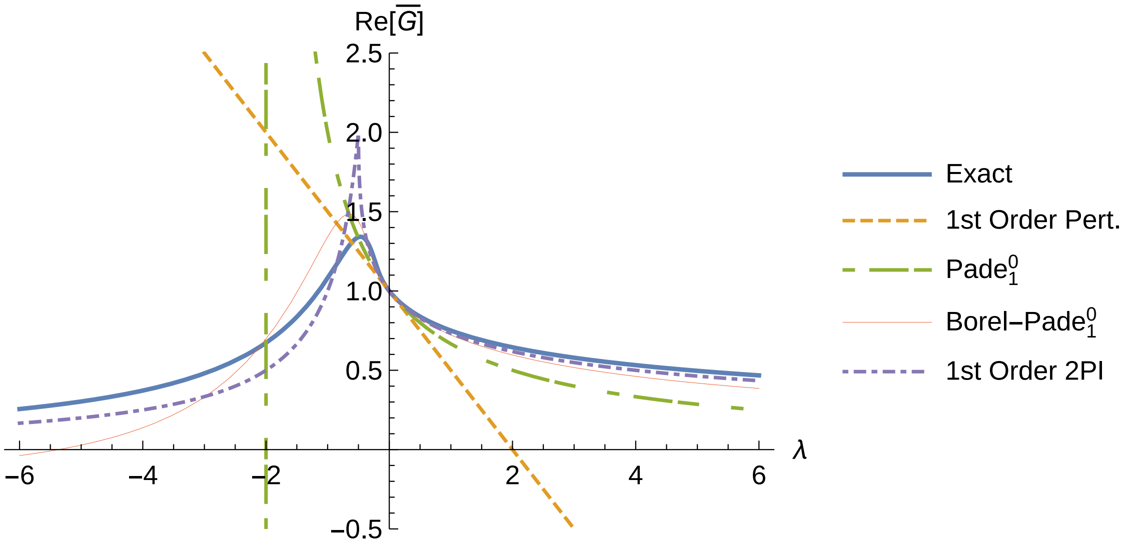


Figure 2.11: Comparison of (the real part of) the exact Green function $\bar{G}$ to each of the approximations discussed to first non-trivial order in each case, for $m = 1$.

root for a given value of λ is tricky in general and not pursued further here. However, one can see on general grounds that truncations of (2.127) give a singular perturbation theory in λ: at $\lambda = 0$ the physical solution starts at G_0 and the spurious solutions flow in from infinity as inverse powers of λ as λ increases to finite values.[8] At strong coupling the roots generically approach each other and one must carefully track them through the complex plane. The possibility that a resummation of the 2PI series would remove some or all of these spurious solutions is examined using two potential methods in the next section with mixed results.

2.5.6 Hybrid 2PI-Padé

Since the series of 2PI diagrams is asymptotic, one may use a series summation method to improve convergence. Note that this is logically independent of the resummations embodied in the 2PI approximation itself. This section studies the use of Padé summation of the action term $\gamma_{2\mathrm{PI}}$ or the equation of motion term $\partial_G \gamma_{2\mathrm{PI}}$. First, consider Padé summation of the action which matches the series expansion up to order $N + M$:

$$\Gamma[G] = \frac{1}{2} \ln\left(G^{-1}\right) + \frac{1}{2} m^2 G + \frac{\sum_{n=0}^{N} A_n^{(\gamma)} \left(\lambda G^2\right)^n}{\sum_{n=0}^{M} B_n^{(\gamma)} \left(\lambda G^2\right)^n} + \text{const.} \tag{2.135}$$

[8] Also note that at least one of the spurious solutions (and always an odd number of them) are real since the coefficients in (2.127) are real and complex solutions must occur in conjugate pairs.

The equation of motion becomes

$$
\begin{aligned}
G^{-1} &= m^2 + 2\frac{\mathrm{d}}{\mathrm{d}G}\frac{\sum_{n=0}^{N} A_n^{(\gamma)}\left(\lambda G^2\right)^n}{\sum_{n=0}^{M} B_n^{(\gamma)}\left(\lambda G^2\right)^n} \\
&= m^2 + \frac{4\sum_{n=0}^{N}\sum_{k=0}^{M}(n-k)\,A_n^{(\gamma)}B_k^{(\gamma)}\lambda^{n+k}G^{2(n+k)-1}}{\left[\sum_{n=0}^{M} B_n^{(\gamma)}\left(\lambda G^2\right)^n\right]^2}.
\end{aligned} \tag{2.136}
$$

Multiplying by the denominator this becomes a polynomial equation of degree $2\,(N+M)$ as expected. This equation will have $2\,(N+M)-1$ spurious solutions as does the ordinary 2PI approximation of matching order. To take a specific example, consider γ_{2PI} up to $\mathcal{O}\left(\lambda^4\right)$ and the matching $(2,2)$- and $(1,3)$-Padé approximants:

$$
\gamma_{\text{2PI}} = \frac{G^2\lambda}{8} - \frac{G^4\lambda^2}{48} + \frac{G^6\lambda^3}{48} - \frac{5G^8\lambda^4}{128} + \cdots \tag{2.137}
$$

$$
\approx \frac{\frac{G^2\lambda}{8} + \frac{113G^4\lambda^2}{480}}{1 + \frac{41G^2\lambda}{20} + \frac{7G^4\lambda^2}{40}} \tag{2.138}
$$

$$
\approx \frac{G^2\lambda}{8\left(1 + \frac{G^2\lambda}{6} - \frac{5G^4\lambda^2}{36} + \frac{113G^6\lambda^3}{432}\right)}. \tag{2.139}
$$

The solution resulting from any of these versions of γ_{2PI} cannot be written in closed form, however numerical solutions are shown in Figure 2.12 for moderately small λ (for λ outside of this range, different roots become relevant). The hybrid solutions are more accurate than the fourth order 2PI solution, although the level of improvement depends strongly on the type of Padé approximant employed. The trade-off is a loss of accuracy at large λ, as shown in Figure 2.13 for the best roots found. Each approximation decays with the correct leading $\lambda^{-1/2}$ power, however they differ by $\mathcal{O}\left(1\right)$ factors which are not negligible. Remarkably, the best approximation of those shown at large λ is the $\mathcal{O}\left(\lambda\right)$ 2PI approximation. For the greater computational expense the $(2,2)$-hybrid approximation is not noticeably an improvement at large coupling. The apparent clustering of the $\mathcal{O}\left(\lambda^4\right)$ 2PI and $(1,3)$-hybrid approximations away from the exact value as $\lambda \to \infty$ stems from the minus sign of the leading term in $\partial_G\gamma_{\text{2PI}}$ in these approximations, suggesting that one should not use approximations with this property.

Next try Padé summing the equation of motion:

$$
G^{-1} = m^2 + 2\frac{G^{-1}\sum_{n=1}^{N} A_n^{(\partial\gamma)}\left(\lambda G^2\right)^n}{\sum_{n=0}^{M} B_n^{(\partial\gamma)}\left(\lambda G^2\right)^n}, \tag{2.140}
$$

where the explicit factor of G^{-1} on the right hand side simply ensures the self-energy is an odd function of G as it must be. This equation of motion is of degree $\max\left(2N, 2M+1\right)$, which for typical choices of N and M results in a rough halving of the number of spurious solutions. To obtain an analytic result consider the first non-trivial approximant, i.e.

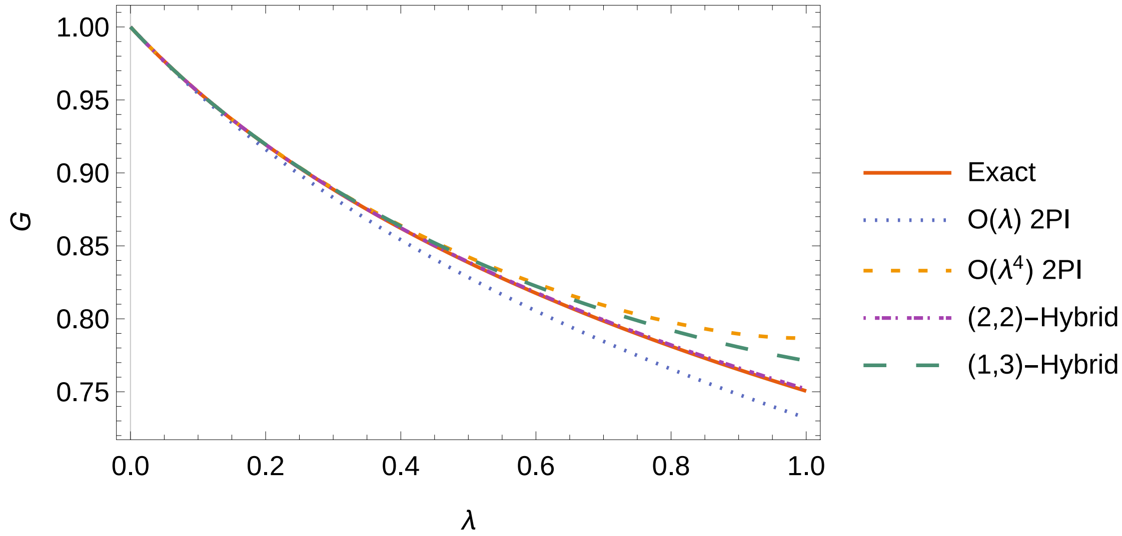


Figure 2.12: Comparison of exact $\bar{G}$, first and fourth order 2PI and $(2,2)$- and $(1,3)$-2PI-Padé hybrid solutions for $m = 1$ at small coupling. The $(2,2)$-hybrid solution lies almost on top of the exact solution.

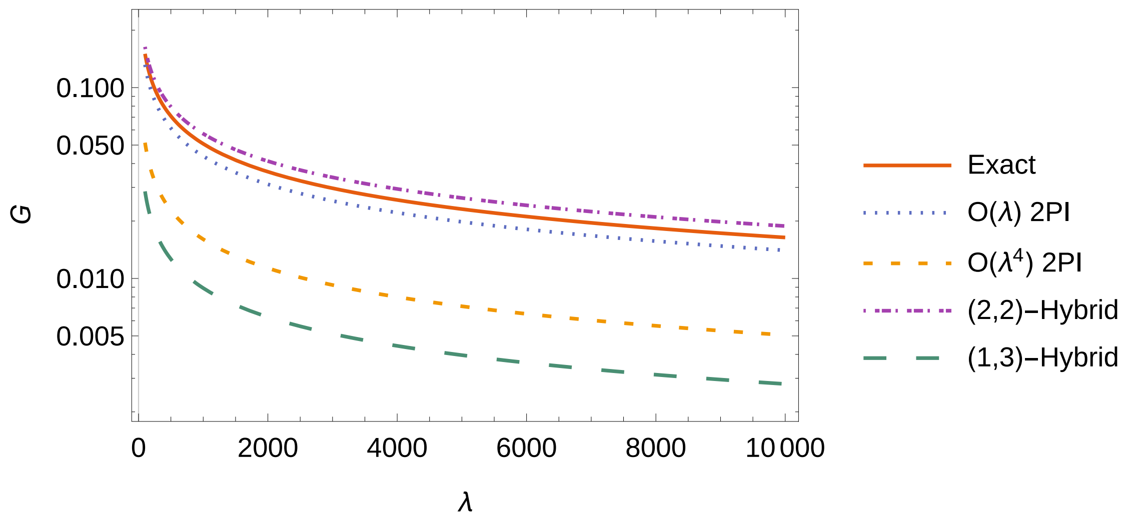


Figure 2.13: Comparison of exact $\bar{G}$, first and fourth order 2PI and $(2,2)$- and $(1,3)$-2PI-Padé hybrid solutions at large coupling for $m = 1$.

$N = M = 1$. The resulting equation of motion is

$$G^{-1} = m^2 + \frac{1}{2}\frac{\lambda G}{1 + \frac{1}{3}\lambda G^2}, \tag{2.141}$$

which agrees with the usual 2PI equation of motion up to terms of order $\mathcal{O}\left(\lambda^3\right)$, however it only has two spurious solutions instead of three. The physical solution obtained by matching at small λ is

$$G_{\text{hy}} = \frac{1}{6m^2}\left(-1 + \frac{36m^4 - \lambda}{X^{1/3}} - \frac{X^{1/3}}{\lambda}\right), \tag{2.142}$$

where

$$X = \lambda^3 - 378\lambda^2 m^4 + 18\lambda m^2\sqrt{\lambda\left(-2\lambda^2 + 144m^8 + 429\lambda m^4\right)}.$$

Indeed, this matches perturbation theory up to $\mathcal{O}\left(\lambda^3\right)$ terms. However, the large λ behaviour in this approximation is pathological:

$$G_{\text{hy}} \to -\frac{1}{2m^2} + \frac{18m^2}{\lambda} + \mathcal{O}\left(\lambda^{-3/2}\right), \tag{2.143}$$

as $\lambda \to \infty$. This reflects the existence of an unphysical branch cut on the positive λ axis starting at $\lambda/m^4 = \frac{3}{4}\left(143 + 19\sqrt{57}\right) \approx 214$, which G_{hy} has in addition to the expected cuts on the negative axis. The existence of this cut also renders the derivation of the spectral function invalid, meaning that G_{hy} cannot be written in the form (2.109). It is not known at this stage whether other forms of Padé summed equations of motion lack these pathologies. Further development of hybrid approximation schemes is beside the main line of this thesis but may be a worthy topic for future work.

2.6 Summary

This chapter has investigated n-particle irreducible effective actions (nPIEA) in quantum field theory. Green functions can be computed in a perturbation expansion in terms of Feynman diagrams or can be efficiently computed using the nPIEA, which were investigated and explicit derivations given for $n = 1, 2$ and 3 in the case of a generic quartic scalar field theory. Finally, to illustrate the advantages of the nPIEA method over resummation methods, the 2PIEA was applied to an exactly solvable toy model and contrasted with Borel, Padé and Borel-Padé summation.

The study of the toy model revealed several interesting features. The perturbation theory has zero radius of convergence due to a branch cut on the negative coupling (λ) axis, a fact which is invisible to perturbation theory. The theory is Borel summable, with the Borel sum giving the exact answer. However, Borel summation alone is not usually very helpful in practice. Padé approximants well describe the Green function at weak coupling, though not at strong coupling. However, the combination of Borel and Padé approximation yields an effective global approximation scheme for the Green function. The two point Green function of this theory admits a nice integral representation in terms of the spectral function (i.e. discontinuity of the branch cut) which shows that the Padé approximants improve perturbation theory by allowing the spectral function to be approximated as a sum of delta functions and the Borel-Padé method gives a continuous, albeit erratic and inaccurate, approximation to the spectral function. The 2PI approximation scheme surpasses perturbation theory, Padé and Borel-Padé approximants already at the leading non-trivial truncation. Like the Borel-Padé method, 2PI approximations can develop branch points and represent the spectral function by a continuous distribution. However, the 2PI approximation is quantitatively superior at the leading truncation. These results are not entirely surprising because while, e.g., the Borel-Padé method is a widely applicable general "black box" method, the self-consistent 2PI equations of motion are derived within a *particular* field theory of interest. This gives the 2PI method "insider information," from which it should be able to construct a better approximation. This comes at the cost of spurious solutions which must be eliminated and the added difficulty of finding the 2PI effective action in the first place. Finally, a hybrid 2PI-Padé scheme was introduced using Padé approximants to partially resum the 2PI diagram series. The quality of the result depends strongly on the type of Padé approximant used, with the best result found for the diagonal approximant. This hybrid approximation performs considerably better than the comparable 2PI approximation at weak coupling, though not noticeably better at strong coupling.

nPIEAs are the subject of a rich literature and have found applications in diverse areas from early universe cosmology to nano-electronics (e.g. [17, 64]). The virtues of approximation schemes built on nPI effective actions are often explained in terms of a resummation of an infinite series of perturbative Feynman diagrams which are encapsulated in the non-perturbative Green function Δ and proper vertex functions $V, \cdots, V^{(n)}$, from which the nPI diagram series is built. However, nPIEAs are not resummation schemes: they are a self-consistent variational principle. The definition of the nPIEA in terms of the Legendre transform is crucial for the self-consistency of the scheme. The immediate practical consequence is that any modification of the nPIEA which does not derive from a consistent modified variational principle is very likely to be inconsistent. So, for instance, the consistency of recent attempts to improve the symmetry properties of 2PIEAs [1] is guaranteed by the existence of a suitable constrained variational principle, however, ad hoc attempts to modify the equations of motion to satisfy symmetries will fail.

From the nPIEA one obtains a set of coupled integro-differential equations of motion for the 1- through n-point correlation functions which are similar in spirit to the Schwinger-Dyson or BBGKY equations. However, the nPI equations of motion are automatically closed and require no further approximation than a truncation of the diagram series to be used in practice. Further, time evolution of the equations of motion is well behaved compared to other schemes. The reason for this is that the non-linear nature of the equations of motion eliminates the secularities present in perturbation theory (see, e.g. [16, 17]). This gives nPIEAs an appealing degree of mechanisation and suitability especially for non-equilibrium time evolution problems. There are two types of outstanding difficulties however, both due to the non-linear nature of the equations of motion. The first is the renormalisation procedure, which has not been discussed in detail here. A renormalisation procedure based on a Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ) type of analysis will be discussed in Chapter 4 where it is applied to the symmetry improved 3PIEA. The renormalisation of nPIEA remains an open problem in general, however. The second problem is violation of symmetries by truncations of nPIEA. Discussion of this problem and attempts at its resolution form the bulk of the rest of this thesis.

Chapter 3

Symmetries, Ward Identities and Truncations in nPIEA

3.1 Synopsis

The previous chapter introduced nPIEA as a technique suitable for studies in non-perturbative QFT. This chapter explores the symmetry properties of nPIEA using the 1PIEA and 2PIEA as specific examples. These cases cover all of the conceptual points required for general nPIEA, with mere increases in technical complexity for $n > 2$. This chapter demonstrates that global symmetries are not generically preserved by truncations of nPIEA. The causes of this can be understood in terms of the difference between 1PI and 2PI (resp. nPI) Ward identities. It can also be understood in terms of the resummation of perturbative Feynman diagrams: when an nPIEA is truncated some subset of perturbative diagrams is summed to infinite order, but the complementary subset is left out entirely. The pattern of resummations does not guarantee that the cancellations between perturbative diagrams needed to maintain the symmetry are kept. In the case of scalar field theories with $\mathrm{O}(N) \to \mathrm{O}(N-1)$ breaking, the result is that the final $\mathrm{O}(N-1)$ symmetry is maintained, but, at the Hartree-Fock level of approximation, the non-linearly realised $\mathrm{O}(N)/\mathrm{O}(N-1)$ is lost, the Goldstone theorem is violated (the $N-1$ Goldstone bosons are massive), and the symmetry restoration phase transition is first order in contradiction with the second order transition expected on the basis of universality arguments. A similar problem arises in gauge theories, where the violation of gauge invariance in the l-loop truncation is due to the missing $(l+1)$-loop diagrams (see, e.g. [102–105] for a discussion of the gauge fixing problem). These results are known in the literature, but the demonstration given here that the 2PI Ward identity is incompatible with the corresponding 1PI Ward identity is (to the author's knowledge) new and shows that this result is independent of the renormalisation scheme used. Here the 2PIEA for the $\mathrm{O}(N)$ scalar field theory is renormalised and solved in the Hartree-Fock approximation at finite temperature, reproducing independently results known in the literature. The solution is then contrasted with the so-called "external propagator" and "large N" methods which have been advocated because they solve some of the phenomenological problems, though not entirely satisfactorily. This sets the stage for the examination of symmetry improvement techniques in the remainder of the thesis.

3.2 Global symmetries in the 1PIEA

Consider the action (2.53) for a generic ϕ^4 scalar field theory with N fields $\phi_1, \cdots, \phi_N$, reproduced here:

$$S[\phi] = \frac{1}{2}\phi_a\left(-\Box\delta_{ab} - m^2_{ab}\right)\phi_b - \frac{1}{3!}g_{abc}\phi_a\phi_b\phi_c - \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d. \tag{3.1}$$

Without loss of generality m^2_{ab}, g_{abc} and λ_{abcd} are all real and totally symmetric in their indices. It is assumed also that m^2_{ab}, g_{abc} and λ_{abcd} are all local (i.e. proportional to appropriate delta functions) and homogeneous in spacetime. It is possible in principle to relax this restriction, but this only adds unnecessary complication for the present discussion.

The derivative term has an $\mathrm{O}(N)$ symmetry under rotations of the fields $\phi_a \to R_{ab}\phi_b$, where R is a rotation (i.e. orthogonal) matrix obeying $R^T = R^{-1}$. In infinitesimal form the shift of ϕ_a can be written

$$\delta\phi_a = iT_{ab}\phi_b = i\epsilon_A T^A_{ab}\phi_b, \tag{3.2}$$

where $T_{ab} = -T_{ba}$ is a generator of rotations which in the second equality has been expanded in a basis T^A_{ab} where $A = 1, \cdots, N(N-1)/2$ runs over the linearly independent generators. When an explicit basis of generators is required one can take $T^{jk}_{ab} = i\left(\delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}\right)$ where $A = (j,k)$ is thought of as an (antisymmetric) multi-index. Note that the implicit integration convention can be maintained if $T^A_{ab}(x,y) \propto \delta(x-y)$ contains a spacetime delta function, though in this notation one must remember that the upper indices do not have corresponding spacetime arguments since they merely label the particular generator.

The mass term in general breaks the symmetry. Using rotations to diagonalise m^2_{ab} one finds that it consists of k blocks

$$m^2_{ab} = \left(\begin{array}{ccc|ccc|c|ccc}
m_1^2 & & 0 & & & & & & & \\
 & \ddots & & & 0 & & 0 & & 0 & \\
0 & & m_1^2 & & & & & & & \\
\hline
 & & & m_2^2 & & 0 & & & & \\
 & 0 & & & \ddots & & 0 & & 0 & \\
 & & & 0 & & m_2^2 & & & & \\
\hline
 & 0 & & & 0 & & \ddots & & 0 & \\
\hline
 & & & & & & & m_k^2 & & 0 \\
 & 0 & & & 0 & & 0 & & \ddots & \\
 & & & & & & & 0 & & m_k^2
\end{array}\right) \tag{3.3}$$

where each eigenvalue m_i^2 is n_i-fold degenerate. Obviously $\sum_{i=1}^k n_i = N$. One then finds that the mass term breaks the symmetry into within-block symmetries $\mathrm{O}(N) \to \mathrm{O}(n_1) \times \mathrm{O}(n_2) \times \cdots \times \mathrm{O}(n_k)$. In the case of a non-degenerate eigenvalue the corresponding group factor is $\mathrm{O}(1) \cong \mathbb{Z}_2$ for the discrete symmetry $\phi_a \to -\phi_a$.

This thesis focuses only on the continuous symmetries. Further, there is clearly nothing essential lost by focusing on a single block, since the more general case is a direct product of blocks, each block having a simple (in the sense of group theory) symmetry group. That is, there is no reason not to focus on an irreducible representation of a simple symmetry group, at least initially. The general case can be built up from there. Therefore, restrict to the case of a single block where $m^2_{ab} = m^2\delta_{ab}$. This retains the full $\mathrm{O}(N)$ symmetry. The cubic coupling transforms like a symmetric rank three tensor, but there is no non-trivial invariant tensor of that form. Thus $g_{abc} = 0$ to maintain the symmetry. Further, the quartic coupling transforms like a symmetric rank four tensor, but the only possibility is $\delta_{ab}\delta_{cd}$ + permutations. Thus, without loss of generality one can consider the general $\mathrm{O}(N)$ symmetric action

$$\begin{aligned} S[\phi] &= \frac{1}{2}\phi_a\left(-\Box - m^2\right)\phi_a - \frac{\lambda}{4!}\left(\phi_a\phi_a\right)^2 && (3.4)\\ &= \frac{1}{2}\phi_a\left(-\Box\right)\phi_a - \frac{\lambda}{4!}\left(\phi_a\phi_a - v^2\right)^2, && (3.5)\end{aligned}$$

where $\lambda v^2/6 = -m^2$ and the first and second lines differ by an irrelevant constant term. For later reference the classical vertex functions for the model are

$$V_{0abc}(x,y,z) = -\frac{\lambda}{3}\left(\delta_{ab}\varphi_c + \delta_{ca}\varphi_b + \delta_{bc}\varphi_a\right)\delta(x-y)\,\delta(x-z), \tag{3.6}$$

$$W_{abcd}(x,y,z,w) = -\frac{\lambda}{3}\left(\delta_{ab}\delta_{cd} + \delta_{ca}\delta_{bd} + \delta_{bc}\delta_{ad}\right)\delta(x-y)\,\delta(x-z)\,\delta(x-w). \tag{3.7}$$

Consider now the partition functional

$$Z[J] = \int \mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}\left(S[\phi] + J_a\phi_a\right)\right], \tag{3.8}$$

and make the change of integration variables (3.2). As discussed above, the classical action is invariant. Also the functional measure $\mathcal{D}[\phi]$ is invariant[1]. Thus the partition functional changes by

$$\delta Z = \int \mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}\left(S[\phi] + J_a\phi_a\right)\right]\frac{i}{\hbar}J_b\delta\phi_b, \tag{3.9}$$

but since this is simply a change of integration variables $\delta Z = 0$. Thus

$$0 = J_b T^A_{bc}\frac{\delta}{\delta J_c}Z[J], \tag{3.10}$$

which must hold independent of J.

[1]Non-invariance of the functional measure arises in a number of physically important cases, none of which are relevant here: chiral transformations of fermion fields, non-linear field redefinitions, and scale and diffeomorphism transformations of spacetime (see, e.g. [18]).

The physical meaning of this can be understood by considering, for example, the quadratic term in $Z[J] = 1 + \left(\frac{i}{\hbar}\right) J_a G_a^{(1)} + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 J_a J_b G_{ab}^{(2)} + \mathcal{O}\left(J^3\right)$:

$$\begin{aligned} 0 &= J_d T_{dc}^A \frac{\delta}{\delta J_c} \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 J_a J_b G_{ab}^{(2)} \\ &= \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \left[J_d T_{da}^A J_b + J_a J_d T_{db}^A\right] G_{ab}^{(2)} \\ &\propto \left(\delta J_a J_b + J_a \delta J_b\right) G_{ab}^{(2)}, \end{aligned} \tag{3.11}$$

where the last line *defines* $\delta J_a = i\epsilon_A T_{ab}^A J_b$. The shift above is simply that produced by a rotation of J in $Z[J]$. The fact the shift vanishes implies that $G_{ab}^{(2)}$ transforms like a symmetric rank two tensor under rotations, which is obvious in hindsight considering that $G_{ab}^{(2)} = \langle \mathrm{T}_C [\phi_a \phi_b]\rangle$. A similar analysis applies to each Green function with the result that $Z[J]$ is invariant under $\mathrm{O}(N)$ rotations of J. Likewise $W[J] = -i\hbar \ln Z[J]$. The fact that $W[J]$ is invariant and that $\varphi_a = G_a^{(1)}$ and J_a both transform like vectors (i.e. that the relationship between φ_a and J_a is maintained under a common rotation) implies that $\Gamma[\varphi] = W[J] - J_a\varphi_a$ is also invariant under rotations. Note that invariance holds *off-shell*, that is, before equations of motion are imposed on φ. In other words the functional form of $\Gamma[\varphi]$ is invariant, not just its value at a solution of the equation of motion.

The invariance of $\Gamma[\varphi]$ under rotations allows one to derive a number of identities called *Ward identities* (WIs) for the proper correlation functions. Start by finding the shift of Γ under a rotation

$$\frac{\delta \Gamma}{\delta \varphi_a} i\epsilon_A T_{ab}^A \varphi_b = 0. \tag{3.12}$$

This is the *master* WI. Since this holds off-shell one can simplify it by taking derivatives with respect to φ and *then* imposing equations of motion, which are (recall (2.29))

$$\frac{\delta \Gamma}{\delta \varphi_a} = -J_a. \tag{3.13}$$

One can also extract the common factors $i\epsilon_A$ and define the identity $\mathcal{W}_{a_1\cdots a_m}^A = 0$ where $\mathcal{W}_{a_1\cdots a_m}^A$ is the left hand side of (3.12) after taking m field derivatives, applying equations of motion and extracting factors. Proceeding this way one obtains all of the WIs, the first few of which are

$$\mathcal{W}^A = -J_a T_{ab}^A \varphi_b, \tag{3.14}$$

$$\mathcal{W}_c^A = \Delta_{ca}^{-1} T_{ab}^A \varphi_b - J_a T_{ac}^A, \tag{3.15}$$

$$\mathcal{W}_{cd}^A = V_{dca} T_{ab}^A \varphi_b + \Delta_{ca}^{-1} T_{ad}^A + \Delta_{da}^{-1} T_{ac}^A, \tag{3.16}$$

$$\mathcal{W}_{cde}^A = W_{edca} T_{ab}^A \varphi_b + V_{dca} T_{ae}^A + V_{eca} T_{ad}^A + V_{eda} T_{ac}^A, \tag{3.17}$$

and so on. These are non-trivial identities relating the (full) propagator and various proper vertex functions. The most important identities for the following development are (3.15) and (3.16), though all of them are on equal footing conceptually. This thesis restricts consideration to the lowest few WIs merely for technical simplicity. The methods developed herein can be extended straightforwardly. Note that after (3.15) the WIs do not depend explicitly on the source. This is useful in the discussion of linear response for symmetry improved effective actions in Chapter 6.

The physical meaning of these identities can be best seen in simple cases. Setting the source to zero corresponds most often to the physical limit (i.e. when the external source is considered as a formal tool, not a physical quantity in and of itself). In this case the first WI is trivial and (3.15) becomes

$$0 = \Delta_{ca}^{-1} T_{ab}^{A} \varphi_b. \tag{3.18}$$

Due to translation invariance the vacuum state has φ_a = constant and $\Delta_{ca}^{-1}(x, y) = \Delta_{ca}^{-1}(x - y)$, so that (3.18) can be simplified using the Fourier transform

$$\Delta_{ca}^{-1}(x, y) = \int_p \mathrm{e}^{-ip(x-y)} \Delta_{ca}^{-1}(p), \tag{3.19}$$

which gives

$$0 = \Delta_{ca}^{-1}(p = 0)\, T_{ab}^{A} \varphi_b = -M_{ca}^2 T_{ab}^{A} \varphi_b, \tag{3.20}$$

recalling that the zero mode of the inverse propagator is (minus) the mass-squared matrix $M_{ca}^2 = M_{ac}^2$. Choosing without loss of generality a basis where $\varphi_b = (0, \cdots, 0, v) = v\delta_{bN}$ gives

$$M_{ca}^2 v = 0, \;\; a \neq N. \tag{3.21}$$

There are thus two regimes: the symmetric regime where $v = 0$, and the broken symmetry regime where $v \neq 0$ and $M_{ca}^2 = 0$ if either $c, a \neq N$ (it hold for both indices because M_{ca}^2 is symmetric). This identity is called *Goldstone's theorem* which says that "in any theory with a spontaneously broken continuous symmetry, each broken symmetry generator corresponds to a massless excitation.[2]" The massive mode is called (loosely[3]) the Higgs boson and the massless modes are called Goldstone bosons.

Under the same conditions the identity (3.16) can be simplified by introducing the Fourier transformed three point vertex function

$$V_{dca}(x, y, z) = \int_{pq} \mathrm{e}^{-ip(x-z)-iq(y-z)} V_{dca}(p, q, -p-q), \tag{3.22}$$

so that (3.16) becomes

$$0 = V_{dca}(p, -p, 0)\, T_{aN}^{A} v + \Delta_{ca}^{-1}(p)\, T_{ad}^{A} + \Delta_{da}^{-1}(p)\, T_{ac}^{A}. \tag{3.23}$$

[2]There is a caveat that this only holds in more than two spacetime dimensions. In two or fewer dimensions spontaneous breaking of continuous symmetries is impossible due to infrared divergences and $v = 0$ always. This is discussed in more detail in Chapter 4.

[3]Strictly speaking a Higgs or Brout-Englert-Higgs boson only exists in theories where the broken symmetries are gauge symmetries. (The history of the "Higgs" mechanism is famously convoluted, with important contributions coming from a number of people. See, e.g. [106–110] for the original literature.) A more accurate term for the case of a global symmetry would be "the radial mode boson" or the like, but the usage of "the Higgs" as a shorthand in this context is near universal.

Substituting in different values of c, d and A gives a slew of identities:

$$0 = \Delta^{-1}_{cg}(p), \qquad A = (g,d), d \neq N,\ c \neq g,d,N, \tag{3.24}$$

$$0 = \Delta^{-1}_{gg}(p) - \Delta^{-1}_{dd}(p), \qquad A = (g,d), c,d \neq N,\ \text{no sum on } d,g, \tag{3.25}$$

$$0 = \Delta^{-1}_{Na}(p), \qquad A = \left(g,g'\right), d = N, c = g', a = g, \tag{3.26}$$

$$0 = V_{dcg}(p,-p,0)\, v, \qquad A = (g,N), c,d \neq N, \tag{3.27}$$

$$0 = V_{NNg}(p,-p,0)\, v, \qquad A = (g,N), c = d = N, \tag{3.28}$$

$$0 = V_{Ncg}(p,-p,0)\, v - \Delta^{-1}_{NN}(p)\,\delta_{gc} + \Delta^{-1}_{cg}(p), \quad A = (g,N), d = N, c \neq N, \tag{3.29}$$

where $g, g' \neq N$ and $g \neq g'$. The first three identities are satisfied iff $\Delta^{-1}_{ab}(p)$ has the form

$$\Delta^{-1}_{ab} = \begin{cases} \Delta^{-1}_{G} & a = b \neq N, \\ \Delta^{-1}_{H} & a = b = N, \end{cases} \tag{3.30}$$

where $\Delta_{G/H}$ are the Goldstone/Higgs propagators respectively, with corresponding masses $m^2_{G/H}$. Note that the free propagators are

$$\Delta^{-1}_{0G/H} = p^2 - m^2_{G/H}. \tag{3.31}$$

The last three identities have differing effects in the symmetric and broken symmetry regimes. In the symmetric regime there is *no constraint* on V (that is, none coming from *these* WI; the higher WI will give further constraints causing $V = 0$), but the last identity requires $\Delta^{-1}_{H} = \Delta^{-1}_{G}$. In the broken symmetry regime the first two identities are satisfied by

$$V_{abc}(x,y,z) = \begin{cases} 0 & \text{odd number of indices} \neq N, \\ V_N(x,y,z) & a = b = c = N, \\ \bar{V}(x;y,z) & a = N,\ b = c \neq N, \\ & \text{permutations}, \end{cases} \tag{3.32}$$

where the subtle point ("permutations") is that the position arguments of $\bar{V}(x,y,z)$ are *not* symmetric: by convention the first argument refers to the index which equals N. The other two arguments are symmetric. The final identity requires

$$\bar{V}(p;-p,0)\, v - \Delta^{-1}_{H}(p) + \Delta^{-1}_{G}(p) = 0. \tag{3.33}$$

To understand this physically write $\bar{V}(x,y,z) = -\lambda v/3 \times \delta(x-y)\,\delta(x-z) + \delta\bar{V}$ where the first term is the tree level value coming from

$$V_{0abc} = -\frac{\lambda}{3}\left(\delta_{ab}\varphi_c + \delta_{ca}\varphi_b + \delta_{bc}\varphi_a\right), \tag{3.34}$$

and $\delta\bar{V}$ represents all loop corrections to $\bar{V}$. Then using (2.51) the WI becomes

$$-\frac{\lambda v^2}{3}+\delta\bar{V}\left(p;-p,0\right)v+m_H^2+\Sigma_H\left(p\right)-m_G^2-\Sigma_G\left(p\right)=0. \tag{3.35}$$

Matching terms of common order in $\hbar$ gives

$$m_H^2 = \frac{\lambda v^2}{3}+m_G^2, \tag{3.36}$$

$$\delta\bar{V}\left(p;-p,0\right)v = \Sigma_G\left(p\right)-\Sigma_H\left(p\right). \tag{3.37}$$

The first equation is the tree level relation between the masses of the Higgs and Goldstone bosons. The second equation is a relation between the vertex loop corrections with a soft (zero momentum) Goldstone leg and the self-energies of the Goldstone and Higgs bosons. In the symmetric regime $v=0$ and $m_H^2=m_G^2$ and $\Sigma_H=\Sigma_G$ as expected.

3.3 Global symmetries in the 2PIEA

Correlation functions derived from the 2PIEA obey different Ward identities than those derived from the 1PIEA. Since understanding the difference between the two types of WI is crucial to understanding the phenomenology of 2PI approximations in theories with global or gauge symmetries, the 2PI version of the WIs are derived for the model $\mathrm{O}\left(N\right)$ field theory above. Applying the same type of analysis as above (i.e. changing variables in the functional integral) to the 2PIEA $\Gamma\left[\varphi,\Delta\right]$ one finds that Γ is invariant and φ_a and Δ_{ab} transform as a vector and symmetric rank two tensor respectively. As before Γ is invariant off-shell.

The shift of Γ gives the 2PI version of the master Ward identity [1]

$$\frac{\delta\Gamma}{\delta\varphi_a}i\epsilon_A T^A_{ab}\varphi_b+\frac{\delta\Gamma}{\delta\Delta_{ab}}i\epsilon_A\left(T^A_{ac}\Delta_{cb}+T^A_{bc}\Delta_{ac}\right)=0. \tag{3.38}$$

The second term accounts for the additional transformation of Δ in the 2PIEA formalism. Taking derivatives with respect to φ and Δ, stripping factors of $i\epsilon_A$ and applying the 2PI equations of motion, recall (2.47)-(2.48) reproduced here for convenience

$$\frac{\delta\Gamma\left[\varphi,\Delta\right]}{\delta\varphi} = -J-K\varphi, \tag{3.39}$$

$$\frac{\delta\Gamma\left[\varphi,\Delta\right]}{\delta\Delta} = -\frac{1}{2}i\hbar K, \tag{3.40}$$

gives a set of 2PI WIs which are now complicated by the fact that there are multiple types of correlation function of the same rank appearing. For example, there are now two different two point functions

$$\frac{\delta^2\Gamma}{\delta\varphi_a\delta\varphi_b},\ \frac{\delta\Gamma}{\delta\Delta_{ab}}, \tag{3.41}$$

and three different four point functions

$$\frac{\delta^4\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c\delta\varphi_d}, \ \frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\Delta_{cd}}, \ \frac{\delta^2\Gamma}{\delta\Delta_{ab}\delta\Delta_{cd}}, \tag{3.42}$$

which are, in general truncations, all different.

To illustrate the various 1PI and 2PI functions recall the equivalence relationship between the 1PIEA and 2PIEA

$$\Gamma[\varphi] = \Gamma[\varphi, \Delta[\varphi]], \tag{3.43}$$

where $\Delta[\varphi]$ is the solution of $\delta\Gamma[\varphi,\Delta]/\delta\Delta = 0$ at fixed φ. Using this one has (recall the Dyson equation (2.51): $\Delta^{-1} = \Delta_0^{-1} - \Sigma$)

$$\frac{\delta^2\Gamma[\varphi,\Delta[\varphi]]}{\delta\varphi_a\delta\varphi_b} = \Delta^{-1}_{ab}[\varphi] = \Delta^{-1}_{0ab}[\varphi] - \Sigma_{ab}[\varphi,\Delta[\varphi]], \tag{3.44}$$

where

$$\Sigma_{ab}[\varphi,\Delta[\varphi]] = \frac{2i}{\hbar}\left(\frac{\delta\Gamma_2[\varphi,\Delta]}{\delta\Delta_{ab}}\right)_{\Delta=\Delta[\varphi]}. \tag{3.45}$$

From these one can relate derivatives of $\Gamma[\varphi]$ with respect to φ to mixed derivatives of $\Gamma[\varphi,\Delta]$ with respect to φ and Δ. For example,

$$V_{abc} = \frac{\delta^3\Gamma[\varphi,\Delta[\varphi]]}{\delta\varphi_a\delta\varphi_b\delta\varphi_c} = V_{0abc}[\varphi] - \frac{\delta\Sigma_{ab}[\varphi,\Delta[\varphi]]}{\delta\varphi_c}. \tag{3.46}$$

Being careful to use the chain rule, one obtains

$$\begin{aligned} V_{abc} &= V_{0abc}[\varphi] - \left(\frac{\delta\Sigma_{ab}[\varphi,\Delta]}{\delta\varphi_c}\right)_{\Delta=\Delta[\varphi]} - \left(\frac{\delta\Delta_{de}}{\delta\varphi_c}\frac{\delta\Sigma_{ab}[\varphi,\Delta]}{\delta\Delta_{de}}\right)_{\Delta=\Delta[\varphi]} \\ &= V_{0abc}[\varphi] - \left(\frac{\delta\Sigma_{ab}}{\delta\varphi_c}\right)_{\Delta=\Delta[\varphi]} + V_{cde}\Delta_{df}[\varphi]\Delta_{eg}[\varphi]\left(\frac{\delta\Sigma_{ab}}{\delta\Delta_{fg}}\right)_{\Delta=\Delta[\varphi]}, \end{aligned} \tag{3.47}$$

where the second line is reached using $\delta\Delta = -\Delta\left(\delta\Delta^{-1}\right)\Delta$ as an intermediate step. Continuing in this way one can find a set of identities relating derivatives of the 1PIEA and the 2PIEA evaluated at $\Delta = \Delta[\varphi]$.

It is possible to also find identities which do not require $\Delta = \Delta[\varphi]$ by taking derivatives of the 2PI equations of motion and using relations involving $W[J,K]$, such as $\delta J_a/\delta\varphi_b = (\delta\varphi_b/\delta J_a)^{-1} = \left(\delta^2 W[J,K]/\delta J_b\delta J_a\right)^{-1} = -\Delta^{-1}_{ba}$ and so on. This allows one to find that, for example,

$$\Delta^{-1}_{ba} = \frac{\delta^2\Gamma[\varphi,\Delta]}{\delta\varphi_b\delta\varphi_a} + \frac{2i}{\hbar}\frac{\delta^2\Gamma[\varphi,\Delta]}{\delta\varphi_b\delta\Delta_{ad}}\varphi_d + \frac{2i}{\hbar}\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_{ab}}, \tag{3.48}$$

regardless of the values of the external sources. If the derivatives are taken holding $K = 0$ fixed (so that $\Delta = \Delta[\varphi]$) the second and third term vanish, reproducing $\Delta^{-1}_{ba} = \delta^2\Gamma[\varphi,\Delta[\varphi]]/\delta\varphi_b\delta\varphi_a$.

The profusion of different derivatives of Γ and identities between them obscures the physical significance of nearly all of them. The remainder of this section focuses solely on the simplest WI, the 2PI analogue of Goldstone's theorem. This will allow direct comparison with the results of the previous section. Proceed by taking $\delta/\delta\varphi_c$ of the master 2PI WI (3.38) and using the previous result in the absence of external sources, giving

$$0 = \Delta_{ca}^{-1} T_{ab}^A \varphi_b + \frac{\delta^2 \Gamma}{\delta\varphi_c \delta\Delta_{ab}} \left(T_{ad}^A \Delta_{db} + T_{bd}^A \Delta_{ad} \right) . \tag{3.49}$$

Note that there is no reason for the second term to vanish in generic truncations of the 2PIEA, though it must vanish in the exact theory since there Δ is equal to the 1PI value for which $\Delta_{ca}^{-1} T_{ab} \varphi_b = 0$. In an approximation however, this no longer holds and the 2PI function Δ no longer faithfully represents the propagator of the Goldstone bosons, which now have an unphysical mass.

To see this explicitly, consider substituting in the explicit saddle point formula for the 2PIEA (2.49) in the Hartree-Fock approximation

$$\Gamma[\varphi, \Delta] = S[\varphi] + \frac{i\hbar}{2} \mathrm{Tr} \ln \Delta^{-1} + \frac{i\hbar}{2} \mathrm{Tr} \left(\Delta_0^{-1} \Delta - 1 \right) + \frac{\hbar^2 \lambda}{24} \left(\Delta_{aa} \Delta_{bb} + 2 \Delta_{ab} \Delta_{ab} \right) + \mathcal{O}\left(\lambda^2\right) , \tag{3.50}$$

which leads to

$$\frac{\delta^2 \Gamma}{\delta\varphi_c(z) \delta\Delta_{ab}(x,y)} = -\frac{i\hbar\lambda}{6} \left(\delta_{ab}\varphi_c + \delta_{ca}\varphi_b + \delta_{bc}\varphi_a \right) \delta^{(4)}(x-y) \delta^{(4)}(x-z) + \mathcal{O}\left(\lambda^2\right) , \tag{3.51}$$

whence

$$\begin{aligned} 0 \;=\; & \int_x \Delta_{ca}^{-1}(z,x) T_{ab}^A \varphi_b(x) \\ & - \frac{i\hbar\lambda}{6} \left(\delta_{ab}\varphi_c(z) + \delta_{ca}\varphi_b(z) + \delta_{bc}\varphi_a(z) \right) \left(T_{ad}^A \Delta_{db}(z,z) + T_{bd}^A \Delta_{ad}(z,z) \right) \\ & + \mathcal{O}\left(\lambda^2\right) , \end{aligned} \tag{3.52}$$

where all spacetime arguments are now made explicit.

Substituting different values of A gives

$$0 = v\left(-m_G^2\right) - \frac{i\hbar\lambda}{3} v \left(\Delta_H(z,z) - \Delta_G(z,z) \right) + \mathcal{O}\left(\lambda^2\right) , \tag{3.53}$$

for $A = (g, N)$ and

$$0 = v \Delta_{Ng}(z,z) + \mathcal{O}(\lambda) , \tag{3.54}$$

for $A = (g, g')$. The second equation is satisfied if the mixed Higgs-Goldstone propagators vanish, but the first equation implies a violation of Goldstone's theorem. Indeed, in the broken symmetry regime

$$m_G^2 = -\frac{i\hbar\lambda}{3} \left(\Delta_H(z,z) - \Delta_G(z,z) \right) + \mathcal{O}\left(\lambda^2\right) \neq 0. \tag{3.55}$$

The propagators evaluated at coincident points are quadratically divergent in four dimensions, so the difference

$$i\Delta_H(z,z) - i\Delta_G(z,z) = i\int_p \frac{m_H^2 - m_G^2}{\left(p^2 - m_H^2\right)\left(p^2 - m_G^2\right)} + \mathcal{T}_H^{\mathrm{th}} - \mathcal{T}_G^{\mathrm{th}}, \tag{3.56}$$

is logarithmically divergent and requires renormalisation. $\mathcal{T}_{H/G}^{\mathrm{th}}$ are the finite thermal Bose-Einstein contributions

$$\mathcal{T}_{H/G}^{\mathrm{th}} = \int_{\boldsymbol{k}} \frac{1}{\sqrt{\boldsymbol{k}^2 + m_{H/G}^2}} \frac{1}{\mathrm{e}^{\beta\sqrt{\boldsymbol{k}^2 + m_{H/G}^2}} - 1}. \tag{3.57}$$

In general, the renormalised form of the difference can be written

$$i\Delta_H(z,z) - i\Delta_G(z,z) = \left(m_H^2 - m_G^2\right)\left[C + F\left(m_H^2, m_G^2\right)\right] + \mathcal{T}_H^{\mathrm{th}} - \mathcal{T}_G^{\mathrm{th}}, \tag{3.58}$$

where C is a real scheme dependent constant and

$$F\left(m_H^2, m_G^2\right) = \int_0^{m_G^2} \mathrm{d}\mu^2 \int_p \frac{i}{\left(p^2 - m_H^2\right)\left(p^2 - \mu^2\right)^2}, \tag{3.59}$$

is finite and equal to zero if $m_G^2 = 0$. Explicitly

$$F\left(m_H^2, x m_H^2\right) = \frac{1}{16\pi^2}\frac{x\ln x}{x-1}. \tag{3.60}$$

In the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$)

$$C = \frac{1}{16\pi^2}\ln\left(\frac{m_H^2}{\mu^2}\right). \tag{3.61}$$

Thus

$$m_G^2 = -\frac{\hbar\lambda}{3}\left[\left(m_H^2 - m_G^2\right)\left(C + F\left(m_H^2, m_G^2\right)\right) + \mathcal{T}_H^{\mathrm{th}} - \mathcal{T}_G^{\mathrm{th}}\right] + \mathcal{O}\left(\lambda^2\right), \tag{3.62}$$

which only admits $m_G^2 = 0$ as a solution at zero temperature if $C = 0$. This can be achieved in $\overline{\mathrm{MS}}$ by choosing the renormalisation point $\mu = m_H$ (a similar choice can be made in any other subtraction scheme). However, once this is done all renormalisation constants are fixed and one has $m_G^2 \neq 0$ at any other temperature in the symmetry breaking regime. As a result Goldstone's theorem is violated at finite temperature. It will also be shown in the next section that the unphysical mass of the Goldstone bosons is responsible for changing the symmetry breaking phase transition from second order to first order. The only other $m_G^2 = 0$ solution also has $m_H^2 = 0$ which is only possible at the critical temperature of the phase transition. To find the critical temperature it is necessary to actually solve the 2PI gap equations, which is undertaken in the next section.

3.4 Renormalisation and solution of the 2PI Hartree-Fock gap equations

This section presents the solution of the 2PI equations of motion in the Hartree-Fock approximation, that is, including only the $\mathcal{O}(\lambda)$ term in the self-energy. The first field theoretical divergences appear at this order. In order to exhibit the solutions in more detail it is then necessary to discuss the renormalisation process. Renormalisation is a necessary process to extract any physical observables from a QFT. In a renormalisable theory all divergences can be absorbed in a redefinition of a fixed finite number of coupling constants, yielding a predictive theory once the couplings are fixed experimentally. In a non-renormalisable theory the divergences cannot be removed by a finite number of coupling constants, but the required infinitude of constants can be organised order by order in powers of an energy scale Λ so that only a finite number are required to give a predictive *effective field theory* accurate to some fixed order in E/Λ for low energy processes $E \ll \Lambda$. However, the number of couplings required grows as E approaches Λ and the theory loses predictivity altogether at the cutoff scale Λ. The scalar $\mathrm{O}(N)$ theory considered above and in the rest of this thesis is renormalisable in $d \leq 4$ spacetime dimensions. Elegant general discussions of renormalisation theory in the perturbative context can be found in [12, 18].

The derivation given previously that $\Gamma[\varphi, \Delta]$ is a scalar under $\mathrm{O}(N)$ rotations also applies order by order in the expansion of Γ in powers of $\hbar$ or λ. That implies that, if $\hbar^j \Gamma^{(j)}$ is the $\mathcal{O}(\hbar^j)$ (resp. λ^j) contribution to Γ, $\Gamma^{(j)}$ is itself an $\mathrm{O}(N)$ scalar. Thus any new divergences appearing at $\mathcal{O}(\hbar^j)$ must also be an $\mathrm{O}(N)$ scalar, and thus can be cancelled by the addition of an $\mathrm{O}(N)$ invariant term to Γ. The non-trivial fact that is key to the renormalisability of the 2PIEA is that the divergences appearing take the form of divergent constants multiplied by operators which are already present in Γ to $\mathcal{O}(\hbar^j)$. A general proof of this statement will not be given here, though arguments that this should be the case can be found in, e.g., [111–113]. On an intuitive level, the non-linearity of the equations of motion for Δ principally affect its long distance behaviour, while the renormalisation problem is one of short distance physics. At short distances, Weinberg's theorem [114] implies that the self-consistent propagators go over to the perturbative propagators, up to logarithmic corrections which do not effect the renormalisability of the theory. Thus a theory is 2PI renormalisable if it is perturbatively renormalisable, which is known to be the case for the scalar $\mathrm{O}(N)$ theory of interest here. Thus the approach taken in the following is simply to trust that the theory will prove to be renormalisable and see that it indeed works.

To renormalise the 2PIEA one first replaces all bare parameters by their renormalised counterparts, which are represented here by the same letters for simplicity:

$$(\phi, \varphi, v) \to Z^{1/2}(\phi, \varphi, v), \tag{3.63}$$

$$m^2 \to Z^{-1} Z_\Delta^{-1}\left(m^2 + \delta m^2\right), \tag{3.64}$$

$$\lambda \to Z^{-2}(\lambda + \delta\lambda), \tag{3.65}$$

$$\Delta \to Z Z_\Delta \Delta, \tag{3.66}$$

where Z, Z_Δ, δm^2 and $\delta\lambda$ are chosen to cancel divergences. Whenever necessary to refer to bare parameters in the remainder of this chapter, the subscript "B" will be used, e.g. m_B^2 etc. Due to the presence of composite operators in the effective action, additional renormalisation constants or "counterterms" are required compared to the standard perturbative renormalisation theory: δm_0^2 and $\delta\lambda_0$ for terms in the bare action, δm_1^2 for one-loop terms, $\delta\lambda_1^A$ for terms of the form $\phi_a\phi_a\Delta_{bb}$, $\delta\lambda_1^B$ for $\phi_a\phi_b\Delta_{ab}$ terms, $\delta\lambda_2^A$ for $\Delta_{aa}\Delta_{bb}$ and $\delta\lambda_2^B$ for $\Delta_{ab}\Delta_{ab}$. The fact that extra counterterms are required to renormalise the 2PIEA is not a problem so long as a sufficient number of renormalisation conditions can be found. Altogether there are nine renormalisation constants which must be eliminated by imposing nine conditions.

The resulting renormalised 2PIEA in the Hartree-Fock approximation is

$$\begin{aligned}
\Gamma[\varphi,\Delta] \;=\;& \frac{1}{2}Z\varphi_a\left(-\Box\right)\varphi_a - \frac{1}{2}Z_\Delta^{-1}\varphi_a\left(m^2+\delta m_0^2\right)\varphi_a - \frac{\lambda+\delta\lambda_0}{4!}\left(\varphi_a\varphi_a\right)^2 \\
&+\frac{i\hbar}{2}\mathrm{Tr}\ln\left(Z^{-1}Z_\Delta^{-1}\Delta^{-1}\right) - \frac{i\hbar}{2}\int_x\Bigg\{\Bigg[\Bigg(ZZ_\Delta\Box_x + m^2 + \delta m_1^2 \\
&+Z_\Delta\frac{\lambda+\delta\lambda_1^A}{6}\varphi_c(x)\varphi_c(x)\Bigg)\delta_{ab} + Z_\Delta\frac{\lambda+\delta\lambda_1^B}{3}\varphi_a(x)\varphi_b(x)\Bigg]\Delta_{ab}(x,y)\Bigg\}_{y\to x} \\
&+Z_\Delta^2\frac{\hbar^2\left(\lambda+\delta\lambda_2^A\right)}{24}\Delta_{aa}\Delta_{bb} + Z_\Delta^2\frac{\hbar^2\left(\lambda+\delta\lambda_2^B\right)}{12}\Delta_{ab}\Delta_{ab} + \mathcal{O}\left(\lambda^2\right),
\end{aligned} \tag{3.67}$$

which, on using the SSB ansatz (3.30), becomes

$$\begin{aligned}
\Gamma[v,\Delta_G,\Delta_H] =& \int_x\left(-\frac{1}{2}Z_\Delta^{-1}\left(m^2+\delta m_0^2\right)v^2 - \frac{\lambda+\delta\lambda_0}{4!}v^4\right) \\
&+(N-1)\frac{i\hbar}{2}\mathrm{Tr}\ln\left(Z^{-1}Z_\Delta^{-1}\Delta_G^{-1}\right) + \frac{i\hbar}{2}\mathrm{Tr}\ln\left(Z^{-1}Z_\Delta^{-1}\Delta_H^{-1}\right) \\
&-(N-1)\frac{i\hbar}{2}\mathrm{Tr}\left[\left(ZZ_\Delta\Box + m^2 + \delta m_1^2 + Z_\Delta\frac{\lambda+\delta\lambda_1^A}{6}v^2\right)\Delta_G\right] \\
&-\frac{i\hbar}{2}\mathrm{Tr}\left[\left(ZZ_\Delta\Box + m^2 + \delta m_1^2 + Z_\Delta\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2\right)\Delta_H\right] \\
&+Z_\Delta^2\frac{\hbar^2}{24}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right](N-1)\Delta_G\Delta_G \\
&+Z_\Delta^2\frac{\hbar^2\left(\lambda+\delta\lambda_2^A\right)}{12}(N-1)\Delta_G\Delta_H \\
&+Z_\Delta^2\frac{\hbar^2}{24}\left[3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B\right]\Delta_H\Delta_H + \mathcal{O}\left(\lambda^2\right),
\end{aligned} \tag{3.68}$$

which matches the expression found in appendix B of the author's paper [2].

The equations of motion following from this action are

$$\begin{aligned}
0 =& -Z_\Delta^{-1}\left(m^2+\delta m_0^2\right)v - \frac{\lambda+\delta\lambda_0}{3!}v^3 - (N-1)\frac{\hbar}{2}\left(Z_\Delta\frac{\lambda+\delta\lambda_1^A}{3}v\right)i\Delta_G(x,x) \\
&-\frac{\hbar}{2}\left(Z_\Delta\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{3}v\right)i\Delta_H(x,x) + \mathcal{O}\left(\lambda^2\right),
\end{aligned} \tag{3.69}$$

for v,

$$\Delta_G^{-1}(x,y) = -\left(ZZ_\Delta\Box_x + m^2 + \delta m_1^2 + Z_\Delta\frac{\lambda+\delta\lambda_1^A}{6}v^2\right)\delta(x-y)$$
$$- Z_\Delta^2\frac{\hbar}{6}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] i\Delta_G(x,x)\,\delta(x-y)$$
$$- Z_\Delta^2\frac{\hbar\left(\lambda+\delta\lambda_2^A\right)}{6} i\Delta_H(x,x)\,\delta(x-y) + \mathcal{O}\left(\lambda^2\right), \tag{3.70}$$

for Δ_G and

$$\Delta_H^{-1}(x,y) = -\left(ZZ_\Delta\Box_x + m^2 + \delta m_1^2 + Z_\Delta\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2\right)\delta(x-y)$$
$$- Z_\Delta^2\frac{\hbar\left(\lambda+\delta\lambda_2^A\right)}{6}(N-1)\, i\Delta_G(x,x)\,\delta(x-y)$$
$$- Z_\Delta^2\frac{\hbar}{6}\left(3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B\right) i\Delta_H(x,x)\,\delta(x-y) + \mathcal{O}\left(\lambda^2\right), \tag{3.71}$$

for Δ_H. Matching these to $\Delta_{G/H}^{-1}(x,y) = -\left(\Box_x + m_{G/H}^2\right)\delta(x-y)$ from (3.31) gives the gap equations

$$m_G^2 = m^2 + \delta m_1^2 + Z_\Delta\frac{\lambda+\delta\lambda_1^A}{6}v^2 + Z_\Delta^2\frac{\hbar}{6}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] i\Delta_G(x,x)$$
$$+ Z_\Delta^2\frac{\hbar\left(\lambda+\delta\lambda_2^A\right)}{6} i\Delta_H(x,x) + \mathcal{O}\left(\lambda^2\right), \tag{3.72}$$
$$m_H^2 = m^2 + \delta m_1^2 + Z_\Delta\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2 + Z_\Delta^2\frac{\hbar\left(\lambda+\delta\lambda_2^A\right)}{6}(N-1)\, i\Delta_G(x,x)$$
$$+ Z_\Delta^2\frac{\hbar}{6}\left(3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B\right) i\Delta_H(x,x) + \mathcal{O}\left(\lambda^2\right), \tag{3.73}$$

and the first renormalisation condition $ZZ_\Delta = 1$.

Once regularised the coincident propagators $\mathcal{T}_{G/H} \equiv i\Delta_{G/H}(x,x)$ split into parts which are infinite $\mathcal{T}_{G/H}^{\infty}$ and finite $\mathcal{T}_{G/H}^{\rm fin}$ in the limit that the regularisation parameter is removed (e.g. $d \to 4$ spacetime dimensions in dimensional regularisation or $\Lambda \to \infty$ in a cutoff regularisation). The split is ambiguous in that a finite value can be shifted from one piece to the other. The choice of split defines the renormalisation scheme. Once the split is made all renormalisation constants are determined by the condition that all divergences cancel in the equations of motion. After the cancellation the finite parts left behind will be

$$0 = -v\left(m^2 + \frac{\lambda}{6}v^2 + (N-1)\frac{\hbar\lambda}{6}\mathcal{T}_G^{\rm fin} + \frac{\hbar\lambda}{2}\mathcal{T}_H^{\rm fin}\right) + \mathcal{O}\left(\lambda^2\right), \tag{3.74}$$
$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{\hbar\lambda}{6}(N+1)\,\mathcal{T}_G^{\rm fin} + \frac{\hbar\lambda}{6}\mathcal{T}_H^{\rm fin} + \mathcal{O}\left(\lambda^2\right), \tag{3.75}$$
$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{\hbar\lambda}{6}(N-1)\,\mathcal{T}_G^{\rm fin} + \frac{\hbar\lambda}{2}\mathcal{T}_H^{\rm fin} + \mathcal{O}\left(\lambda^2\right), \tag{3.76}$$

It remains to determine the renormalisation constants. By dimensional analysis one can generically write $\mathcal{T}^{\infty}_{G/H} = c_0\Lambda^2 + c_1 m^2_{G/H} \ln\left(\Lambda^2/\mu^2\right)$ where Λ is a large energy cutoff scale and μ is an arbitrary subtraction scale (typically chosen to be of the same order of magnitude as v or m_H) which parameterises the arbitrariness of the split. The dimensionless constants c_0 and c_1 depend on the renormalisation scheme. For example, in dimensional regularisation with $\overline{\mathrm{MS}}$ subtraction $c_0 = 0$, $c_1 = -1/16\pi^2$ and $\Lambda^2 = 4\pi\mu^2 \exp\left(\frac{1}{\epsilon} - \gamma + 1\right)$ where the physical limit is $\epsilon = 2 - d/2 \to 0$ corresponding to $\Lambda \to \infty$. In a straightforward cutoff regularisation $c_0 = -c_1 = 1/16\pi^2$. The approach with c_0 and c_1 left unspecified has the virtue of being scheme agnostic. In either case the finite parts of the coincident propagators are take to be

$$\mathcal{T}^{\mathrm{fin}}_{G/H} = \frac{m^2_{G/H}}{16\pi^2} \ln\left(\frac{m^2_{G/H}}{\mu^2}\right) + \mathcal{T}^{\mathrm{th}}_{G/H}, \tag{3.77}$$

where the thermal contribution is

$$\mathcal{T}^{\mathrm{th}}_{G/H} = \int_{\boldsymbol{k}} \frac{1}{\sqrt{\boldsymbol{k}^2 + m^2_{G/H}}} \frac{1}{\mathrm{e}^{\beta\sqrt{\boldsymbol{k}^2 + m^2_{G/H}}} - 1}. \tag{3.78}$$

Substituting $\mathcal{T}^{\infty}_{G/H}$ and (3.75)-(3.76) in (3.69)-(3.73) and rearranging one can identify the divergences proportional to the various powers of $1, v, \mathcal{T}^{\mathrm{fin}}_G, \mathcal{T}^{\mathrm{fin}}_H$ which are kinematically distinct and so must vanish separately. This tedious and unenlightening computation is best left to a computer algebra package. The constant Z_Δ is redundant in the Hartree-Fock approximation and can simply be set to one. There are ten independent equations for the eight remaining constants, thus the fact that a solution exists is a non-trivial consistency check. The details of the computation of the renormalisation constants can be found in Appendix C. The resulting renormalisation constants are

$$Z = Z_\Delta = 1, \tag{3.79}$$

$$\delta m_0^2 = \delta m_1^2 = -\frac{\hbar\lambda}{6}(N+2)\left(c_0\Lambda^2 + c_1 m^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right)\right)\frac{\delta\lambda_1^A + \lambda}{\delta\lambda_1^B + \lambda}, \tag{3.80}$$

$$\delta\lambda_1^A = \delta\lambda_2^A = \left(1 + \frac{3(N+2)}{6 + c_1(N+2)\lambda\hbar\ln\left(\frac{\Lambda^2}{\mu^2}\right)}\right)\delta\lambda_1^B, \tag{3.81}$$

$$\delta\lambda_1^B = \delta\lambda_2^B = \left(-1 + \frac{3}{3 + c_1\lambda\hbar\ln\left(\frac{\Lambda^2}{\mu^2}\right)}\right)\lambda, \tag{3.82}$$

$$\delta\lambda_0 = \delta\lambda_1^A + 2\delta\lambda_1^B. \tag{3.83}$$

The final renormalisation choice is to take $\mu^2 = m_H^2$, which results in the tree level (i.e. classical) relations $m^2 = -\lambda v^2/6$, $m_G^2 = 0$ and $m_H^2 = \lambda v^2/3$ holding at zero temperature (recall this is also the choice that makes the constant $C = 0$ in the 2PI WI (3.62)). It is convenient to denote the zero temperature values with an over-bar, so that $\bar{v}^2 = 3\bar{m}_H^2/\lambda = -6m^2/\lambda$ and $\bar{m}_G^2 = 0$.

Using (3.76), (3.74) can be simplified to

$$0 = v\left(m_H^2 - \frac{\lambda}{3}v^2\right) + \mathcal{O}\left(\lambda^2\right). \tag{3.84}$$

In the symmetric regime $v = 0$ and $m_H^2 - m_G^2 = \frac{\hbar\lambda}{3}\left(\mathcal{T}_H^{\text{fin}} - \mathcal{T}_G^{\text{fin}}\right)$ which has the solution $m_H^2 = m_G^2 = m^2 + \frac{\hbar\lambda}{6}(N+2)\,\mathcal{T}_G^{\text{fin}} + \mathcal{O}\left(\lambda^2\right)$. The critical temperature $T_\star$ occurs when $m_H^2 = m_G^2 = 0$, for which $\mathcal{T}_G^{\text{fin}} = T_\star^2/12$, giving

$$T_\star = \sqrt{\frac{12\bar{v}^2}{\hbar\,(N+2)}} + \mathcal{O}\left(\lambda^2\right). \tag{3.85}$$

A numerical solution of the full gap equations is shown in Figure 3.1. The parameters chosen are $N = 4$, $\bar{v} = 93$ MeV and $\bar{m}_H = 500$ MeV, which are representative of the *linear sigma model* effective theory of pions in the limit of unbroken chiral symmetry [115]. Because the "Higgs" mode of this model corresponds to the poorly constrained σ meson[4], a range of values for $\bar{m}_H$ can be used. The value used here is chosen to enable direct comparison with [116], since that paper brings in the concept of symmetry improvement which will be discussed at length in the following chapters of this thesis. The linear sigma model is known to possess a continuous (i.e. second order) phase transition, analogous to the Ising ferromagnet, with a critical temperature $T_\star \approx 132$ MeV on the basis of lattice computation (see, e.g. [117, 118]). However Figure 3.1 shows the existence of a broken symmetry phase extending to temperatures greater than $T_\star$. Physically this signals the existence of a discontinuous (first order) phase transition: if the temperature is carefully raised from zero to $> T_\star$ the system remains in the broken symmetry phase, now superheated, until a perturbation causes bubbles of symmetric phase to suddenly form and expand with the release of latent heat, eventually filling the whole space with symmetric phase. The upper spinodal point (the maximum temperature which the metastable phase can attain) is seen to be ≈ 210 MeV. The value of $v_\star \approx 80$ MeV in the broken phase at the critical temperature is a common measure of the strength of a first order phase transition, and since $v_\star \approx \bar{v}$ the phase transition is *strongly* first order. Also, the Goldstone theorem clearly is violated in that m_G, the blue curve, is not zero at finite temperature. However, one can check that, by subtracting (3.74) from (3.75) that the 2PI WI (3.62) *is* satisfied. Further, it will be shown in the next chapter that the symmetry improvement method, which imposes 1PI style WIs on the 2PI solution, restores the second order phase transition in the Hartree-Fock approximation. Thus the unphysical artefacts of the 2PIEA in the Hartree-Fock approximation can be understood as a result of the difference between the 1PI and 2PI Ward identities.

[4]The σ meson is very broad and identifying it as a real particle is controversial, though there is definitely a pole which can be observed in pion scattering amplitudes (see "NOTE ON SCALAR MESONS BELOW 2 GEV" from the Review of Particle Physics [8], available separately at http://pdg.lbl.gov/2015/reviews/rpp2015-rev-scalar-mesons.pdf).

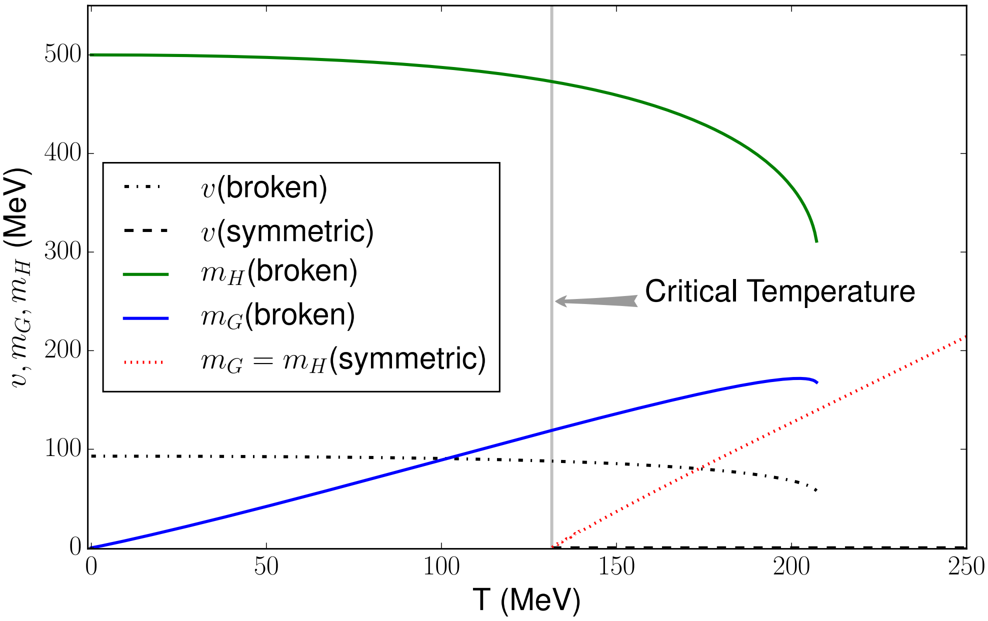


Figure 3.1: Solution of the 2PI Hartree-Fock equations of motion (3.74)-(3.76) for $N = 4$, $\bar{v} = 93$ MeV and $\bar{m}_H = 500$ MeV, values representative of the linear sigma model effective theory of pions in the limit of unbroken chiral symmetry. $m_G \neq 0$ unphysically at finite temperatures in the broken symmetry phase. The broken symmetry phase persists to temperatures substantially greater than the critical temperature $T_\star \approx 132$ MeV, signalling the presence of an unphysical meta-stable phase and a first order phase transition.

3.5 The External Propagator method

It is possible to derive correlation functions from the 2PIEA which do obey 1PI style WIs. The idea is that one first computes the self-consistent, or "internal," φ and Δ as solutions of the 2PI equations of motion, and then from these constructs a resummed 1PIEA whose functional derivatives are the "external" propagators and vertices. This method is used frequently in the literature (see e.g. [68, 112, 119]) and was definitively studied by [119]. Since $\Gamma[\varphi] = \Gamma[\varphi, \Delta[\varphi]]$ where $\Delta[\varphi]$ is a solution of the *exact* 2PI equations of motion, the natural definition of the resummed 1PIEA is

$$\tilde{\Gamma}[\varphi] = \tilde{\Gamma}\left[\varphi, \tilde{\Delta}[\varphi]\right], \tag{3.86}$$

where $\tilde{\Gamma}$ is the *truncated* (not exact) 2PIEA and $\tilde{\Delta}[\varphi]$ is a solution of the *truncated* equation of motion

$$\left.\frac{\delta\tilde{\Gamma}[\varphi, \Delta]}{\delta\Delta}\right|_{\Delta=\tilde{\Delta}[\varphi]} = 0, \tag{3.87}$$

which is no longer equal to the exact $\Delta[\varphi]$. The solution of the resummed 1PI equation of motion is the same as the truncated 2PI solution because

$$\frac{\delta \tilde{\Gamma}[\varphi]}{\delta \varphi} = \left(\frac{\delta \tilde{\Gamma}[\varphi, \Delta]}{\delta \varphi} \right)_{\Delta = \tilde{\Delta}[\varphi]} + \frac{\delta \tilde{\Delta}[\varphi]}{\delta \varphi} \left(\frac{\delta \tilde{\Gamma}[\varphi, \Delta]}{\delta \Delta} \right)_{\Delta = \tilde{\Delta}[\varphi]} = \left(\frac{\delta \tilde{\Gamma}[\varphi, \Delta]}{\delta \varphi} \right)_{\Delta = \tilde{\Delta}[\varphi]}, \quad (3.88)$$

by the definition of $\tilde{\Delta}[\varphi]$.

Now, as long as the truncation of the 2PIEA is $\mathrm{O}(N)$ invariant[5], $\tilde{\Delta}[\varphi]$ will transform as a symmetric rank-2 tensor and $\tilde{\Gamma}[\varphi]$ will be a scalar. Thus the derivation of the 1PI WIs carries through without modification for $\tilde{\Gamma}[\varphi]$. Thus the external propagator, defined as

$$\Delta_{\mathrm{ext}}^{-1} = \frac{\delta^2 \tilde{\Gamma}[\varphi]}{\delta \varphi \delta \varphi}, \quad (3.89)$$

obeys the Goldstone theorem

$$0 = \left(\Delta_{\mathrm{ext}}^{-1} \right)_{ab} T^A_{bc} \varphi_c, \quad (3.90)$$

as desired. Similarly defined higher order external vertices all obey the appropriate 1PI type WIs. Clearly these definitions and properties can be extended also to higher nPIEA with minimal modification.

This technique elegantly accomplishes its main aim: to define correlation functions satisfying 1PI WIs while including some effects of the 2PI resummation in propagators. Unfortunately, there are several drawbacks to the method which may prompt one to consider alternative solutions to the problem of symmetries in 2PIEA. First, the method is not fully self-consistent in that the external propagators are not involved in the 2PI solution at all. The internal propagators appearing in the 2PI graphs are still unphysical and do not obey the 1PI WIs. As a result of unphysically massive Goldstone bosons in internal propagators, the external propagators are useless to predict decay rates, reaction thresholds, absorptive parts of amplitudes, finite width effects and the unitarity of the theory is violated. Also an unphysical first order phase transition is still obtained in the Hartree-Fock approximation [119]. Furthermore, the physical quantities (being represented by external rather than internal Green functions) are a further step removed from the underlying formalism of the theory. This greatly complicates the renormalisation procedure, which now requires the solution of non-linear coupled Bethe-Salpeter integral equations in order to find the renormalisation counter-terms.

3.6 Contrast to the Large N method

It is worth contrasting the results obtained above (and in the next chapter) with those for another commonly used non-perturbative technique in QFT: the large N expansion. The large N expansion is a systematic expansion of field theories in powers of $1/N$, where N is the number of fields. Surprisingly there are a number of field theories where this limit exists and leads to non-trivial insights. While the method is not always quantitatively useful in the main practical applications, which are quantum chromodynamics with $N = 3$ and the

[5] An example of a non-invariant truncation would be keeping the $\Delta_G \Delta_G$ term, and only that term, in the Hartree-Fock diagram. Any loop or large N truncation will be automatically invariant.

linear sigma model for the light mesons $\left(\pi^0, \pi^\pm, \sigma\right)$ or Higgs[6] $\left(h, a, \phi^\pm\right)$ with $N = 4$, it is responsible for numerous theoretical insights and has rendered some incalculable models calculable and yielded exact non-perturbative results in the case of supersymmetric and low dimensional field theories.

The main idea of the large N approximation is that in the limit of a large number of fields, composites of the fields such as $\phi_a\phi_a$ become self-averaging by the central limit theorem and their behaviour is essentially classical. The method is not the same as a simple loop expansion, however, as loops often contain sums over field components which give factors of N, partially compensating small couplings. Excellent introductory lectures on the method are found in [39] and an extensive modern review in [15]. The application of the large N method to the $\mathrm{O}(N)$ model has a long history as can be seen from the previous references. Particularly relevant to this thesis are the connection to the 2PI method, which was published by Petropoulos [115] in essentially the form given here, and the contrast with the symmetry improved 2PI method which was originally studied by Mao [116]. This section will only consider the leading order of the large N expansion.

To see how the large N method works in the present case consider again the 2PI equations of motion (3.74)-(3.76) reproduced here:

$$0 = -v\left(m^2 + \frac{\lambda}{6}v^2 + (N-1)\frac{\hbar\lambda}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar\lambda}{2}\mathcal{T}_H^{\mathrm{fin}}\right) + \mathcal{O}\left(\lambda^2\right), \tag{3.91}$$

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{\hbar\lambda}{6}(N+1)\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar\lambda}{6}\mathcal{T}_H^{\mathrm{fin}} + \mathcal{O}\left(\lambda^2\right), \tag{3.92}$$

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{\hbar\lambda}{6}(N-1)\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar\lambda}{2}\mathcal{T}_H^{\mathrm{fin}} + \mathcal{O}\left(\lambda^2\right). \tag{3.93}$$

One now makes a systematic expansion of these equations in powers of $1/N$. Obviously in order to obtain a sensible limit it is necessary to send $\lambda \to 0$ as well, otherwise the $\mathcal{T}_G^{\mathrm{fin}}$ terms dominate and cannot be matched against any other terms. However, in order to obtain a non-trivial limit λ cannot vanish too fast, so one is forced to take $g \equiv \lambda N$ a constant in the $N \to \infty$ limit. In order to obtain a broken symmetry phase it is also necessary to take $v^2 \sim \mathcal{O}(N)$ which can be motivated, for example, by a consideration of the central limit theorem with $v^2 = \langle\phi_a\phi_a\rangle = \left\langle\phi_1^2\right\rangle + \cdots + \left\langle\phi_N^2\right\rangle \sim N$. To that end define $\hat{v} = v/\sqrt{N}$ which is fixed in the large N limit. The equations of motion become

$$0 = -\sqrt{N}\hat{v}\left(m^2 + g\frac{\hat{v}^2}{6} + \frac{\hbar g}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{g}{N}\left(-\frac{\hbar}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar}{2}\mathcal{T}_H^{\mathrm{fin}}\right)\right) + \mathcal{O}\left(\lambda^2\right), \tag{3.94}$$

$$m_G^2 = m^2 + g\frac{\hat{v}^2}{6} + \frac{\hbar g}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{g}{N}\left(\frac{\hbar}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar}{6}\mathcal{T}_H^{\mathrm{fin}}\right) + \mathcal{O}\left(\lambda^2\right), \tag{3.95}$$

$$m_H^2 = m^2 + g\frac{\hat{v}^2}{2} + \frac{\hbar g}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{g}{N}\left(-\frac{\hbar}{6}\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar}{2}\mathcal{T}_H^{\mathrm{fin}}\right) + \mathcal{O}\left(\lambda^2\right). \tag{3.96}$$

Now an important technicality. While $\lambda \to 0$ naively sends the $\mathcal{O}\left(\lambda^2\right)$ terms to zero,

[6] The pseudoscalar a and charged $\phi^\pm$ Higgs components are absorbed by the Z and $W^\pm$ gauge bosons respectively and are not observed as scalar states.

these terms contain loops which contain sums over $\mathrm{O}(N)$ indices which partially compensate the limit as $N \to \infty$. It is not immediately obvious that the $\mathcal{O}(\lambda^2)$ terms can be consistently neglected in the equations of motion. It is a non-trivial fact, the proof of which is too much of a combinatorial aside to give here (see below for another method which bypasses this technicality altogether), that the maximum amount of compensation is $\mathcal{O}(\lambda^2) \to \mathcal{O}(g^2/N)$, so that the higher order graphs can be consistently neglected to leading order in g/N. As a result,

$$0 = -\sqrt{N}\hat{v}\left(m^2 + g\frac{\hat{v}^2}{6} + \frac{\hbar g}{6}\mathcal{T}_G^{\mathrm{fin}}\right) + \mathcal{O}\left(\frac{g}{N}\right), \tag{3.97}$$

$$m_G^2 = m^2 + g\frac{\hat{v}^2}{6} + \frac{\hbar g}{6}\mathcal{T}_G^{\mathrm{fin}} + \mathcal{O}\left(\frac{g}{N}\right), \tag{3.98}$$

$$m_H^2 = m^2 + g\frac{\hat{v}^2}{2} + \frac{\hbar g}{6}\mathcal{T}_G^{\mathrm{fin}} + \mathcal{O}\left(\frac{g}{N}\right). \tag{3.99}$$

Note that the first equation of motion is now

$$0 = -v m_G^2 + \mathcal{O}\left(\frac{g}{N}\right), \tag{3.100}$$

i.e. Goldstone's theorem holds to leading order in the $1/N$ and g expansion. Then in the broken phase $m_G^2 = 0$ and

$$m_H^2 = g\frac{\hat{v}^2}{3} + \mathcal{O}\left(\frac{g}{N}\right), \tag{3.101}$$

and the critical temperature

$$T_\star = \sqrt{\frac{12\bar{v}^2}{\hbar N}} + \mathcal{O}\left(\frac{1}{N}\right), \tag{3.102}$$

such that

$$m_H^2 = \bar{m}_H^2\left(1 - \frac{T^2}{T_\star^2}\right) + \mathcal{O}\left(\frac{g}{N}\right). \tag{3.103}$$

Unlike the 2PIEA, a second order phase transition is found and Goldstone's theorem is satisfied. However, the large N estimate for the critical temperature is larger than the 2PI value by a factor $\sqrt{(N+2)/N}$ which is $\sqrt{3/2} \approx 1.22$ for $N = 4$, accurate to the expected $\mathcal{O}(1/N) = 25\%$. This makes the critical temperature ≈ 162 MeV for the numerical values used previously. This difference is the due to the neglect of the Higgs boson thermal contribution, which is indeed swamped by Goldstone modes as $N \to \infty$. Note however that Goldstone's theorem is lost again at higher orders of the $1/N$ expansion.

There is another way to see the above results that does not rely on any special argument that the Hartree-Fock truncation is consistent to leading order in $1/N$. Start with the partition functional

$$Z[J] = \int \mathcal{D}[\phi] \exp\left[\frac{i}{\hbar}\left(\frac{1}{2}\phi_a\left(-\Box - m^2\right)\phi_a - \frac{\lambda}{4!}\left(\phi_a\phi_a\right)^2 + J_a\phi_a\right)\right], \tag{3.104}$$

and note that multiplying by a J-independent overall factor of the form

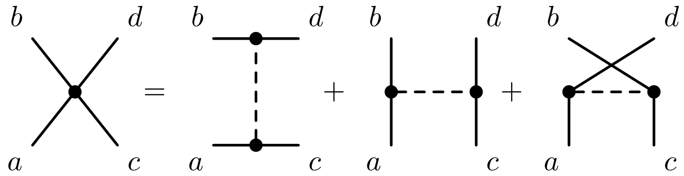
$$\text{const.} = \int \mathcal{D}[\sigma] \exp\left[\frac{i}{\hbar}\frac{3}{2\lambda}\left(\sigma - \frac{\lambda}{3!}\phi_a\phi_a\right)^2\right], \tag{3.105}$$

has no physical effect. (Also note that $\langle\sigma\rangle = \frac{\lambda}{3!}\langle\phi_a\phi_a\rangle \sim g\hat{v}^2 \sim \mathcal{O}(1)$ as $N \to \infty$.)

The new term has the effect of introducing a new field σ and shifting the action to

$$S_{\text{eff}}[\phi, \sigma] = \frac{1}{2}\phi_a\left(-\Box - m^2\right)\phi_a + J_a\phi_a + \frac{3}{2\lambda}\sigma^2 - \frac{1}{2}\sigma\phi_a\phi_a, \tag{3.106}$$

which is now quadratic in the ϕ_as, meaning the functional integral over them can be performed in closed form. In terms of Feynman diagrams the effect of introducing σ (the "Hubbard-Stratonovich transformation") is to "open up" four point vertices into a pair of three point vertices with σ exchange in all possible ways, that is



$$, \tag{3.107}$$

where the dashed line is the σ propagator, equal to $\lambda/3$. The vertices conserve $\mathrm{O}(N)$ flavour, e.g. the first term above is proportional to $\delta_{ac}\delta_{bd}$, and so on. This reorganisation of Feynman diagrams dramatically simplifies the combinatorics of the theory.

For the present purposes it is convenient to write $\phi_a = (\pi_1, \cdots, \pi_{N-1}, h)$ where πs are the Goldstone modes and h is the Higgs mode, and only integrate over the πs. It is also convenient to take $J_a = (0, \cdots, 0, J)$. Then

$$S_{\text{eff}}[\pi, h, \sigma] = \frac{1}{2}\pi_a\left(-\Box - m^2 - \sigma\right)\pi_a + \frac{1}{2}h\left(-\Box - m^2\right)h + Jh + \frac{3}{2\lambda}\sigma^2 - \frac{1}{2}\sigma h^2. \tag{3.108}$$

Performing the functional integral over the πs gives an effective action for the h and σ fields, given by $\exp\left[\frac{i}{\hbar}S_{\text{eff}}[h,\sigma]\right] = \int \mathcal{D}[\pi]\exp\left[\frac{i}{\hbar}S_{\text{eff}}[\pi, h, \sigma]\right]$, which gives

$$S_{\text{eff}}[h, \sigma] = \frac{i\hbar}{2}(N-1)\,\mathrm{Tr}\ln\left(-\Box - m^2 - \sigma\right) + \frac{1}{2}h\left(-\Box - m^2\right)h + Jh + \frac{3N}{2g}\sigma^2 - \frac{1}{2}\sigma h^2. \tag{3.109}$$

Considering that $\sigma \sim \mathcal{O}(1)$, $h \sim \mathcal{O}\left(\sqrt{N}\right)$ and one can take $J \sim \mathcal{O}\left(\sqrt{N}\right)$, then the entire action is proportional to N in the large N limit, i.e. $S_{\text{eff}}[h, \sigma] \to N\tilde{S}_{\text{eff}}[h, \sigma]$ as $N \to \infty$, where

$$\tilde{S}_{\text{eff}}[h, \sigma] = \frac{i\hbar}{2}\mathrm{Tr}\ln\left(-\Box - m^2 - \sigma\right) + \frac{1}{2N}h\left(-\Box - m^2\right)h + \frac{Jh}{N} + \frac{3}{2g}\sigma^2 - \frac{1}{2N}\sigma h^2, \tag{3.110}$$

is $\mathcal{O}(1)$. The effect of the large N limit in the functional integral

$$\int \mathcal{D}[h, \sigma]\exp\left[\frac{i}{\hbar}N\tilde{S}_{\text{eff}}[h, \sigma]\right], \tag{3.111}$$

is the same as $\hbar \to 0$, i.e. the classical limit. To leading order in $1/N$, then, the dynamics are determined by the *classical* solution of the equations of motion following from $\tilde{S}_{\text{eff}}[h, \sigma]$. These are

$$\left(\Box + m^2 + \sigma\right) h = J, \tag{3.112}$$

$$\frac{i\hbar}{2} \mathrm{Tr}\left(-\Box - m^2 - \sigma\right)^{-1} = \frac{3}{g}\sigma - \frac{1}{2N}h^2. \tag{3.113}$$

Looking for uniform solutions $h(x) = v$, $\sigma(x) = \sigma$ in the physical limit $J = 0$,

$$\left(m^2 + \sigma\right) v = 0, \tag{3.114}$$

$$\frac{i\hbar}{2} \int_k \frac{1}{k^2 - (m^2 + \sigma)} = \frac{3}{g}\sigma - \frac{1}{2N}v^2. \tag{3.115}$$

In the broken phase the first equation gives $\sigma = -m^2$, and one recognises that after renormalisation the second equation is just the large N gap equation obtained previously for the Goldstone with $m_G^2 = m^2 + \sigma$ as the Goldstone mass. Substituting $m_G^2 - m^2$ for σ above one indeed recovers the large N gap equation and the Goldstone theorem $m_G^2 v = 0$ in both phases. The advantage of this method via transformations of the functional integral rather than a direct expansion of the equations of motion is that the systematics of the large N expansion are clearer. In particular, it is now obvious that the gap equations obtained are strictly valid to order $\mathcal{O}(1/N)$ without having to worry about the combinatorics of higher order self-energy graphs. The disadvantage of this method is that the connection to the 2PIEA is less clear.

3.7 Summary

This chapter explored the symmetry properties of nPIEAs using the 1PIEA and 2PIEA as specific examples. The symmetry properties of a QFT are embodied in a set of identities relating the various correlation functions. In particular, the Ward identities (WIs) encode the underlying global $\mathrm{O}(N)$ symmetry of the theory studied here in terms of the propagator and proper vertices derived from the 1PIEA. The simplest consequence of these identities is Goldstone's theorem: the existence of massless bosons corresponding to each generator of a broken continuous symmetry. The global symmetries are not generically preserved by truncations of higher nPIEAs because the nPIEAs obey different WIs for each n. Phenomenological consequences were studied by solving the 2PI equations of motion in the Hartree-Fock approximation. The Goldstone bosons were unphysically massive and the phase transition was incorrectly predicted to be first order. These results are in contrast to the so-called "external propagator" and "large N" methods which solve some of the phenomenological problems, though not entirely satisfactorily. The external propagator obeys Goldstone's theorem, but the order of the phase transition is still incorrect, as are reaction thresholds and decay rates. The leading order large N approximation correctly predicts a second order phase transition and massless Goldstone bosons, but gives an incorrect transition temperature due to the neglect of Higgs boson thermal contributions. Also, Goldstone's theorem is lost at next-to-leading order in $1/N$. This review sets the stage for the examination of symmetry improvement techniques in the remainder of the thesis.

Chapter 4

Symmetry Improvement of nPIEA through Lagrange Multipliers

4.1 Synopsis

Chapter 2 introduced nPIEAs as powerful tools for computations in QFT, particularly used for their ability to handle non-equilibrium situations through time contour methods. The previous chapter discussed global symmetries and Ward identities in nPIEA, showing that the solutions of truncated nPI equations of motion do not obey the 1PI Ward identities. Several studies have attempted to find a remedy for this problem (see, e.g., [1] and references therein). The last chapter discussed the external propagator method, which has been frequently used in the literature and does yield massless Goldstone bosons. However, the external propagator is not the propagator used in loop graphs, so the loop corrections still contain massive Goldstone bosons leading to incorrect thresholds, decay rates and violations of unitarity. In order to avoid these problems a manifestly self-consistent scheme must be used. One such scheme is the large N approximation, which was also briefly reviewed. The Goldstone theorem and second order phase transition do hold to leading order in the large N approximation, but these attractive features are lost at higher orders. Another approach which is not discussed in detail here is to abandon the nPIEA formalism entirely, using instead the Schwinger-Dyson equations. The Schwinger-Dyson equations require a closure ansatz which can be chosen to respect appropriate Ward identities. However, this choice still involves some degree of arbitrariness and self-consistency is not guaranteed.

This chapter discusses the symmetry improvement method introduced by Pilaftsis and Teresi to circumvent these difficulties [1] for the 2PIEA. The idea is simply to impose the desired Ward identities directly on the free correlation functions. This is consistently implemented by using Lagrange multipliers. The remarkable point is that the resulting equations of motion can be put into a form that completely eliminates the Lagrange multiplier field. They achieve this by taking a limit in which the Lagrange multiplier vanishes from all but one of the equations of motion, and this remaining equation of motion is replaced with the constraint to obtain a closed system. Here the symmetry improved 2PIEA (SI-2PIEA) is reviewed and extended to general $\mathrm{O}(N)$ theories. An ambiguity of the constraint scheme is pointed out which was not recognised in the original literature. This ambiguity has no influence on the equilibrium results of this chapter, but is relevant to the discussion of Chapter 6. Following that the method is generalised to the 3PIEA (reporting on work published by the author [2]). The generalisation is non-trivial, requiring a careful consideration of the variational procedure leading to an infinity of possible schemes. A new principle is introduced to choose between the schemes and is called the *d'Alembert formalism* by analogy to the constrained variational problem in mechanics.

The motivation for extending symmetry improvement to the 3PIEA is threefold. First, the 3PIEA is known to be the required starting point to obtain a self-consistent non-equilibrium kinetic theory of gauge theories. The accurate calculation of transport coefficients and thermalisation times in gauge theories requires the use of nPIEA with $n \geq 3$ (see, e.g. [16, 68, 69] and references therein for discussion). The fundamental reason for this is that the 3PIEA includes medium induced effects on the three-point vertex at leading order. The 2PIEA in gauge theory contains a dressed propagator but not a dressed vertex, leading not only to an inconsistency of the resulting kinetic equation but also to a spurious gauge dependence analogous to the failure of global symmetries in truncations as discussed in chapter 3. Thus this work is a stepping stone towards a fully self-consistent, non-perturbative and manifestly gauge invariant treatment of out of equilibrium gauge theories.

Second, nPIEAs allow one to accurately describe the initial value problem with 1- to n-point connected correlation functions in the initial state. For example, the 2PIEA allows one to solve the initial value problem for initial states with a Gaussian density matrix. However, the physical applications one has in mind typically start from a near thermal equilibrium state which is not well approximated by a Gaussian density matrix. This leads to problems with renormalisation, unphysical transient responses and thermalisation to the wrong temperature [120]. This is addressed in [63, 120, 121] by the addition of an infinite set of non-local vertices which only have support at the initial time. Going to $n > 2$ allows one to better describe the initial state, thereby reducing the need for additional non-local vertices.

Lastly, the infinite hierarchy of nPIEA is the natural home for the 2PIEA and provides the clearest route for systematic improvements over existing treatments. Thus investigating symmetry improvement of 3PIEA is a well motived step in the development of non-perturbative QFT.

4.2 Symmetry Improvement of the 2PIEA

The essence of symmetry improvement is to impose the WIs derived for the 1PI correlation functions on the nPI correlation functions. Effectively, one changes

$$\mathcal{W}^A_{a_1\cdots a_j}\left(\Delta_{1\mathrm{PI}}, V_{1\mathrm{PI}}, \cdots, V^{(n)}_{1\mathrm{PI}}\right) \to \mathcal{W}^A_{a_1\cdots a_j}\left(\Delta_{n\mathrm{PI}}, V_{n\mathrm{PI}}, \cdots, V^{(n)}_{n\mathrm{PI}}\right), \tag{4.1}$$

where $\mathcal{W}^A_{a_1\cdots a_j}$ for $j = 1, \cdots, n-1$ are the WIs and the arguments change but not the functional form. Note that higher order WIs cannot be enforced on an nPIEA since only the 1- through n-point functions are free. Thus, in the SI-2PIEA a single constraint is enforced, namely (3.15), which recall is

$$0 = \mathcal{W}_c^A = \Delta_{ca}^{-1} T_{ab}^A \varphi_b - J_a T_{ac}^A, \tag{4.2}$$

where Δ is thought of as the 2PI function. Only the case $J_a = 0$ will be considered here. Pilaftsis and Teresi [1] considered the case of a translationally invariant $J_a \neq 0$ to define a symmetry improved effective potential. The situation with a general $J_a \neq 0$ is discussed in Chapter 6.

Symmetry improvement imposes (3.15) as a constraint on the allowable values of φ and Δ in the 2PIEA. This proceeds through the introduction of Lagrange multiplier fields $\ell_A^d(x)$ and a shift of the action $\Gamma \to \Gamma - \mathcal{C}$ where $\mathcal{C} \sim \ell\mathcal{W}$ is the constraint. Note however that Pilaftsis and Teresi introduced symmetry improvement non-covariantly, leading to two possible covariant symmetry improvement schemes that reduce to theirs in equilibrium. The first is to take

$$\mathcal{C} = \frac{i}{2} \ell_A^c \mathcal{W}_c^A. \tag{4.3}$$

The second follows the author's previous paper [2] by including a transverse projector $P_{ab}^{\perp}(x) = \delta_{ab} - \varphi_a(x)\varphi_b(x)/\varphi^2(x)$:

$$\mathcal{C}' = \frac{i}{2} \ell_A^c P_{cd}^{\perp} \mathcal{W}_d^A, \tag{4.4}$$

to ensure that only Goldstone modes are involved in the constraint. The choice between $\mathcal{C}$ and $\mathcal{C}'$ turns out to make no difference in equilibrium. However, the two constraints lead to different schemes beyond equilibrium, both of which are pathological as discussed in the author's paper [4] and in Chapter 6. For now the simple constraint (4.3) is taken.

The equations of motion following from the symmetry improved effective action are

$$\mathcal{W}_c^A = 0, \tag{4.5}$$

$$\frac{\delta\Gamma}{\delta\varphi_d(z)} = \frac{i}{2}\int_x \ell_A^c(x)\,\Delta_{ca}^{-1}(x,z)\,T_{ad}^A - J_d(z) - \int_w K_{de}(z,w)\,\varphi_e(w), \tag{4.6}$$

$$\frac{\delta\Gamma}{\delta\Delta_{de}(z,w)} = \frac{i}{2}\int_x \ell_A^c(x)\int_y \frac{\delta\Delta_{ca}^{-1}(x,y)}{\delta\Delta_{de}(z,w)} T_{ab}^A \varphi_b(y) - \frac{1}{2} i\hbar K_{de}(z,w), \tag{4.7}$$

where the last equation simplifies to

$$\frac{\delta\Gamma}{\delta\Delta_{de}(z,w)} = -\frac{i}{2}\int_x \ell_A^c(x)\,\Delta_{cd}^{-1}(x,z)\int_y \Delta_{ea}^{-1}(w,y)\,T_{ab}^A \varphi_b(y) - \frac{1}{2} i\hbar K_{de}(z,w), \tag{4.8}$$

on using the identity $\delta\Delta_{ca}^{-1}/\delta\Delta_{de} = -\Delta_{cd}^{-1}\Delta_{ea}^{-1}$. In the present discussion only the equilibrium case with $J = K = 0$ is relevant (this is reviewing [1, 2], the more general case is discussed in [4] and Chapter 6). The only non-trivial WIs are $\mathcal{W}_c^{gN} = -ivm_G^2 P_{cg}^{\perp}$, so that the constraint enforces $vm_G^2 = 0$, i.e. the Goldstone mass vanishes if $v \neq 0$ as expected. Using homogeneity $\ell_A^c(x) = \ell_A^c$, and the SSB ansatz (3.30) the other equations of motion become

$$\frac{\partial\Gamma/\mathrm{V}T}{\partial v} = \ell^c_{cN} m_G^2, \tag{4.9}$$

$$\frac{\delta\Gamma}{\delta\Delta_G(z,w)} = \ell^c_{cN} v m_G^4, \tag{4.10}$$

$$\frac{\delta\Gamma}{\delta\Delta_H(z,w)} = 0, \tag{4.11}$$

where $\mathrm{V}T$ is the volume of spacetime[1]. The symmetry improvement constraint is a singular one: one must require $\ell \to \infty$ as $v \int \Delta_G^{-1} \to 0$. The reason for this is that $m_G^2 \to 0$ so the right hand sides vanish unless $\ell^c_{cN} \to \infty$. Applying the constraint with $v \neq 0$ directly in the equations of motion would give zero right hand sides, reducing to the standard 2PI formalism. This is valid in the full theory because the Ward identity is satisfied. However, this is impossible in the case where the 2PI effective action is truncated at finite loop order because the actual Ward identity obeyed by the 2PIEA is *not* $\mathcal{W}^A_c$ as discussed in Chapter 3.

The divergence is regulated by setting $v m_G^2 = \eta m^3$ and taking the limit $\eta \to 0$ such that $\eta \ell^c_{cN}/v = \ell_0$ is a constant. This gives

$$\frac{\partial\Gamma/\mathrm{V}T}{\partial v} = \ell_0 m^3, \tag{4.12}$$

$$\frac{\delta\Gamma}{\delta\Delta_G(z,w)} = 0. \tag{4.13}$$

(Note that one can consistently set

$$\ell^a_{bN} = -\ell^a_{Nb} = P^{\perp}_{ab}\left(\frac{1}{N-1}\ell^c_{cN}\right), \tag{4.14}$$

and all other components of ℓ^c_A to zero.) Thus the propagator equations of motion are unmodified and the vev equation is modified by the presence of a homogeneous force that acts to push v away from the minimum of the effective potential to the point where $m_G^2 = 0$. In practice, in the symmetry broken phase, one simply discards the vev equation of motion and solves the propagator ones in conjunction with the Ward identity, which suffices to give a closed system. In the symmetric phase $v = 0$ and the Ward identity is trivial, but Γ also does not depend linearly on v, hence one can take the previous equations of motion with $\ell_0 = 0$. Note that one can keep a non-zero m_G^2 in the intermediate stages of the computation to serve as an infrared regulator.

To recap the procedure: first define a symmetry improved effective action using Lagrange multipliers and compute the equations of motion. Second, note that the equations of motion are singular when the constraints are applied. Third, regulate the singularity by slightly violating the constraint. Fourth, pass to a suitable limit where violation of the constraint tends to zero while requiring the limiting procedure to be universal in the

[1] Recall that $(\delta/\delta\varphi(x))\int_y \varphi(y)\,J(y) \equiv J(x)$, however, for a translation invariant $\varphi(x) = v$ and $J(x) = J$, $\partial_v \int_y \varphi(x)\,J(x) = \partial_v \int_x vJ = \mathrm{V}TJ$ so that $\delta/\delta\varphi = (\mathrm{V}T)^{-1}\,\partial_v$ for constant fields.

sense that no additional data (arbitrary forms of the Lagrange multiplier fields) need be introduced into the theory.

4.3 Renormalisation and solution of the SI-2PI Hartree-Fock gap equations

The renormalisation of the SI-2PIEA in the Hartree-Fock approximation follows the same procedure as in section 3.4 with minor alterations. The gap equations (3.72)-(3.76) are the same, as are the finite versions (3.75)-(3.76), reproduced here:

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{\hbar\lambda}{6}(N+1)\,\mathcal{T}_G^{\text{fin}} + \frac{\hbar\lambda}{6}\mathcal{T}_H^{\text{fin}} + \mathcal{O}\left(\lambda^2\right), \tag{4.15}$$
$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{\hbar\lambda}{6}(N-1)\,\mathcal{T}_G^{\text{fin}} + \frac{\hbar\lambda}{2}\mathcal{T}_H^{\text{fin}} + \mathcal{O}\left(\lambda^2\right), \tag{4.16}$$

however the vev equation (3.74) is replaced by Goldstone's theorem

$$0 = v m_G^2, \tag{4.17}$$

which does not require renormalisation (more precisely, the WI is multiplicatively renormalised, however since $\mathcal{W}_c^A = 0$ an overall factor makes no difference). The counter-terms δm_0^2 and $\delta\lambda_0$ do not enter into the renormalisation of the equations of motion and Z_Δ is again redundant. Altogether there are six remaining constants Z, δm_1^2, and $\delta\lambda_{1,2}^{A,B}$ which are constrained by eight independent equations. That there is a solution is a non-trivial check on the calculations and one find (for details refer to the Mathematica [99] notebook in Appendix C)

$$Z = Z_\Delta = 1, \tag{4.18}$$
$$\delta m_1^2 = -\frac{\hbar\lambda}{6}(N+2)\left(c_0\Lambda^2 + c_1 m^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right)\right)\frac{\delta\lambda_1^A + \lambda}{\delta\lambda_1^B + \lambda}, \tag{4.19}$$
$$\delta\lambda_1^A = \delta\lambda_2^A = \left(1 + \frac{3(N+2)}{6 + c_1(N+2)\lambda\hbar\ln\left(\frac{\Lambda^2}{\mu^2}\right)}\right)\delta\lambda_1^B, \tag{4.20}$$
$$\delta\lambda_1^B = \delta\lambda_2^B = \left(-1 + \frac{3}{3 + c_1\lambda\hbar\ln\left(\frac{\Lambda^2}{\mu^2}\right)}\right)\lambda, \tag{4.21}$$

which is actually identical to the expressions for the counter-terms in the unimproved 2PIEA case. This should accord with intuition: symmetry improvement is an infrared modification of the theory which does not alter the ultraviolet divergence structure at all. However, such a result is not automatically assured in a self-consistent approximation scheme such as nPIEA, since the non-linearity of the equations of motion results in a coupling of scales. Furthermore, section 4.5 shows that the SI-3PIEA truncated to two loops is *not* equivalent to the SI-2PIEA. Instead the Higgs propagator equation of motion is modified as are the counter-terms. Full self-consistency of the theory is then lost with various physical effects that will be discussed.

Remarkably there is an exact analytical solution for the SI-2PI HF approximation. Subtracting $3m_G^2$ from the m_H^2 equation to cancel the $\lambda v^2/2$ and $\hbar\lambda\mathcal{T}_H^{\rm fin}/2$ terms, and using that $m_G^2 = 0$ in the symmetry breaking regime, one finds

$$m_H^2 = \bar{m}_H^2 - \frac{\hbar\lambda}{3}(N+2)\,\mathcal{T}_G^{\rm fin} + \mathcal{O}\left(\lambda^2\right), \tag{4.22}$$

where $\mathcal{T}_G^{\rm fin} = T^2/12$ since $m_G^2 = 0$. The critical temperature occurs when $m_H^2 = 0$, giving

$$T_\star = \sqrt{\frac{12\bar{v}^2}{\hbar\,(N+2)}} + \mathcal{O}\left(\lambda^2\right), \tag{4.23}$$

which is the same as in the unimproved case (3.85). However, unlike the unimproved case, the SI-2PI solution has a second order phase transition. Indeed, using the previous expression for $T_\star$ in m_H gives

$$m_H^2 = \bar{m}_H^2\left(1 - \frac{T^2}{T_\star^2}\right) + \mathcal{O}\left(\lambda^2\right), \tag{4.24}$$

which is the expression obtained from the classic Landau theory of second order phase transitions [122]. The value of v is found by using this result in the m_G^2 equation, obtaining

$$v^2 = \bar{v}^2 - \hbar\left((N+1)\,\frac{T^2}{12} + \mathcal{T}_H^{\rm th}\right) + \mathcal{O}\left(\lambda\right), \tag{4.25}$$

where m_H is the mass appearing in $\mathcal{T}_H^{\rm th}$. At the critical temperature

$$v^2 = 0 + \mathcal{O}\left(\lambda\right), \tag{4.26}$$

as expected. v^2 smoothly interpolates between $\bar{v}^2$ at $T = 0$ and zero at $T = T_\star$, then remains at zero at all higher temperatures. There is no unphysical metastable phase with broken symmetry at high temperature. This agrees with the behaviour of the $\mathrm{O}\left(N\right)$ model in lattice simulations (see, e.g. [117, 118]).

The solution of the SI-2PI gap equations in the Hartree-Fock approximation is shown in Figures 4.1, 4.2 and 4.3 using the parameters $N = 4$, $\bar{v} = 93$ MeV and $\bar{m}_H = 500$ MeV to enable direct comparison to the results of Chapter 3 (which have been reproduced here for ease of comparison). One sees that in the SI-2PI HF approximation the unphysical metastable phase is removed and replaced by a continuous (2nd order) phase transition. Furthermore the Goldstone theorem is satisfied, as expected, in the broken phase, in contrast to the unimproved 2PI solution. This behaviour has been noted before in the literature ([1] in the case $N = 2$, [116] in the case $N = 4$, and [2] and here for general N).

These results can be contrasted with the large N approximation discussed in section 3.6 and [116]. A second order phase transition is found and Goldstone's theorem is satisfied in both schemes. However, the large N estimate for the critical temperature is larger than the symmetry improved value by a factor $\sqrt{(N+2)/N}$ which is $\sqrt{3/2} \approx 1.22$ for $N = 4$,

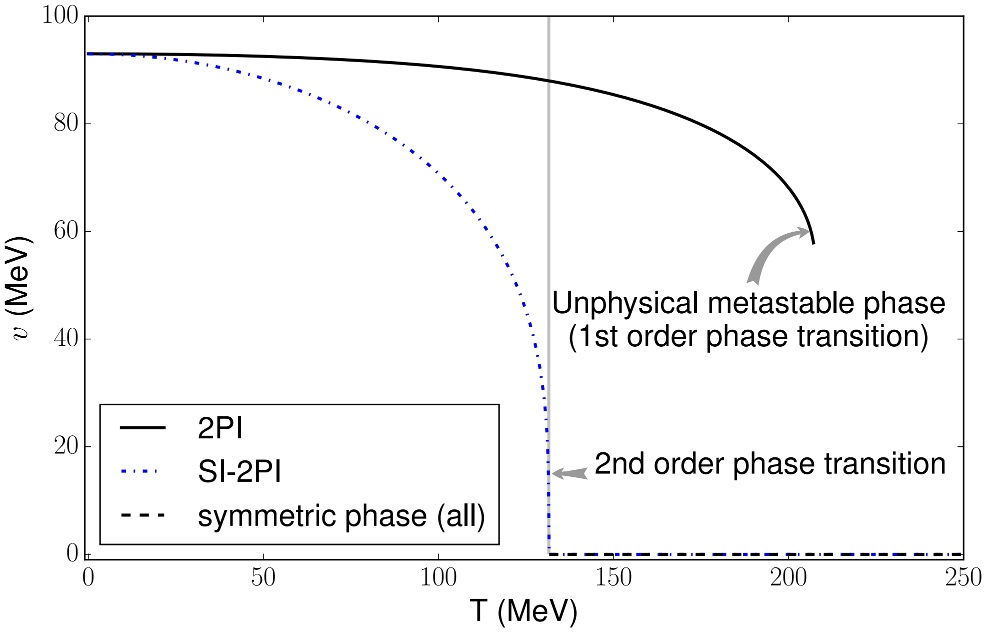


Figure 4.1: Field expectation value v from the unimproved 2PI (solid) and SI-2PI (dash-dot) in the Hartree-Fock approximation with parameters $N = 4$, $\bar{v} = 93$ MeV and $\bar{m}_H = 500$ MeV.

accurate to the expected $\mathcal{O}(1/N) = 25\%$. This makes the critical temperature ≈ 162 MeV for the numerical values used previously. This difference is the due to the neglect of the Higgs boson thermal contribution, which is indeed swamped by Goldstone modes as $N \to \infty$. Thus the leading large N approximation has several qualitative similarities to the symmetry improved 2PIEA, and is quantitatively consistent with it to the expected $\mathcal{O}(1/N)$ accuracy.

There are some important qualitative differences however. First, the Goldstone theorem is lost at higher orders in $1/N$, while the SI-2PIEA explicitly enforces it at all orders. Second, the contribution of the Higgs to the phase transition thermodynamics is totally ignored in the large N approximation, while it is included fully self-consistently in the SI-2PIEA. Third, the large N approximation only resums $\langle\phi_a\phi_a\rangle$ and is not fully self-consistent in the sense that all possible Feynman diagram topologies with skeletons of a given order are fully resummed, but the SI-2PIEA computes all of the $\Delta_{ab} \propto \langle\phi_a\phi_b\rangle$ fully self-consistently and hence is a more accurate method when applicable. Finally, the large N approximation is more natural in the sense that it does not rely upon any *ad hoc* modification of the effective action, unlike the artificially imposed Ward identities of the SI-2PIEA. This means that when a large N limit exists in a given field theory, it is fairly well behaved. However, the fact that the SI-2PIEA is constrained leads to difficulties in certain truncations (see section 4.4) and out of equilibrium (see Chapter 6).

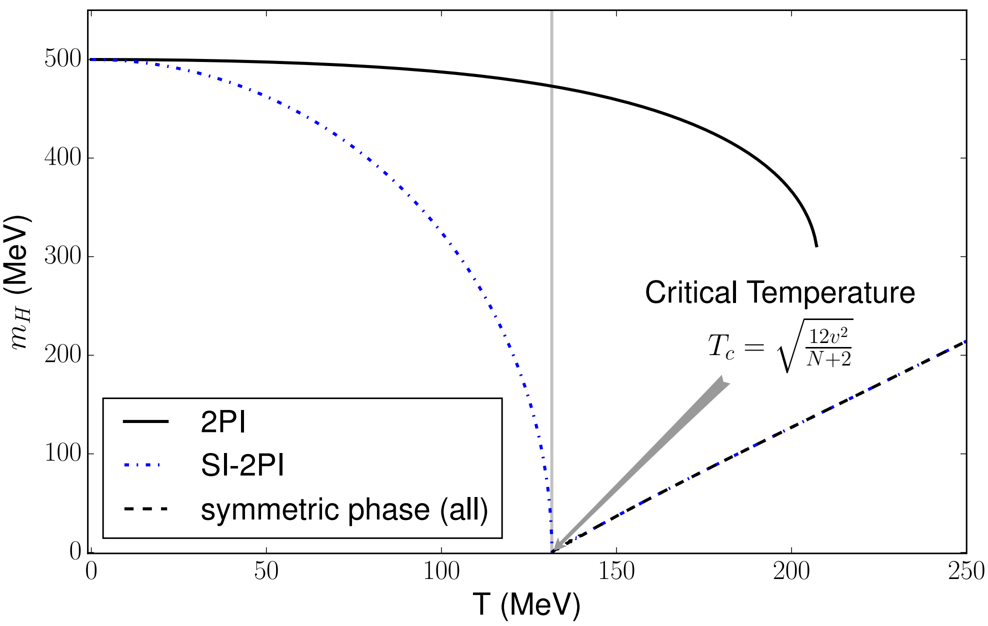


Figure 4.2: Higgs mass m_H^2 from the unimproved 2PI (solid) and SI-2PI (dash-dot) in the Hartree-Fock approximation with parameters $N = 4$, $\bar{v} = 93$ MeV and $\bar{m}_H = 500$ MeV.

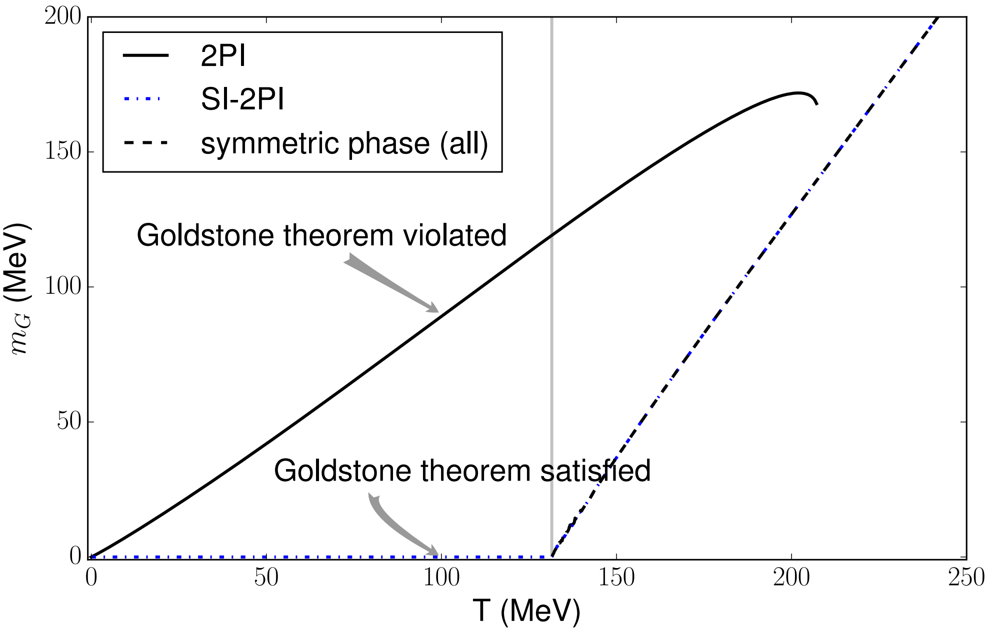


Figure 4.3: Goldstone mass m_G^2 from the unimproved 2PI (solid) and SI-2PI (dash-dot) in the Hartree-Fock approximation with parameters $N = 4$, $\bar{v} = 93$ MeV and $\bar{m}_H = 500$ MeV. Note that the Goldstone theorem is satisfied, i.e. $m_G^2 = 0$, in the broken phase for the SI-2PI solution but not the unimproved 2PI solution.

4.4 The two loop solution (or lack thereof)

In order to investigate higher orders in the SI-2PIEA it is necessary to discuss more generally the problem of renormalising nPIEA. Self-consistent equations of motion are in general far harder to renormalise than perturbation theory and several methods of renormalisation have been introduced in the literature, though the general problem is still open. The 2PIEA is known to be renormalisable, either by an implicit construction involving Bethe-Salpeter integral equations or an explicit algebraic Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ) style construction which has non-trivial consistency requirements (but which has been shown to be equivalent to the Bethe-Salpeter method) [112, 123–125]. A recent discussion of the general theory can be found in Chapter 4 of [111], which however is light on specific examples. The BPHZ style procedure of [113, 123, 124] was adapted to symmetry improved 2PIEA by [1] and to 3PIEA for three dimensional pure glue QCD (without symmetry improvement) by [104] and is the one used here.

The essence of the procedure is quite simple. Consider for example the quadratically divergent integral $\int_q i\Delta_{G/H}(q)$. Since $\Delta_{G/H}(q)$ is determined self-consistently this is a complicated integral which must be evaluated numerically. However, the UV behaviour of the propagator should approach q^{-2} as $q \to \infty$. Indeed, Weinberg's theorem [114] implies that the self-consistent propagators have this form up to powers of logarithms [113][2]. Now one can add and subtract an integral with the same UV asymptotic behaviour as the divergent integral:

$$\int_q i\Delta_{G/H}(q) = \int_q \left[i\Delta_{G/H}(q) - \frac{i}{q^2 - \mu^2 + i\epsilon} \right] + \int_q \frac{i}{q^2 - \mu^2 + i\epsilon}, \tag{4.27}$$

where μ is an arbitrary mass subtraction scale (not a cutoff scale). The first term is now only logarithmically divergent and the second term can be evaluated analytically in a chosen regularisation scheme such as dimensional regularisation. A further subtraction of this kind can render the first term finite.

The renormalised propagators can be written as

$$\Delta^{-1}_{G/H} = p^2 - m^2_{G/H} - \Sigma_{G/H}(p), \tag{4.28}$$

$$\Sigma_{G/H}(p) = \Sigma^a_{G/H}(p) + \Sigma^0_{G/H}(p) + \Sigma^r_{G/H}(p), \tag{4.29}$$

where $m_{G/H}$ is the physical mass and the (renormalised) self-energies have been separated into pieces according to their asymptotic behaviour: $\Sigma^a_{G/H}(p) \sim p^2 (\ln p)^{d_1}$, $\Sigma^0_{G/H} \sim (\ln p)^{d_2}$ and $\Sigma^r_{G/H} \sim p^{-2}$ as $p \to \infty$ respectively. The pole condition requires

$$\Sigma_{G/H}\left(p^2 = m^2_{G/H}\right) = 0. \tag{4.30}$$

[2] Though note that renormalisation group theory shows the true large-momentum behaviour of the propagators is a power law with an anomalous dimension. This implies that a truncated nPIEA does not effect a resummation of large logarithms, and that a 2PI (or indeed nPI) expansion is generally only valid within a finite energy range about the renormalisation point.

It is useful to introduce the auxiliary propagator $\Delta^{\mu}_{G/H} = \left(p^2 - \mu^2 - \Sigma^{a}_{G/H}(p)\right)^{-1}$. Then $\Delta_{G/H}$ can be expanded in $\Delta^{\mu}_{G/H}$ as

$$\begin{aligned}\Delta_{G/H}(p) = \Delta^{\mu}_{G/H}(p) + \left[\Delta^{\mu}_{G/H}(p)\right]^2 \left(m^2_{G/H} - \mu^2 + \Sigma^{0}_{G/H} + \Sigma^{r}_{G/H}\right) \\ + \mathcal{O}\left(\left[\Delta^{\mu}_{G/H}(p)\right]^3 \left[\Sigma^{0}_{G/H}(p)\right]^2\right). \end{aligned} \tag{4.31}$$

Proceeding systematically in this manner allows one to extract the leading order asymptotics of diagrams as $p \to \infty$. In the two loop truncation it turns out that $\Sigma^{a}_{G/H}(p) = 0$ so one can simply use a common Δ^{μ}.

Substituting the expressions for bare fields and parameters in terms of the renormalised fields and parameters according to

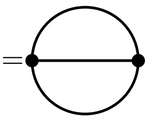
$$(\phi, \varphi, v) \to Z^{1/2}(\phi, \varphi, v), \tag{4.32}$$

$$m^2 \to Z^{-1} Z_{\Delta}^{-1}\left(m^2 + \delta m^2\right), \tag{4.33}$$

$$\lambda \to Z^{-2}(\lambda + \delta\lambda), \tag{4.34}$$

$$\Delta \to Z Z_{\Delta} \Delta, \tag{4.35}$$

with separate counter-terms $\delta m^2_{0,1}, \delta\lambda_0, \delta\lambda^{A,B}_{1,2}$ as done before for the Hartree-Fock truncation in section 3.4, with an additional counter-term $\delta\lambda$ for the sunset diagram[3]

$\Phi_2 =$ , (4.36)

gives the renormalised effective action in momentum space

$$\begin{aligned}\Gamma = \int_x \left(-Z_{\Delta}^{-1}\frac{m^2 + \delta m_0^2}{2} v^2 - \frac{\lambda + \delta\lambda_0}{4!} v^4\right) + \frac{i\hbar}{2}(N-1)\,\mathrm{Tr}\ln\left(\Delta_G^{-1}\right) + \frac{i\hbar}{2}\mathrm{Tr}\ln\left(\Delta_H^{-1}\right) \\ - \frac{i\hbar}{2}(N-1)\int_k \left(-Z Z_{\Delta} k^2 + m^2 + \delta m_1^2 + Z_{\Delta}\frac{\lambda + \delta\lambda_1^A}{6} v^2\right)\Delta_G(k) \\ - \frac{i\hbar}{2}\int_k \left(-Z Z_{\Delta} k^2 + m^2 + \delta m_1^2 + Z_{\Delta}\frac{3\lambda + \delta\lambda_1^A + 2\delta\lambda_1^B}{6} v^2\right)\Delta_H(k) \\ + \frac{\hbar^2}{24}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] Z_{\Delta}^2 (N-1)\left(\int_k \Delta_G(k)\right)^2 \\ + \frac{\hbar^2}{24}\left[3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B\right] Z_{\Delta}^2 \left(\int_k \Delta_H(k)\right)^2 + \frac{\hbar^2}{24} 2\left(\lambda + \delta\lambda_2^A\right) Z_{\Delta}^2 (N-1)\int_k \Delta_G(k)\int_q \Delta_H(q) \\ + \frac{\hbar^2}{4}\left[\frac{(\lambda + \delta\lambda) v}{3}\right]^2 Z_{\Delta}^3 (N-1)\int_{kl} \Delta_H(k)\Delta_G(l)\Delta_G(k+l) \\ + \frac{\hbar^2}{12}\left[(\lambda + \delta\lambda) v\right]^2 Z_{\Delta}^3 \int_{kl} \Delta_H(k)\Delta_H(l)\Delta_H(k+l). \end{aligned} \tag{4.37}$$

[3]Note that the majority of the quantum field theory literature does not call this the sunset diagram, instead using that to refer to the self-energy term with two four point vertices. This thesis follows the convention that has appeared in the symmetry improvement literature, e.g. [1].

The equations of motion following from this are

$$\begin{aligned}\Delta_G^{-1}(k) = ZZ_\Delta k^2 - m^2 - \delta m_1^2 - Z_\Delta \frac{\lambda + \delta\lambda_1^A}{6} v^2 \\ - \frac{\hbar}{6}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] Z_\Delta^2 \mathcal{T}_G \\ - \frac{\hbar}{6}\left(\lambda + \delta\lambda_2^A\right) Z_\Delta^2 \mathcal{T}_H + i\hbar\left[\frac{(\lambda+\delta\lambda)v}{3}\right]^2 Z_\Delta^3 \mathcal{I}_{HG}(k),\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}\Delta_H^{-1} = ZZ_\Delta k^2 - m^2 - \delta m_1^2 - Z_\Delta \frac{3\lambda + \delta\lambda_1^A + 2\delta\lambda_1^B}{6} v^2 \\ - \frac{\hbar}{6}\left[3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B\right] Z_\Delta^2 \mathcal{T}_H - \frac{\hbar}{6}\left(\lambda + \delta\lambda_2^A\right) Z_\Delta^2 (N-1)\mathcal{T}_G \\ + \frac{i\hbar}{2}\left[\frac{(\lambda+\delta\lambda)v}{3}\right]^2 Z_\Delta^3 (N-1)\mathcal{I}_{GG}(k) + \frac{i\hbar}{2}\left[(\lambda+\delta\lambda)v\right]^2 Z_\Delta^3 \mathcal{I}_{HH}(k),\end{aligned} \tag{4.39}$$

where the (divergent) integrals are

$$\mathcal{T}_{G/H} = i\int_q \Delta_{G/H}(q), \tag{4.40}$$

$$\mathcal{I}_{AB}(k) = \int_q \Delta_A(q)\,\Delta_B(k-q), \tag{4.41}$$

where $A, B = H, G$ in the last equation.

$\mathcal{I}_{AB}(p)$ can be rendered finite by a single subtraction

$$\mathcal{I}_{AB}(p) = \mathcal{I}^\mu + \mathcal{I}_{AB}^{\text{fin}}(p), \tag{4.42}$$

where $\mathcal{I}^\mu = \int_q \left[i\Delta^\mu(q)\right]^2$. Since the propagators are written with the physical masses explicit, it is crucial to also subtract a portion of the finite piece $\mathcal{I}_{HG}^{\text{fin}}\left(m_{G/H}\right)$ so that the pole of the propagator is fixed at the physical mass of the Goldstone/Higgs propagator respectively. Making this subtraction separately allows a universal $\mathcal{I}^\mu$.

The tadpole integrals $\mathcal{T}_{G/H}$ require two subtractions each since $\int_q i\left[\Delta^\mu(q)\right]^2 \Sigma_{G/H}^0(q)$ is logarithmically divergent. To that end introduce

$$\Sigma^\mu(q) = -i\hbar\left[\frac{(\lambda+\delta\lambda)v}{3}\right]^2 Z_\Delta^3\left[\int_\ell i\Delta^\mu(\ell)\, i\Delta^\mu(q+\ell) - \mathcal{I}^\mu\right], \tag{4.43}$$

which is asymptotically the same as $\Sigma_{G/H}^0(q)$, so that $\int_q i\left[\Delta^\mu(q)\right]^2\left[\Sigma_{G/H}^0(q) - \Sigma^\mu(q)\right]$ is finite. For later convenience write

$$\int_q i\left[\Delta^\mu(q)\right]^2 \Sigma^\mu(q) = \hbar\left[\frac{(\lambda+\delta\lambda)v}{3}\right]^2 Z_\Delta^3 c^\mu. \tag{4.44}$$

Then

$$\mathcal{T}_{G/H} = \mathcal{T}^\mu - i\left(m_{G/H}^2 - \mu^2\right)\mathcal{I}^\mu + \hbar\left[\frac{(\lambda+\delta\lambda)v}{3}\right]^2 Z_\Delta^3 c^\mu + \mathcal{T}_{G/H}^{\text{fin}}, \tag{4.45}$$

where $\mathcal{T}^\mu = \int_q i\Delta^\mu(q)$. Note that $\mathcal{T}^\mu$ and c^μ are real and $\mathcal{I}^\mu$ is imaginary, so that all of the subtractions can be absorbed into real counter-terms. By dimensional analysis these can be written in terms of regularisation scheme dependent constants as

$$\mathcal{T}^\mu = c_0\Lambda^2 + c_1\mu^2 \ln\frac{\Lambda^2}{\mu^2}, \tag{4.46}$$

$$\mathcal{I}^\mu = c_2 \ln\frac{\Lambda^2}{\mu^2}, \tag{4.47}$$

$$c^\mu = a_0\left(\ln\frac{\Lambda^2}{\mu^2}\right)^2 + a_1\ln\frac{\Lambda^2}{\mu^2}. \tag{4.48}$$

The counter-terms are found by eliminating $m^2_{G/H}$ and demanding that the divergences proportional to different powers of v^2 and $\mathcal{T}^{\text{fin}}_{G/H}$ separately vanish. Further renormalisation conditions are the pole conditions $\Delta_G^{-1}(m_G) = 0$ and $\Delta_H^{-1}(m_H) = 0$ and that the counter-terms are momentum independent. This gives eight equations for the seven constants $Z, Z_\Delta, \delta m_1^2, \delta\lambda_1^A, \delta\lambda_2^A, \delta\lambda_2^B$ and $\delta\lambda$, however one of them is redundant and a solution exists. Details of this calculation are included in a Mathematica [99] notebook in Appendix C. After all subtractions the finite equations of motion are

$$\Delta_G^{-1}(k) = k^2 - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}(N+1)\lambda\mathcal{T}_G^{\text{fin}} - \frac{\hbar}{6}\lambda\mathcal{T}_H^{\text{fin}} + i\hbar\left(\frac{\lambda v}{3}\right)^2\left[\mathcal{I}_{HG}^{\text{fin}}(k) - \mathcal{I}_{HG}^{\text{fin}}(m_G)\right], \tag{4.49}$$

and

$$\begin{aligned}\Delta_H^{-1} = {} & k^2 - m^2 - \frac{\lambda}{2}v^2 - \frac{\hbar}{6}\lambda(N-1)\mathcal{T}_G^{\text{fin}} - \frac{\hbar}{2}\lambda\mathcal{T}_H^{\text{fin}} \\ & + \frac{i\hbar}{2}\left(\frac{\lambda v}{3}\right)^2(N-1)\left[\mathcal{I}_{GG}^{\text{fin}}(k) - \mathcal{I}_{GG}^{\text{fin}}(m_H)\right] + \frac{i\hbar}{2}(\lambda v)^2\left[\mathcal{I}_{HH}^{\text{fin}}(k) - \mathcal{I}_{HH}^{\text{fin}}(m_H)\right],\end{aligned} \tag{4.50}$$

and, of course, $vm_G^2 = 0$.

Note that due to the momentum dependence of the self-energies these equations are significantly more complicated than their Hartree-Fock counter-parts. No analytical solutions of these equations are possible (to the author's knowledge). Numerical solutions have been investigated in [1, 6], which have found dramatically differing results using somewhat different methods. These results will be discussed but not reproduced here. Pilaftsis and Teresi [1] renormalised using the $\overline{\text{MS}}$ scheme at zero temperature in four dimensional Euclidean space, then solved the gap equations by an iterative relaxation method in momentum space using a lattice in $|k_E|$, using rotational invariance to simplify the numerics, then performing a numerical analytic continuation back to Minkowski space using Padé approximants. They used parameter values relevant to Higgs boson physics, with $m_H = 125$ GeV and $v = 246$ GeV (though with an unphysical value of $N = 2$!). The solutions they found were well behaved at all momentum scales included in the calculation and had physically reasonable self-energies, including imaginary parts representing quantitatively accurate decay rates and reaction thresholds for processes such as $H \to GG$ and decays of off-shell states such as $G^\star \to HG$ and $H^\star \to HH^\star \to HGG$. In short, all of the results found in [1] indicate that the two loop SI-2PIEA approximation is physically well behaved.

In contrast, Markó et al. [6] used several different varieties of cutoff regularisation with three different parameter sets (chosen for illustrative, not physical, purposes, and all with $N = 2$) at zero and finite temperature in $d = 4$ Euclidean space to argue that, contrary to physical intuition and renormalisation group based experience, the IR behaviour of the SI-2PI propagators is highly sensitive to the UV structure of the regulator used to render the theory finite. The general conclusion of their investigation, not rigorously proven though supported by semi-analytical and numerical arguments, is that for any "sufficiently smooth" UV regulator[4], the SI-2PI equations of motion lack a solution in the physical infinite volume limit. Their general argument proceeds in two steps. First: since $m_G^2 = 0$ in the broken phase by construction, $\Delta_G(k)$ must have an anomalous behaviour $\sim k^{-2\alpha}$ with $\alpha < 1$ as $k^2 \to 0$, otherwise the term $\mathcal{I}_{GG}^{\text{fin}}(k)$ in the Higgs equation of motion diverges in the IR and the only solution is $v = 0$. Second: if $m_H^2 \neq 0$ and the UV regulator does not produce an anomalous dimension in Δ_G, an anomalous dimension in Δ_G is incompatible with the Goldstone equation of motion which can be rewritten as

$$\Delta_G^{-1}(k) = k^2 + i\hbar\left(\frac{\lambda v}{3}\right)^2\left[\mathcal{I}_{HG}^{\text{fin}}(k) - \mathcal{I}_{HG}^{\text{fin}}(0)\right], \tag{4.51}$$

using $\Delta_G^{-1}(k=0) = m_G^2 = 0$ to eliminate the constant and $\mathcal{T}_{G/H}^{\text{fin}}$ terms. This equation is incompatible with an anomalous dimension because $\mathcal{I}_{HG}^{\text{fin}}(k)$ has a regular (i.e. analytic) dependence on k in neighbourhood of zero[5]. Hence, in the two loop truncation of the SI-2PIEA *either* a UV regulator is chosen which produces an anomalous scaling behaviour of Δ_G at large distances, in which case the large distance behaviour of the theory is sensitive to the specific UV regulator chosen, *or* there is no physically reasonable solution for a broken symmetry phase in the infinite volume limit. While [6] does not directly address the issue, the likely reason these effects are not seen in the analysis of [1] is the latter's use of dimensional regularisation, which *does* directly alter the IR properties of the theory[6]. One should also note that there was no conclusive (numerical) observation in [6] of loss of solution for sufficiently small λ with a smooth UV regulator. Also, the behaviour of the IR regulator at higher loop orders remains an open problem.

This situation is unfortunate, although not necessarily disastrous, since there are situations in which the UV cutoff is physical (e.g. in condensed matter physics) and/or the infinite volume limit is merely an idealisation which may not be strictly required. For example, the universe could be placed in a box with no observable consequences so long as the box is sufficiently large compared to the Hubble volume. This provides a natural IR regulator which, depending on the parameter values, can allow a physically reasonable

[4]Which they define to be any regulator that does not induce an anomalous dimension of the Goldstone propagator in the IR, or in other words any UV regulator which does not directly alter the IR physics.

[5]This follows because the momentum k can be routed through the Higgs propagator which is regular in the IR, regardless of the form of Δ_G there. This is also the reason for the assumption $m_H^2 = 0$ in the argument.

[6]Dimensional regularisation requires careful treatment of the infinite volume limit since IR divergences automatically vanish in the standard version of the scheme, see e.g. [18].

solution to exist. The essential message is that the IR and UV sensitivity of solutions of the SI-2PI gap equations should be checked on a case by case basis to establish the plausibility of the results obtained.

4.5 Symmetry Improvement of the 3PIEA

This section examines the extension of symmetry improvement to the 3PIEA, closely following the work published by the author in [2]. In this section the equations of motion are derived but not solved. Solutions for the (over simplified) Hartree-Fock truncation are presented in section 4.8. Unfortunately, due to the added numerical difficulty, solutions of the more interesting (and physically informative) two and three loop truncations are left for future work. For ease of reference the explicit forms of the 3PIEA and its equations of motion reproduced from (2.74) and following are

$$\Gamma^{(3)}[\varphi, \Delta, V] = S[\varphi] + \frac{i\hbar}{2}\mathrm{Tr}\ln\Delta^{-1} + \frac{i\hbar}{2}\mathrm{Tr}\left(\Delta_0^{-1}\Delta - 1\right) + \Gamma_3[\varphi, \Delta, V], \tag{4.52}$$

where

$$\Gamma_3 = \Phi_1 - \Phi_2 + \frac{\hbar^2}{3!}V_{0abc}\Delta_{ad}\Delta_{be}\Delta_{cf}V_{def} + \Phi_3 + \Phi_4 + \Phi_5 + \mathcal{O}\left(\hbar^4\right), \tag{4.53}$$

where the Φs are

$$\begin{aligned}
\Phi_1 &= -\frac{\hbar^2}{8}W_{abcd}\Delta_{ab}\Delta_{cd}, &(4.54)\\
\Phi_2 &= \frac{\hbar^2}{12}V_{abc}V_{def}\Delta_{ad}\Delta_{be}\Delta_{cf}, &(4.55)\\
\Phi_3 &= \frac{i\hbar^3}{4!}V_{abc}V_{def}V_{ghi}V_{jkl}\Delta_{ad}\Delta_{bg}\Delta_{cj}\Delta_{eh}\Delta_{fk}\Delta_{il}, &(4.56)\\
\Phi_4 &= -\frac{i\hbar^3}{8}W_{abcd}V_{efg}V_{hij}\Delta_{ae}\Delta_{bf}\Delta_{ch}\Delta_{di}\Delta_{gj}, &(4.57)\\
\Phi_5 &= \frac{i\hbar^3}{48}W_{abcd}W_{efgh}\Delta_{ae}\Delta_{bf}\Delta_{cg}\Delta_{dh}. &(4.58)
\end{aligned}$$

The equations of motion are

$$\begin{aligned}
\frac{\delta\Gamma^{(3)}}{\delta\varphi_a} &= -J_a - K^{(2)}_{ab}\varphi_b - \frac{1}{2}K^{(3)}_{abc}\left(i\hbar\Delta_{bc} + \varphi_b\varphi_c\right), &(4.59)\\
\frac{\delta\Gamma^{(3)}}{\delta\Delta_{ab}} &= -\frac{i\hbar}{2}K^{(2)}_{ab} - \frac{i\hbar}{2}K^{(3)}_{abc}\varphi_c\\
&\quad -\frac{\hbar^2}{3!2}\left(K^{(3)}_{acd}\Delta_{ce}\Delta_{df}V_{bef} + K^{(3)}_{bcd}\Delta_{ce}\Delta_{df}V_{aef}\right) &(4.60)\\
\frac{\delta\Gamma^{(3)}}{\delta V_{abc}} &= -\frac{1}{3!}\hbar^2\Delta_{ad}\Delta_{be}\Delta_{cf}K^{(3)}_{def}, &(4.61)
\end{aligned}$$

where the left hand sides are

$$\frac{\delta\Gamma^{(3)}}{\delta\varphi_a} = \frac{\delta S}{\delta\varphi_a} + \frac{i\hbar}{2}V_{0abc}\Delta_{cb} + \frac{\hbar^2}{3!}W_{abcd}\Delta_{be}\Delta_{cf}\Delta_{dg}V_{efg} + \mathcal{O}\left(\hbar^3\right), \tag{4.62}$$

$$\frac{\delta\Gamma^{(3)}}{\delta\Delta_{ab}} = -\frac{i\hbar}{2}\left(\Delta^{-1}_{ab} + \mathrm{Tr}\Delta^{-1}_{0ab} - \Sigma_{ab}\right) + \mathcal{O}\left(\hbar^4\right), \tag{4.63}$$

$$\begin{aligned}\frac{\delta\Gamma^{(3)}}{\delta V_{abc}} &= \frac{\hbar^2}{3!}\left(V_{0def} - V_{def}\right)\Delta_{ad}\Delta_{be}\Delta_{cf} + \frac{i\hbar^3}{3!}V_{def}V_{ghi}V_{jkl}\Delta_{ad}\Delta_{bg}\Delta_{cj}\Delta_{eh}\Delta_{fk}\Delta_{il}\\ &- \frac{i\hbar^3}{4}W_{hijd}V_{efg}\left(\frac{1}{3}\Delta_{ah}\Delta_{bi}\Delta_{ej}\Delta_{df}\Delta_{cg} + \text{cyclic permutations }\ (a\to b\to c\to a)\right)\\ &+ \mathcal{O}\left(\hbar^4\right),\end{aligned} \tag{4.64}$$

It is now necessary to impose a WI on the three point vertex function. To that end introduce the symmetry improved 3PIEA

$$\tilde{\Gamma}^{(3)} = \Gamma^{(3)} - \frac{i}{2}\int_x \ell^d_A(x)\,\mathcal{W}^A_c(x)\left[P^\perp(\varphi,x)\right]_{cd} - Bf\left[\mathcal{W}^A_{cd}\right], \tag{4.65}$$

where the second term is the same as the 2PI symmetry improvement term (note that the projected version of the constraint is used to match [2]) and the third term contains the extended symmetry improvement where, recall (3.16), the WI is

$$\mathcal{W}^A_{cd} = V_{dca}T^A_{ab}\varphi_b + \Delta^{-1}_{ca}T^A_{ad} + \Delta^{-1}_{da}T^A_{ac}, \tag{4.66}$$

B is the new Lagrange multiplier and $f\left[\mathcal{W}^{A(1)}_{cd}\right]$ is an arbitrary functional which vanishes if and only if its argument vanishes. Substituting the SSB ansatz (3.30) gives

$$\tilde{\Gamma}^{(3)} = \Gamma^{(3)} - \ell\mathcal{W}_1 - Bf\left[\mathcal{W}_2\right], \tag{4.67}$$

where

$$\mathcal{W}_1 = \int_z \Delta^{-1}_G(x,z)\,v, \tag{4.68}$$

$$\mathcal{W}_2 = \int_z \bar{V}(x,y,z)\,v + \Delta^{-1}_G(x,y) - \Delta^{-1}_H(x,y), \tag{4.69}$$

are the WIs in terms of the ansatz variables.

The equations of motion are

$$\frac{\partial\Gamma^{(3)}/VT}{\partial v} = \ell_0 m^3 + B\int_{xz}\frac{\delta f}{\delta\mathcal{W}_2(x,y)}\bar{V}(x,y,z), \tag{4.70}$$

$$\frac{\delta\Gamma^{(3)}}{\delta\Delta_G(r,s)} = -B\int_{xy}\frac{\delta f}{\delta\mathcal{W}_2(x,y)}\Delta^{-1}_G(x,r)\,\Delta^{-1}_G(s,y), \tag{4.71}$$

$$\frac{\delta\Gamma^{(3)}}{\delta\Delta_H(r,s)} = B\int_{xy}\frac{\delta f}{\delta\mathcal{W}_2(x,y)}\Delta^{-1}_H(x,r)\,\Delta^{-1}_H(s,y), \tag{4.72}$$

$$\frac{\delta\Gamma^{(3)}}{\delta\bar{V}(r,s,t)} = vB\int_{xyz}\frac{\delta f}{\delta\mathcal{W}_2(x,y)}\delta^{(4)}(x-r)\,\delta^{(4)}(y-s)\,\delta^{(4)}(z-t), \tag{4.73}$$

$$\frac{\delta\Gamma^{(3)}}{\delta V_N(r,s,t)} = 0, \tag{4.74}$$

$$\mathcal{W}_1 = 0, \tag{4.75}$$

$$\mathcal{W}_2 = 0, \tag{4.76}$$

where the limiting procedure to eliminate ℓ and $\mathcal{W}_1$ is already taken as in the 2PI case. A factor of $1 = \int_z \delta^{(4)}(z-t)$ has been inserted in (4.73) for later convenience. Now a limiting procedure must be found such that the right hand sides of two of (4.71), (4.72) and (4.73) vanish. The remaining equation must be chosen so that it can be replaced by the constraint (4.76) and still give a closed system. (4.73) cannot be eliminated because there is not enough information to reconstruct $\bar{V}$ from $\Delta_{G/H}$ using $\mathcal{W}_2$ (the integral in $\mathcal{W}_2$ irreversibly loses information about $\bar{V}$). Thus one must eliminate either Δ_G or Δ_H, or else artificially restrict the form of $\bar{V}$.

The desired simplification of the equations of motion can be achieved without restricting $\bar{V}$ under the assumption that $\delta f/\delta\mathcal{W}_2(x,y)$ is a spacetime independent constant. Note that this is not required by Poincaré invariance (only the weaker condition $\delta f/\delta\mathcal{W}_2(x,y) = g\left(|x-y|^2\right)$ is mandated). This assumption will be justified in section 4.6 with the introduction of the so-called *d'Alembert formalism*. Proceed now by adopting it for the sake of argument.

Computing the left hand side of (4.73) using the explicit form of the 3PIEA (2.74) and displaying only the two loop terms explicitly gives

$$\begin{aligned}\frac{\delta\Gamma^{(3)}}{\delta\bar{V}(r,s,t)} = &-\frac{\hbar^2}{2}(N-1)\int_{xyz}\left(\bar{V}(x,y,z) + \frac{\lambda v}{3}\delta(x-y)\,\delta(x-z)\right)\\ &\times\Delta_H(x,r)\,\Delta_G(y,s)\,\Delta_G(z,t) + \mathcal{O}\left(\hbar^3\right).\end{aligned} \tag{4.77}$$

Without symmetry improvement one sets this quantity to zero, giving an equation equivalent to the usual 3PI vertex equation of motion. This equation is now modified by the symmetry improvement to

$$\begin{aligned}&-\frac{\hbar^2}{2}(N-1)\int_{xyz}\left(\bar{V}(x,y,z) + \frac{\lambda v}{3}\delta(x-y)\,\delta(x-z)\right)\\ &\qquad\times\Delta_H(x,r)\,\Delta_G(y,s)\,\Delta_G(z,t) + \mathcal{O}\left(\hbar^3\right)\\ &\qquad= vB\int_{xyz}\frac{\delta f}{\delta\mathcal{W}_2(x,y)}\delta^{(4)}(x-r)\,\delta^{(4)}(y-s)\,\delta^{(4)}(z-t).\end{aligned} \tag{4.78}$$

Convolving with the inverse propagators $\Delta_H^{-1}\Delta_G^{-1}\Delta_G^{-1}$ gives

$$\begin{aligned}&-\frac{\hbar^2}{2}(N-1)\left(\bar{V}(a,b,c) + \frac{\lambda v}{3}\delta(a-b)\,\delta(a-c)\right) + \mathcal{O}\left(\hbar^3\right)\\ &\qquad= vB\int_{xyz}\frac{\delta f}{\delta\mathcal{W}_2(x,y)}\Delta_H^{-1}(x,a)\,\Delta_G^{-1}(y,b)\,\Delta_G^{-1}(z,c),\end{aligned} \tag{4.79}$$

which can be recognised as

$$\left[B\int_{xy}\frac{\delta f}{\delta\mathcal{W}_2\left(x,y\right)}\Delta_H^{-1}\left(x,a\right)\Delta_G^{-1}\left(y,b\right)\right]\mathcal{W}_1. \tag{4.80}$$

The right hand side now vanishes due to (4.75). With the regulator $vm_G^2=\eta m^3$ in place the symmetry improvement term in the $\bar{V}$ equation of motion vanishes faster than the naive $B\delta f/\delta\mathcal{W}_2$ scaling manifest in (4.73). Schematically, the right hand side scales as $B\left(\delta f/\delta\mathcal{W}_2\right)m_H^2m_G^4v\sim B\left(\delta f/\delta\mathcal{W}_2\right)m_H^8\eta^2/v$. So long as $B\delta f/\delta\mathcal{W}_2$ does not blow up as fast as η^{-2} as $\eta\to 0$ the symmetry improvement has no effect on $\bar{V}$.

Now consider the Goldstone propagator. Substituting $\Gamma^{(3)}$ into (4.71) the symmetry improved equation of motion for Δ_G is found to be

$$\Delta_G^{-1}\left(r,s\right)=\Delta_{0G}^{-1}\left(r,s\right)-\tilde{\Sigma}_G\left(r,s\right), \tag{4.81}$$

where the 3PI symmetry improved self-energy $\tilde{\Sigma}_G$ is

$$\tilde{\Sigma}_G\left(r,s\right)\equiv\frac{2i}{\hbar\left(N-1\right)}\left[\frac{\delta\Gamma_3}{\delta\Delta_G\left(r,s\right)}+B\int_{xy}\frac{\delta f}{\delta\mathcal{W}_2\left(x,y\right)}\Delta_G^{-1}\left(x,r\right)\Delta_G^{-1}\left(s,y\right)\right]. \tag{4.82}$$

Substituting in $\Gamma_3^{(3)}$ to two loop order,

$$\begin{aligned}\tilde{\Sigma}_G\left(r,s\right)=&\frac{i\hbar\lambda}{6}\mathrm{Tr}\left[\left(N+1\right)\Delta_G+\Delta_H\right]\delta^{(4)}\left(r-s\right)\\&-i\hbar\int_{abcd}\left(\bar{V}\left(a,b,r\right)+\frac{2\lambda v}{3}\delta^{(4)}\left(a-r\right)\delta^{(4)}\left(b-r\right)\right)\bar{V}\left(c,d,s\right)\Delta_H\left(a,c\right)\Delta_G\left(b,d\right)\\&+\frac{2i}{\hbar\left(N-1\right)}B\int_{xy}\frac{\delta f}{\delta\mathcal{W}_2\left(x,y\right)}\Delta_G^{-1}\left(x,r\right)\Delta_G^{-1}\left(s,y\right)+\mathcal{O}\left(\hbar^2\right).\end{aligned} \tag{4.83}$$

The first term corresponds to the Hartree-Fock diagram, the second term corresponds to the sunset diagrams and the third term is the symmetry improvement term. The equation of motion for Δ_H can be written in the same form with a suitable definition for a symmetry improved self energy $\tilde{\Sigma}_H\left(r,s\right)$, where the symmetry improvement term now has the form $\sim B\int\left(\delta f/\delta\mathcal{W}_2\right)\Delta_H^{-1}\Delta_H^{-1}$.

If $\delta f/\delta\mathcal{W}_2$ is constant, then $\tilde{\Sigma}_G$ and $\tilde{\Sigma}_H$ scale as

$$\left(B\delta f/\delta\mathcal{W}_2\right)m_G^4\sim\left(B\delta f/\delta\mathcal{W}_2\right)m_H^6\eta^2/v^2, \tag{4.84}$$

and

$$\left(B\delta f/\delta\mathcal{W}_2\right)m_H^4\sim\left(B\delta f/\delta\mathcal{W}_2\right)m_H^4\eta^0, \tag{4.85}$$

respectively. Thus, by choosing a regulator such that $B\delta f/\delta\mathcal{W}_2$ goes to a finite limit, the equations of motion for $\bar{V}$ and Δ_G are unmodified and the equation of motion for Δ_H is modified by a finite term. This is the desired limiting procedure. Adopting it gives the final set of SI-3PI equations of motion:

$$\frac{\delta\Gamma^{(3)}}{\delta\Delta_G(r,s)} = 0, \tag{4.86}$$

$$\frac{\delta\Gamma^{(3)}}{\delta\bar{V}(r,s,t)} = 0, \tag{4.87}$$

$$\frac{\delta\Gamma^{(3)}}{\delta V_N(r,s,t)} = 0, \tag{4.88}$$

$$\mathcal{W}_1 = 0, \tag{4.89}$$

$$\mathcal{W}_2 = 0. \tag{4.90}$$

That is the vev and Higgs propagator equations of motion are replaced by WIs, but the Goldstone propagator and vertex equations of motion are unmodified.

4.6 Justifying the d'Alembert formalism

The assumption that $\delta f/\delta\mathcal{W}_2$ can be consistently taken to be constant requires explanation. Constrained Lagrangian problems are generally under-specified unless some principle like d'Alembert's principle (that the constraint forces are "ideal", i.e. they do no work on the system) is invoked to specify the constraint forces. Note that while it is usually stated that enforcing constraints through Lagrange multipliers is equivalent to applying d'Alembert's principle, this is no longer automatically the case if the constraints involve a singular limit as happens in the field theory case. This leads to a real ambiguity in the procedure which requires the analyst to input physical information to resolve it. In the case of mechanical systems the analyst is expected to be able to furnish the correct form of the constraints by inspection of the system. However, the interpretation of "work" and "constraint force" in the field theory case is subtle and the appropriate generalisation is not obvious. Here it is argued, by way of a simple mechanical analogy, that the procedure which leads to the maximum simplification of the equations of motion is the correct field theory analogue of d'Alembert's principle in mechanics.

d'Alembert's principle is empirically verifiable for a given mechanical system, but here it forms part of the *definition* of the SI-3PIEA approximation scheme, which can be referred to as the "d'Alembert formalism." The result of Section 4.5 was a set of unambiguous f-independent equations of motion and constraint at some fixed order of the loop expansion, say l-loops. The use of any other limiting procedure requires the analyst to specify a spacetime function's worth of data ahead of time, representing the "work" that the constraint forces do. The resulting equations of motion represent a different formulation of the system and will have a different solution depending on the choice of "work" function.

Imagine that one is competing against another analyst to find the most accurate solution for a particular system. It is possible that the competing smart analyst could choose a work function that results in a more accurate solution, also working at l-loops. However, one could still beat the other analyst by working in the d'Alembert formalism but at higher loop order. A reasonable conjecture is that the optimum choice of work function (in the sense of guaranteeing the optimum accuracy of the resulting solution of the l-loop equations) is merely a clever repackaging of information contained in $> l$-loop corrections. (This conjecture is given without proof. Indeed it is hard to see if any alternative to the d'Alembert formalism is practicable.) Additionally the d'Alembert formalism has the virtue of being a definite procedure requiring little cleverness from the analyst, at the cost of potentially having a sub-optimal accuracy for a given loop order.

To illustrate the connection with a mechanics problem consider a classical particle in 2D constrained to $x^2 + y^2 = r^2$. The motion is uniform circular:

$$\begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos(\omega t + \phi) \\ \sin(\omega t + \phi) \end{pmatrix}. \tag{4.91}$$

The action is

$$S = \int L \mathrm{d}t - \lambda f[w], \tag{4.92}$$

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right), \tag{4.93}$$

where the constraint is $w(t) = x(t)^2 + y(t)^2 - r^2 = 0$ and $f[w] = 0$ if $w(t) = 0$. The equations of motion are

$$m\ddot{x}(t) = -2\lambda \frac{\delta f}{\delta w(t)} x(t), \tag{4.94}$$

$$m\ddot{y}(t) = -2\lambda \frac{\delta f}{\delta w(t)} y(t). \tag{4.95}$$

In this mechanics problem one could set $f[w] = \int w(t) \mathrm{d}t$ and carry through the problem in the standard way without any complications. But to mimic the field theory case, where a limiting procedure is required, take $f[w] = \int w(t)^2 \mathrm{d}t$. In this case

$$\frac{\delta f}{\delta w(t)} = 2w(t) \to 0 \text{ as } w \to 0. \tag{4.96}$$

This requires $\lambda \to \infty$ such that $\lambda \delta f / \delta w$ approaches a finite limit. Importantly, it must approach a t independent limit, otherwise an unspecified function of time enters the equations of motion: $m\ddot{x}(t) = -k(t) x(t)$, etc. This limit can be achieved by restricting the class of variations considered. Let $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, where $\delta r(t) = r(t) - r$ parametrises deviations from the constraint surface. Then $w(t) = 2r\delta r(t) + \mathcal{O}\left(\delta r^2\right)$. One needs $\dot{w}(t) = 0$ which is obviously satisfied by $\delta r(t) = \delta r$.

The present section is arguing that one only consider variations of this restricted form. The variations along the constraint surface (i.e. variations of $\theta(t)$) are unrestricted as they should be. Only variations orthogonal to the constraint surface are restricted. This is equivalent to the classical d'Alembert principle. To see this compute the second derivative of r^2 to obtain $\dot{\theta}^2 - k = \frac{\ddot{r}}{r}$. When the constraint is enforced $\ddot{r} = 0$, hence $\dot{k} \neq 0$ implies $\ddot{\theta} \neq 0$: the constraint forces are causing angular accelerations, doing work on the particle. At constant radius, the centripetal force only changes if the angular velocity changes.

In the field theory case the constraint is (4.76). For any given value of $\bar{V}$ and Δ_G only one value of Δ_H satisfies the constraint, given by

$$\Delta_H^{-1\star}(x,y) = \int_z \bar{V}(x,y,z)\, v + \Delta_G^{-1}(x,y), \tag{4.97}$$

where the $\star$ denotes the constraint solution. This is a holonomic constraint: in principle one could substitute this into the effective action directly and not worry about Lagrange multipliers at all (this is very messy analytically, though it may be numerically feasible). In the d'Alembert formalism, then, one restricts variations of Δ_H^{-1} to be of the form $\Delta_H^{-1\star}(x,y) + \delta k$, where δk is a spacetime independent constant. This guarantees

$$\frac{\delta f}{\delta \mathcal{W}_2(x,y)} = 2\mathcal{W}_2(x,y) = -2\delta k = \text{const}, \tag{4.98}$$

and all the desired simplifications go through. Variations of the other variables are unrestricted. Because the constraint force $B\delta f/\delta\mathcal{W}_2$ disappears from the Δ_G, $\bar{V}$ and V_N equations of motion the constraint "does no work" on these variables, and the other variables (v and Δ_H) are determined solely by the constraint equations. This seems a fitting field theory analogy for the classical d'Alembert principle.

4.7 Renormalisation of the SI-3PIEA

4.7.1 General comments

Before examining the renormalisation problem for the SI-3PIEA in detail a few general comments are in order. Detailed considerations follow in Sections 4.7.2 and 4.7.3 for the two and three loop truncations at zero temperature respectively. The two loop renormalisation of the theory in Section 4.7.2 is non-trivial already even though the vertex equation of motion can be solved trivially. This is because the symmetry improvement breaks the nPIEA equivalence hierarchy by modifying the Higgs equation of motion. The renormalisation of the Hartree-Fock approximation is deferred to Section 4.8 where it is discussed along with its numerical solution at finite temperature.

Generically, modifications of the equations of motion following from the 2PIEA will lead to an inconsistency of the renormalisation procedure since the 2PIEA is self-consistently complete at two loop order (in the action, i.e. one loop order in the equations of motion). However, the wavefunction and propagator renormalisation constants (normally trivial in ϕ^4 at one loop) provide the extra freedom required to obtain consistency in the present case. The theory will be renormalised at three loops in Section 4.7.3. Non-perturbative counter-term calculations are generally much more difficult than the analogous perturbative calculations, hence many of the manipulations are left in a supplemental MATHEMATICA [99] notebook (Appendix C). The results of this section are finite equations of motion for renormalised quantities which must be solved numerically. Except in the case of the Hartree-Fock approximation, the numerical implementation of the equations of motion requires a much greater numerical effort than anything else that has been attempted in this thesis and so is left to future work.

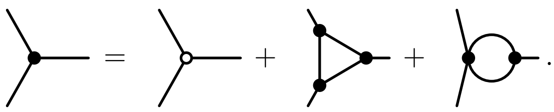
To demonstrate the renormalisability of the SI-3PI equations of motion first consider the symmetric phase since, on physical grounds SSB, an infrared phenomenon, is irrelevant to renormalisability. In the symmetric phase $v = 0$ and the Ward identity $\mathcal{W}_1$ is trivially satisfied, while $\mathcal{W}_2$ requires $\Delta_G = \Delta_H$ as expected. Further, iteration shows that vertex equations have the solution $\bar{V} = V_N = 0$ as expected on general grounds: there is no three point vertex in the symmetric phase. As a result, the symmetry improved 3PIEA in the symmetric phase is equivalent to the ordinary 2PIEA

$$\tilde{\Gamma}^{(3)}\left[\varphi = 0, \Delta, V = 0\right] = \Gamma^{(2)}\left[\varphi = 0, \Delta\right], \tag{4.99}$$

which is known to be renormalisable, as discussed in section 4.4. Thus, only divergences arising from non-zero V pose any new conceptual problems.

Now an analysis similar to the one done for the 2PIEA in section 4.4 is done to isolate the leading asymptotics for V at large momentum. This thesis follows this approach because it maps easily to that used for the SI-2PIEA above and in the literature. However, it is not known whether this procedure works in four dimensions. So far, counter-term renormalisation has not been applied to 3PIEA in four dimensions and it is unknown whether subdivergences, which are ignored here, are fully controlled in this method. In three dimensions the renormalisation theory of the 3PIEA is much simpler and this method goes through. However, the argument given here falls short of a proof of renormalisability of the SI-3PIEA in four dimensions and the general problem remains open.

Suppressing $\mathrm{O}(N)$ indices one can write $V(p_1, p_2, p_3) = \lambda v f\left(\frac{p_1}{v}, \frac{p_2}{v}, \frac{p_3}{v}\right)$ where $p_1 + p_2 + p_3 = 0$. Now $V \to 0$ as $v \to 0$ implies that $f(\chi_1, \chi_2, \chi_3) \sim \chi^{\alpha} (\ln \chi)^{d_3}$, where $\alpha < 1$ and χ is representative of the largest scale among χ_1, χ_2 and χ_3. Now consider the vertex equation of motion (4.61) and (4.64), or



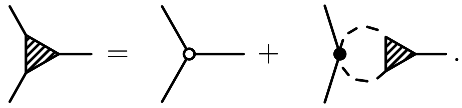
(4.100)

where the open circle vertex denotes the bare function V_{0abc} and a sum over cyclic permutations of the legs of the last diagram is understood. The triangle graph goes like[7] $\int_\ell \ell^{3\alpha-6} (\ln \ell)^{3d_3-3d_1} \sim \chi^{3\alpha-2} (\ln \chi)^{3d_3-3d_1}$ if $\alpha \neq 2/3$ or $(\ln \chi)^{1+3d_3-3d_1}$ if $\alpha = 2/3$, which is dominated by the bubble graph which goes like $\int_\ell \ell^{\alpha-4} (\ln \ell)^{d_3-2d_1} \sim \chi^{\alpha} (\ln \chi)^{d_3-2d_1}$ if $\alpha \neq 0$ or $(\ln \chi)^{1+d_3-2d_1}$ if $\alpha = 0$. Thus to a leading approximation the large momentum behaviour is obtained by dropping the triangle graph from the equation of motion. This can also be seen by taking $v \to 0$ at fixed p_i which suppresses the triangle graph relative to the bubble graph.

Now define auxiliary vertex functions $\bar{V}^{\mu}$ and V_N^{μ} which have the same asymptotic behaviour as $\bar{V}$ and V_N respectively, though depend only on the auxiliary propagators. $\bar{V}^{\mu}$ and V_N^{μ} are defined by taking the equations of motion for $\bar{V}$ and V_N, dropping the triangle graphs, and making the replacements $\bar{V} \to \bar{V}^{\mu}$, $V_N \to V_N^{\mu}$ (represented diagrammatically

[7] Recall that the asymptotic form of the self-energies as $p \to \infty$ is given by terms going like $\Sigma^{a}_{G/H}(p) \sim p^2 (\ln p)^{d_1}$ and $\Sigma^{0}_{G/H} \sim (\ln p)^{d_2}$.

by a shaded blob), and $\Delta_{G/H} \rightarrow \Delta^{\mu}_{G/H}$ (represented by dashed lines), resulting in



(4.101)

This gives a pair of coupled linear integral equations, analogous to the Bethe-Salpeter equations, for $\bar{V}^{\mu}$ and V_N^{μ} which can be solved explicitly by iteration. (Details of this calculation are presented in Section 4.7.3). Unfortunately the problem is only tractable in fewer than 1+3 dimensions, so for the present any analytical results depending on the explicit forms of $\bar{V}^{\mu}$ and V_N^{μ} are confined to this case. In 1+2 or 3 dimensions the problem simplifies dramatically as there are no divergent vertex corrections. In the physically most interesting case of 1+3 dimensions, $\bar{V}^{\mu}$ and V_N^{μ} must be numerically determined at the same time as $\bar{V}$ and V_N and it is unkown whether this renormalisation procedure actually works.

4.7.2 Two loop truncation

The theory simplifies dramatically at two loop order. It follows from (4.64) that at this order $V \rightarrow V_0$ up to a renormalisation (this is just the equivalence hierarchy at work). Substituting this into the action gives, apart from the symmetry improvement terms, the standard 2PIEA. Another simplification is that the logarithmic enhancement of the propagators in the UV due to $\Sigma^a_{G/H}$ vanishes at this level ($\Sigma^a_{G/H}$ is generated by the diagram Φ_5 appearing at three loop order). In this case $\Delta^{\mu}_G = \Delta^{\mu}_H \equiv \Delta^{\mu} = \left(p^2 - \mu^2\right)^{-1}$. However, the reduction is not trivial because now the Higgs equation of motion has been replaced by a Ward identity. The equations of motion reduce to

$$\begin{aligned}
\Delta_G^{-1}(x,y) = &-\left(\partial_\mu \partial^\mu + m^2 + \frac{\lambda}{6} v^2\right) \delta^{(4)}(x-y) \\
&- \frac{i\hbar}{6}(N+1)\lambda \Delta_G(x,x)\,\delta^{(4)}(x-y) \\
&- \frac{i\hbar}{6}\lambda \Delta_H(x,x)\,\delta^{(4)}(x-y) \\
&- \frac{i\hbar}{9}\lambda^2 v^2 \Delta_H(x,y)\,\Delta_G(x,y), \qquad (4.102)
\end{aligned}$$

$$\Delta_H^{-1}(x,y) = -\frac{\lambda v^2}{3}\delta^{(4)}(x-y) + \Delta_G^{-1}(x,y), \qquad (4.103)$$

$$v m_G^2 = 0. \qquad (4.104)$$

The first line is the tree level term, the second and third lines are the Hartree-Fock self-energies, the fourth line is the sunset self-energy, and the last two lines are the Ward identities $\mathcal{W}_2$ and $\mathcal{W}_1$ respectively.

To renormalise the theory one follows the by now standard procedure by introducing the renormalised parameters

$$(\phi, \varphi, v) \to Z^{1/2} (\phi, \varphi, v) \,, \tag{4.105}$$

$$m^2 \to Z^{-1} Z_\Delta^{-1} \left(m^2 + \delta m^2\right) , \tag{4.106}$$

$$\lambda \to Z^{-2} (\lambda + \delta\lambda) \,, \tag{4.107}$$

$$\Delta \to Z Z_\Delta \Delta, \tag{4.108}$$

$$V \to Z^{-3/2} Z_V V, \tag{4.109}$$

and demanding that all kinematically distinct divergences vanish. Details of this computation and explicit expressions for the counter-terms are given in Appendix C. The finite equations of motion are found to be

$$\begin{aligned}\Delta_G^{-1}(p) &= p^2 - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}(N+1)\,\lambda\mathcal{T}_G^{\text{fin}} - \frac{\hbar}{6}\lambda\mathcal{T}_H^{\text{fin}} \\ &\quad + i\hbar\left(\frac{\lambda v}{3}\right)^2 \left[\mathcal{I}_{HG}^{\text{fin}}(p) - \mathcal{I}_{HG}^{\text{fin}}(m_G)\right] ,\end{aligned} \tag{4.110}$$

$$\begin{aligned}\Delta_H^{-1}(p) &= p^2 - m^2 - \frac{\lambda v^2}{3} - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}(N+1)\,\lambda\mathcal{T}_G^{\text{fin}} \\ &\quad - \frac{\hbar}{6}\lambda\mathcal{T}_H^{\text{fin}} + i\hbar\left(\frac{\lambda v}{3}\right)^2 \left[\mathcal{I}_{HG}^{\text{fin}}(p) - \mathcal{I}_{HG}^{\text{fin}}(m_H)\right] .\end{aligned} \tag{4.111}$$

The finite parts $\mathcal{T}_{G/H}^{\text{fin}}$ and $\mathcal{I}_{HG}^{\text{fin}}(p)$ are

$$\mathcal{I}_{HG}^{\text{fin}}(p) = \mathcal{I}_{HG}(p) - \mathcal{I}^\mu, \tag{4.112}$$

$$\mathcal{T}_{G/H}^{\text{fin}} = \mathcal{T}_{G/H} - \mathcal{T}^\mu + i\left(m_{G/H}^2 - \mu^2\right)\mathcal{I}^\mu - \int_q i\left[\Delta^\mu(q)\right]^2 \Sigma^\mu(q) \,, \tag{4.113}$$

where the auxiliary quantities are

$$\mathcal{T}^\mu = \int_q i\Delta^\mu(q) \,, \tag{4.114}$$

$$\mathcal{I}^\mu = \int_q \left[i\Delta^\mu(q)\right]^2 , \tag{4.115}$$

$$\Sigma^\mu(q) = -i\hbar\left(\frac{\lambda v}{3}\right)^2 \left[\int_\ell i\Delta^\mu(\ell)\, i\Delta^\mu(q+\ell) - \mathcal{I}^\mu\right] . \tag{4.116}$$

These equations are the main result of this section. They could presumably be solved numerically using techniques similar to [125], though this is deferred as the analytical results given in section 4.10 indicate this is an unsatisfactory truncation.

4.7.3 Three loop truncation

Now consider the three loop truncation of the SI-3PIEA. The vertex equation of motion (4.100) now includes one loop corrections and, as shown above, the leading asymptotics at large momentum are captured by the auxiliary vertex defined by its equation of motion (4.101). Subtracting these two equations shows that the right hand side is finite (indeed the auxiliary vertices were constructed to guarantee this). Thus the problem of renormalising the vertex equation of motion reduces to the problem of renormalising the auxiliary vertex equation of motion.

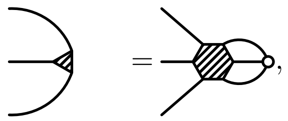
It is temporarily more convenient to go back to the $\mathrm{O}(N)$ covariant form of the equations of motion before introducing the SSB ansatz. Introduce the covariant auxiliary vertex V^{μ}_{abc} which is related to $\bar{V}^{\mu}$ and V^{μ}_{N} by the analogue of (3.32). Iterating the auxiliary vertex equation of motion gives the solution

$$V^{\mu}_{abc} = \mathcal{K}_{abcdef} V_{0def}, \tag{4.117}$$

where the six point kernel $\mathcal{K}_{abcdef}$ obeys the Bethe-Salpeter like equation

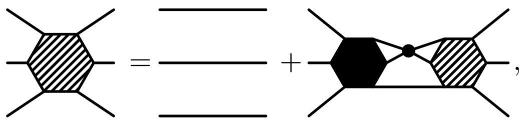
$$\mathcal{K}_{abcdef} = \delta_{ad}\delta_{be}\delta_{cf} + \frac{1}{3!}\sum_{\pi}\left(-\frac{3i\hbar}{2}\right)\delta_{\pi(a)h}W_{\pi(b)\pi(c)kg}\Delta^{\mu}_{ki}\Delta^{\mu}_{gj}\mathcal{K}_{hijdef}, \tag{4.118}$$

where $\sum_{\pi}$ is a sum over permutations of the incoming legs. Graphically this is



(4.119)

where the shaded hexagonal blob is $\mathcal{K}$ which obeys



(4.120)

where the solid hexagonal blob is the sum over all permutations of the lines coming in from the left and connecting to those on the right. (4.118) can be written in a form that makes explicit all divergences (see Appendix A) and replaces the bare vertex W by a four point kernel $\mathcal{K}^{(4)}_{abcd} \sim \lambda/\left(1 + \lambda\mathcal{I}^{\mu}\right)$.

In fewer than four dimensions $\mathcal{K}^{(4)}_{abcd}$ is finite and every correction to the tree level value is asymptotically sub-dominant. Thus the leading term at large momentum is the tree level term and, instead of the full auxiliary vertex as defined above, one can simply take $V^{\mu}_{abc} = V_{0abc}$, dramatically simplifying the renormalisation theory. A similar simplification happens to the auxiliary propagator due to the logarithmic (rather than quadratic) divergence of the Φ_5-generated self-energy in $< 1 + 3$ dimensions. This confirms statements made in the literature (supported by numerical evidence though without proof, to the author's knowledge) to the effect that the asymptotic behaviour of Green functions is free (e.g. [104]).

Unfortunately, the situation is much more difficult in four dimensions and the renormalisation of the nPIEA for $n \geq 3$ in $d > 3$ remains an open problem, both in general and in the present case. The problem can be seen from the behaviour of the auxiliary vertex which is discussed further in Appendix A. For the sake of obtaining analytical results the rest of this section is restricted to $< 1+3$ dimensions. Because of the extra complication of the $d > 3$ case a full numerical implementation is required before one can "get off the ground," so to speak, so the renormalisation of the $1+3$ dimensional case is left to future work.

The counter-terms for $1+2$ dimensions are derived following essentially the same procedure used several times now already. Details are given in Appendix B. The are only two interesting comments about this derivation: the first is that an additional (non-universal) counter-term is required for the sunset graph linear in V; the second is that only δm_1^2 is required to UV-renormalise the theory. This is consistent with the known fact that ϕ^4 theory is *super-renormalisable* in 2+1 dimensions, that is, only a finite number of primitive divergent graphs exist and these graphs only contribute to the mass shift. Every other counter-term is finite and exists solely to maintain the pole condition for the Higgs propagator despite the vertex Ward identity. The resulting finite equations of motion are

$$\Delta_G^{-1} = -\left(\partial_\mu \partial^\mu + m^2 + \frac{\lambda}{6}v^2\right) - \left[\Sigma_G^0(p) - \Sigma_G^0(m_G)\right], \tag{4.121}$$

for the Goldstone propagator,

$$\begin{aligned}\bar{V} = &-\frac{\lambda v}{3} + i\hbar\left[V_N\left(\bar{V}\right)^2(\Delta_H)^2\Delta_G + \left(\bar{V}\right)^3\Delta_H(\Delta_G)^2\right] \\ &+ \frac{i\hbar\lambda}{6}\left[V_N(\Delta_H)^2 + (N+1)\bar{V}(\Delta_G)^2 + 4\bar{V}\Delta_G\Delta_H\right],\end{aligned} \tag{4.122}$$

for the Higgs-Goldstone-Goldstone vertex, and

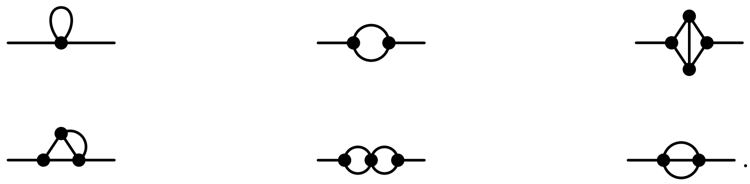
$$\begin{aligned}V_N = &-\lambda v + i\hbar\left[(N-1)\left(\bar{V}\right)^3(\Delta_G)^3 + (V_N)^3(\Delta_H)^3\right] \\ &+ \frac{i\hbar\lambda}{2}\left[(N-1)\bar{V}\Delta_G\Delta_G + 3V_N(\Delta_H)^2\right],\end{aligned} \tag{4.123}$$

for the triple Higgs vertex.

The finite Goldstone self-energy is

$$\begin{aligned}-\Sigma_G^0(p) = &-\frac{\hbar}{6}(N+1)\lambda\left(\mathcal{T}_G - \mathcal{T}^\mu\right) - \frac{\hbar}{6}\lambda\left(\mathcal{T}_H - \mathcal{T}^\mu\right) \\ &- i\hbar\left[-2\frac{\lambda v}{3} - \bar{V}\right]\Delta_H\Delta_G\bar{V} + \hbar^2\left[V_N\left(\bar{V}\right)^3(\Delta_H)^3(\Delta_G)^2 + \left(\bar{V}\right)^4\Delta_H\Delta_H(\Delta_G)^3\right] \\ &+ \frac{\hbar^2\lambda}{3}\left[\bar{V}V_N(\Delta_H)^3\Delta_G + (N+1)\bar{V}\bar{V}\Delta_H(\Delta_G)^3 + 3\bar{V}\bar{V}(\Delta_G)^2\Delta_H\Delta_H\right] \\ &+ \frac{\hbar^2\lambda^2}{18}\left[(N+1)(\Delta_G)^3 + \Delta_H\Delta_H\Delta_G - (N+2)\mathcal{B}^\mu\right],\end{aligned} \tag{4.124}$$

where the BBALL integral is $\mathcal{B}^{\mu} = \int_{qp} \Delta^{\mu}(q)\, \Delta^{\mu}(p)\, \Delta^{\mu}(p+q)$. The graph topologies appearing in this equation are



Finally, the Higgs equation of motion is

$$\Delta_{H}^{-1}(p) = \left(m_{G}^{2} + \Sigma_{G}^{0}(m_{H}) - m_{H}^{2}\right) \frac{\bar{V}(p, -p, 0)}{\bar{V}(m_{H}, -m_{H}, 0)} + \Delta_{G}^{-1}(p). \tag{4.125}$$

The unusual form of this equation is a result of the pole condition $\Delta_{H}^{-1}(m_{H}) = 0$. As discussed previously, numerical implementation of these equations is deferred to future work.

4.8 Solution of the SI-3PI Hartree-Fock approximation

In the Hartree-Fock approximation one drops the $\mathcal{I}_{HG}(p)$ term in the two loop equations of motion, or equivalently drops the sunset diagram. In this case the problem simplifies dramatically because the self-energy is momentum independent. The machinery of the auxiliary propagators introduced previously is now unnecessary and $\mathcal{T}_{G/H} = \mathcal{T}_{G/H}^{\infty} + \mathcal{T}_{G/H}^{\text{fin}}$ can be written as the sum of divergent and finite parts which can be evaluated in closed form. Renormalisation follows through almost exactly as in the unimproved and SI-2PI HF cases, with the resulting renormalisation constants (details in Appendix C)

$$Z = Z_{\Delta} = 1, \tag{4.126}$$

$$\delta m_{1}^{2} = -\frac{(N+2)\,\hbar\lambda\left(c_{0}\Lambda^{2} + c_{1}m^{2}\ln\left(\frac{\Lambda^{2}}{\mu^{2}}\right)\right)}{6 + c_{1}(N+2)\,\hbar\lambda\ln\left(\frac{\Lambda^{2}}{\mu^{2}}\right)}, \tag{4.127}$$

$$\delta\lambda_{1}^{A} = -\frac{c_{1}(N+4)\,\hbar\lambda^{2}\ln\left(\frac{\Lambda^{2}}{\mu^{2}}\right)}{6 + c_{1}(N+2)\,\hbar\lambda\ln\left(\frac{\Lambda^{2}}{\mu^{2}}\right)}, \tag{4.128}$$

$$\delta\lambda_{2}^{A} = \delta\lambda_{2}^{B} = \frac{N+2}{N+4}\delta\lambda_{1}^{A}. \tag{4.129}$$

Note that the undetermined constant $\delta\lambda$ can be consistently set to zero at this order. The finite equations of motion are

$$m_{G}^{2} = m^{2} + \frac{\lambda}{6}v^{2} + \frac{\hbar\lambda}{6}(N+1)\,\mathcal{T}_{G}^{\text{fin}} + \frac{\hbar\lambda}{6}\mathcal{T}_{H}^{\text{fin}}, \tag{4.130}$$

$$m_{H}^{2} = \frac{\lambda v^{2}}{3} + m_{G}^{2}, \tag{4.131}$$

$$v m_{G}^{2} = 0. \tag{4.132}$$

Finally, as before, requiring the zero temperature tree level relation $v^2(T=0) \equiv \bar{v}^2 = -6m^2/\lambda$ sets the renormalisation point $\mu^2 = \bar{m}_H^2 \equiv m_H^2(T=0) = \lambda\bar{v}^2/3$. These are to be contrasted with the SI-2PI equations of motion (4.15)-(4.17) reproduced here

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{\hbar\lambda}{6}(N+1)\,\mathcal{T}_G^{\rm fin} + \frac{\hbar\lambda}{6}\mathcal{T}_H^{\rm fin}, \tag{4.133}$$

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{\hbar\lambda}{6}(N-1)\,\mathcal{T}_G^{\rm fin} + \frac{\hbar\lambda}{2}\mathcal{T}_H^{\rm fin}, \tag{4.134}$$

$$v m_G^2 = 0. \tag{4.135}$$

Note that only the Higgs equation of motion differs from the SI-2PI case as expected. These equations of motion possess a phase transition and a critical point where $m_H^2 = m_G^2 = v^2 = 0$ with the same value of the critical temperature

$$T_\star = \sqrt{\frac{12\bar{v}^2}{\hbar(N+2)}}, \tag{4.136}$$

independent of the formalism used (apart from large N). However, the order of the phase transition differs in the three cases.

Numerical solutions of the SI-3PIEA in the Hartree-Fock approximation are presented in Figures 4.4, 4.5 and 4.6 with the same parameters as before, namely $N = 4$, $v = 93$ MeV and $\bar{m}_H = 500$ MeV. The solution is implemented in PYTHON as an iterative root finder based on *scipy.optimize.root* [126] with an estimated Jacobian or, if that fails to converge, a direct iteration of the gap equations. The Bose-Einstein integrals are precomputed to save time.

Figure 4.4 shows $v(T)$, the order parameter of the phase transition. Below the critical temperature there is a broken phase with $v \neq 0$, but the symmetry is restored when $v = 0$ above the critical temperature. Note, however, that the unimproved 2PIEA and symmetry improved 3PIEA have unphysical metastable broken phases at $T > T_\star$, signalling a first order phase transition. The symmetry improved 2PIEA correctly predicts the second order nature of the phase transition. Though unphysical, the symmetry improved 3PIEA behaviour is much more reasonable than the unimproved 2PIEA in that the strength of the first order phase transition is greatly reduced and the metastable phase ceases to exist at a temperature much closer to the critical temperature than for the unimproved 2PIEA. Figure 4.5 shows the Higgs mass $m_H(T)$. The same phase transition behaviour is seen again, and again all three methods agree in the symmetric phase, giving the usual thermal mass effect. Finally, Figure 4.6 shows the Goldstone boson mass. The unimproved 2PIEA strongly violates the Goldstone theorem, but both symmetry improvement methods satisfy it as expected. Note that the Goldstone theorem is even satisfied in the unphysical metastable phase predicted by the symmetry improved 3PIEA. All three methods correctly predict $m_G = m_H$ in the symmetric phase.

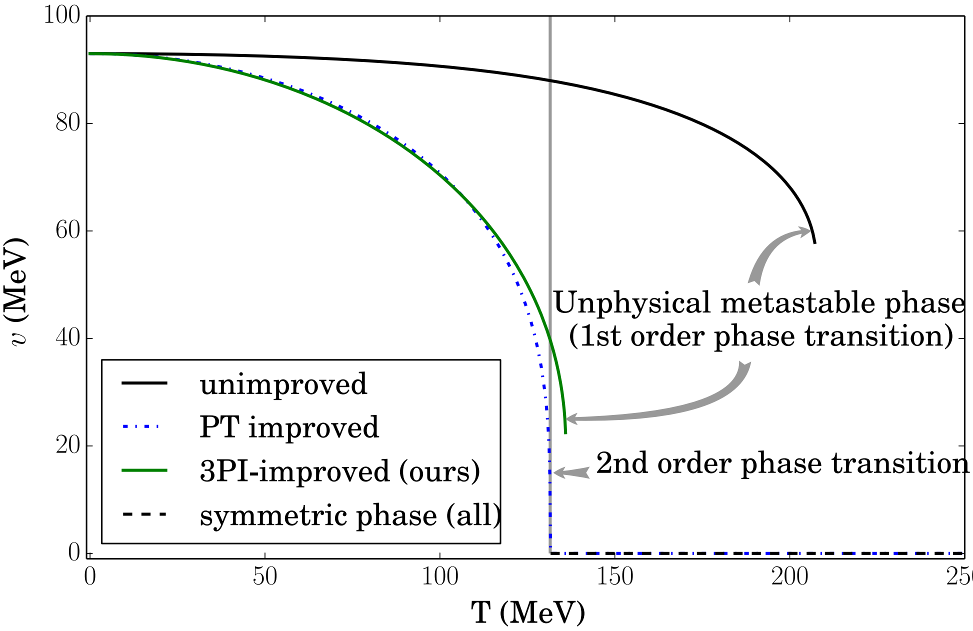


Figure 4.4: Expectation value of the scalar field $v = \langle \phi \rangle$ as a function of temperature T computed in the Hartree-Fock approximation using the unimproved 2PIEA (solid black), the Pilaftsis and Teresi symmetry improved 2PIEA (dash dotted blue) and the symmetry improved 3PIEA (solid green). In the symmetric phase (dashed black) all methods agree. The vertical grey line at $T \approx 131.5$ MeV corresponds to the critical temperature which is the same in all methods.

4.9 Two dimensions and the Coleman-Mermin-Wagner theorem

The Coleman-Mermin-Wagner theorem [127], which has been interpreted as a breakdown of the Goldstone theorem (e.g. [42]), is a general result stating that the spontaneous breaking of a continuous symmetry is impossible in $d = 2$ or $d = 1 + 1$ dimensions. This occurs due to the infrared divergence of the massless scalar propagator in two dimensions. The symmetry improved gap equations satisfy this theorem despite the direct imposition of Goldstone's theorem. Thus symmetry improvement passes a test that any robust quantum field theoretical method must satisfy. (Note that symmetry improvement is not *required* to obtain consistency of nPIEA with the Coleman-Mermin-Wagner theorem, but neither does it ruin it.)

The general statement of the result is that $\int_x \Sigma(x, 0)$ diverges in the IR whenever massless particles appear in loops in $d = 2$. Since by the Dyson equation Goldstone's theorem can be written $0 = \left(m_G^2 + \int_x \Sigma_G(x, 0)\right) v$, this implies $v = 0$ and a mass gap is generated. This can be shown explicitly using the Hartree-Fock gap equations where, in two dimensions,

$$\mathcal{T}_a^{\text{vac}}\left(\overline{\text{MS}}\right) = -\frac{1}{4\pi} \ln\left(\frac{m_a^2}{\mu^2}\right). \tag{4.137}$$

(Note that the renormalisation can be carried through without difficulty in two dimensions.

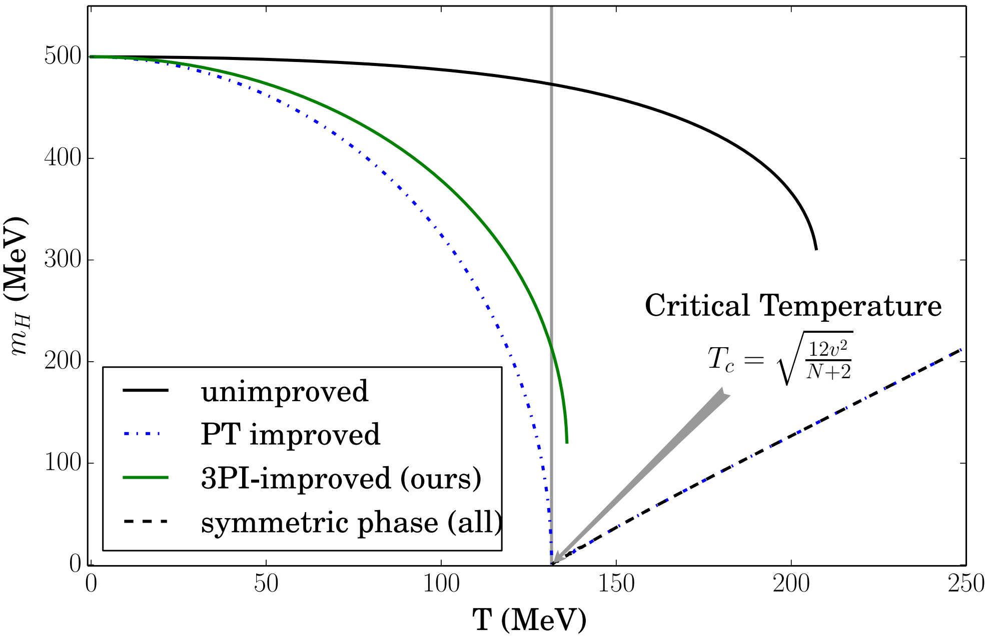


Figure 4.5: The Higgs mass m_H as a function of temperature T computed in the Hartree-Fock approximation using the unimproved 2PIEA (solid black), the Pilaftsis and Teresi symmetry improved 2PIEA (dash dotted blue) and the symmetry improved 3PIEA (solid green). In the symmetric phase (dashed black) all methods agree. The vertical grey line at $T \approx 131.5$ MeV corresponds to the critical temperature which is the same in all methods.

Only the δm_1^2 counter-term is needed.) One must show that the gap equations possess no solution for $m_G^2 = 0$. It is clear that if $v \neq 0$, $\mathcal{T}_G^{\mathrm{vac}}$ diverges as $m_G^2 \to 0$ if μ is taken as a constant, and $\mathcal{T}_H^{\mathrm{vac}}$ diverges if one takes $\mu^2 \propto m_G^2$ as $m_G^2 \to 0$. Either way there is no solution. At finite temperature the Bose-Einstein integral (3.78) also has an infrared divergence as $m_a \to 0$ which does not cancel against the singularity of the vacuum term. It can be shown that the singularity is due to the Matsubara zero mode.

For $v = 0$ on the other hand, the gap equations reduce to

$$m_H^2 = m_G^2 = m^2 - \frac{1}{4\pi}\frac{\hbar}{6}(N+2)\,\lambda \ln\left(\frac{m_G^2}{\mu^2}\right), \tag{4.138}$$

which always has a positive solution. If $m^2 > 0$ then one can choose the renormalisation point $\mu^2 = m_G^2$ so that the tree level relationship $m_G^2 = m^2$ holds. If $m^2 < 0$ a positive mass is dynamically generated and one requires a renormalisation point $\mu^2 > m_G^2 \exp\left(\frac{24\pi|m^2|}{\hbar(N+2)\lambda}\right)$ non-perturbatively large in the ratio $\lambda/\left|m^2\right|$, reflecting the fact that perturbation theory is bound to fail in this case.

4.10 Optical theorem and dispersion relations

In this section the analytic structure of propagators and self-energies is studied in the symmetry improved 3PI formalism. A physical quantity of particular interest is the decay

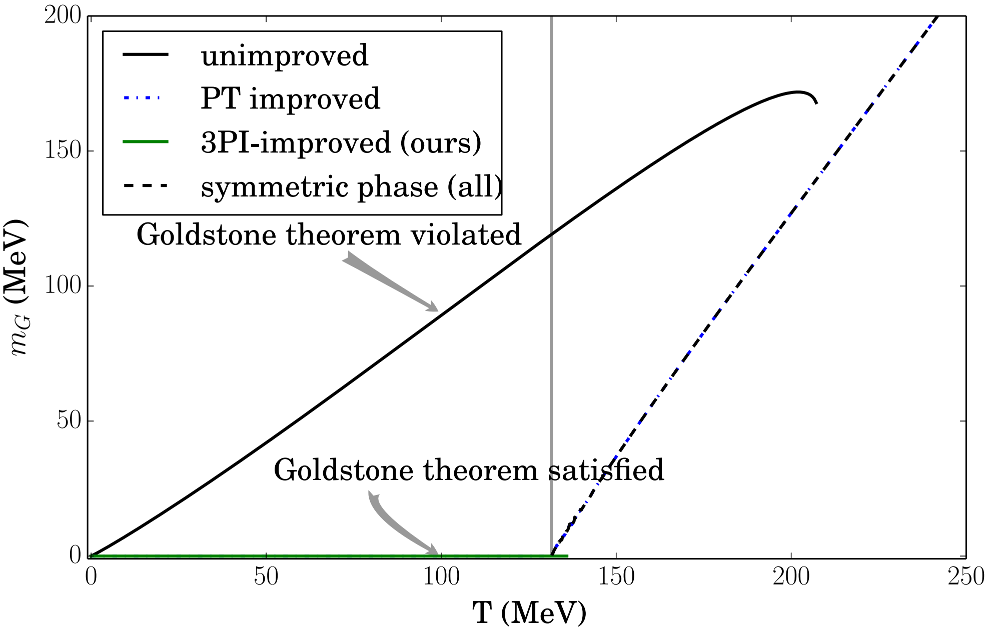


Figure 4.6: The Goldstone mass m_G as a function of temperature T computed in the Hartree-Fock approximation using the unimproved 2PIEA (solid black), the Pilaftsis and Teresi symmetry improved 2PIEA (dash dotted blue) and the symmetry improved 3PIEA (solid green). In the symmetric phase (dashed black) all methods agree. The vertical grey line at $T \approx 131.5$ MeV corresponds to the critical temperature which is the same in all methods.

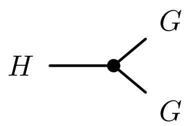
width Γ_H of the Higgs, which is dominated by decays to two Goldstone bosons. Γ_H is given by the optical theorem in terms of the imaginary part of the self-energy evaluated on-shell (see, e.g. [36] Chapter 7):

$$-m_H\Gamma_H = \mathrm{Im}\Sigma_H\left(m_H\right). \tag{4.139}$$

(This is valid so long as $\Gamma_H \ll m_H$, otherwise the full energy dependence of $\Sigma_H\left(p\right)$ must be taken into account.) The standard one loop perturbative result gives (see, e.g., the "Kinematics" review in [8])

$$\Gamma_H = \frac{N-1}{2}\frac{\hbar}{16\pi m_H}\left(\frac{\lambda v}{3}\right)^2, \tag{4.140}$$

which comes entirely from the Goldstone loop sunset graph. Each part of this expression has a simple interpretation in relation to the tree level decay graph



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The $N-1$ is due to the sum over final state Goldstone flavours, the factor of $1/2$ is due to the Bose statistics of the two particles in the final state, the $\hbar/16\pi m_H$ is due to the final state phase space integration and the $(\lambda v/3)^2$ is the absolute square of the invariant decay amplitude.

The Hartree-Fock approximation fails to reproduce this result regardless of the use or not of symmetry improvement. This is because there is no self-energy apart from a mass correction. Thus the Hartree-Fock approximation always predicts that the Higgs is stable. Attempts to repair the Hartree-Fock approximation through the use of an external propagator lead to a non-zero but still incorrect result. This is because an unphysical value of m_G still appears in loops. Satisfactory results are obtained within the symmetry improved 2PI formalism for both on- and off-shell Higgs [1] (so long as the two loop solutions exist). Here it is shown that the symmetry improved 3PIEA can not yield a satisfactory value for Γ_H at the two loop level.

From the Higgs equation of motion (4.125) the two loop truncation gives

$$\begin{aligned}\mathrm{Im}\Sigma_H(p) &= \frac{\hbar}{6}(N+1)\lambda\mathrm{Im}\mathcal{T}_G^{\mathrm{fin}} + \frac{\hbar}{6}\lambda\mathrm{Im}\mathcal{T}_H^{\mathrm{fin}} - \hbar\left(\frac{\lambda v}{3}\right)^2\mathrm{Im}\left[i\mathcal{I}_{HG}^{\mathrm{fin}}(p)\right] \\ &= \frac{\hbar}{6}(N+1)\lambda\mathrm{Im}\mathcal{T}_G + \frac{\hbar}{6}\lambda\mathrm{Im}\mathcal{T}_H - \hbar\left(\frac{\lambda v}{3}\right)^2\mathrm{Im}\left[i\mathcal{I}_{HG}(p)\right], \end{aligned} \tag{4.141}$$

which can be written in terms of the un-subtracted $\mathcal{T}_{G/H}$ and $\mathcal{I}_{HG}$ because all of the subtractions are manifestly real. Now it is shown that $\mathrm{Im}\mathcal{T}_{G/H} = 0$. To do this introduce the Källén-Lehmann spectral representation of the propagators [62]

$$\Delta_{G/H}(q) = \int_0^\infty \mathrm{d}s\frac{\rho_{G/H}(s)}{q^2 - s + i\epsilon}, \tag{4.142}$$

where the spectral densities $\rho_{G/H}(s)$ are real and positive for $s \geq 0$ and obey the sum rule

$$\int_0^\infty \mathrm{d}s\rho_{G/H}(s) = (ZZ_\Delta)^{-1} = 1, \tag{4.143}$$

where the last equality holds at two loop order (the standard formula has been adapted for the current renormalisation scheme)[8].

Then

$$\mathrm{Im}\int_q i\int_0^\infty \mathrm{d}\mu^2\frac{\rho_{G/H}(\mu^2)}{q^2-\mu^2+i\epsilon} = \mathrm{Im}\int_0^\infty \mathrm{d}\mu^2\rho_{G/H}(\mu^2)\mathcal{T}^\mu = 0. \tag{4.144}$$

This allows one to obtain a dispersion relation relating the real and imaginary parts of the self-energies

$$\begin{aligned}0 &= \mathrm{Im}\int_q i\frac{1}{q^2 - m_{G/H}^2 - \Sigma_{G/H}(q)} \\ &= \int_q \frac{q^2 - m_{G/H}^2 - \mathrm{Re}\Sigma_{G/H}(q)}{\left[q^2 - m_{G/H}^2 - \mathrm{Re}\Sigma_{G/H}(q)\right]^2 + \left[\mathrm{Im}\Sigma_{G/H}(q)\right]^2}. \end{aligned} \tag{4.145}$$

[8]Note that this standard theory actually conflicts with the asymptotic $p^2\left(\ln p^2\right)^{d_1}$ form assumed for the self-energy when the Φ_5 graph is included, so that this argument must be refined at the three loop level. The essential problem is that the self-consistent nPI propagator is not resumming large logarithms. However, it seems unlikely that a refinement of the argument to account for this fact will change the qualitative conclusions of this section since, as will be shown shortly, the predicted Γ_H is wrong by group theory factors in addition to the $\mathcal{O}(1)$ factors which could be compensated by a modification of $\rho_{G/H}$.

Finally one must compute $\mathrm{Im}\left[i\mathcal{I}_{HG}\left(p\right)\right]$, which can be written

$$
\begin{aligned}
\mathrm{Im}\left[i\mathcal{I}_{HG}\left(p\right)\right] &= \mathrm{Im}i\int_q\int_0^\infty \mathrm{d}s_1\int_0^\infty \mathrm{d}s_2 \frac{i\rho_H\left(s_1\right)}{q^2-s_1+i\epsilon}\frac{i\rho_G\left(s_2\right)}{\left(p-q\right)^2-s_2+i\epsilon} \\
&= \mathrm{Im}i\int_0^\infty \mathrm{d}s_1\int_0^\infty \mathrm{d}s_2\rho_H\left(s_1\right)\rho_G\left(s_2\right)\int_q \frac{i}{q^2-s_1+i\epsilon}\frac{i}{\left(p-q\right)^2-s_2+i\epsilon} \\
&= \frac{1}{16\pi^2}\int_0^\infty \mathrm{d}s_1\int_0^\infty \mathrm{d}s_2\rho_H\left(s_1\right)\rho_G\left(s_2\right) \\
&\times \mathrm{Im}\int_0^1 \mathrm{d}x \ln\left(\frac{\mu^2}{-x\left(1-x\right)p^2+xs_1+\left(1-x\right)s_2-i\epsilon}\right). \qquad (4.146)
\end{aligned}
$$

The imaginary part of the x integral is only non-zero for $\sqrt{s_1}+\sqrt{s_2}<\sqrt{p^2}$. Denote the region of $s_{1,2}$ integration by Ω. Then the imaginary part of the x integral can be evaluated straightforwardly, giving

$$
\begin{aligned}
\mathrm{Im}\left[i\mathcal{I}_{HG}\left(p\right)\right] = \frac{1}{16\pi p^2}\int_\Omega \mathrm{d}s_1\mathrm{d}s_2\rho_H\left(s_1\right)\rho_G\left(s_2\right) \\
\times\sqrt{p^2-\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\sqrt{p^2-\left(\sqrt{s_1}-\sqrt{s_2}\right)^2}. \qquad (4.147)
\end{aligned}
$$

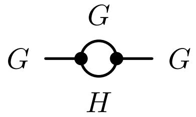
Now, since the each term of the integrand is positive and the square root is $\leq p^2$,

$$
\mathrm{Im}\left[i\mathcal{I}_{HG}\left(p\right)\right] \leq \frac{1}{16\pi}\int_\Omega \mathrm{d}s_1\mathrm{d}s_2\rho_H\left(s_1\right)\rho_G\left(s_2\right) \leq \frac{1}{16\pi}, \qquad (4.148)
$$

using the sum rule for $\rho_{H/G}\left(s\right)$. Finally,

$$
\Gamma_H \leq \frac{\hbar}{16\pi m_H}\left(\frac{\lambda v}{3}\right)^2, \qquad (4.149)
$$

which is smaller than the expected value for all $N>3$. For the $N=2$ and 3 cases one could possibly obtain an (accidentally) reasonable result, depending on the precise form of the spectral functions, but it is clear that one should not generically expect a correct prediction of Γ_H from the symmetry improved 3PIEA at two loops. The source of the problem is the derivation of the two loop truncation where the vertex correction term is dropped, resulting in a truncation of the true Ward identity that keeps the one loop graphs in Σ_G but not in $\bar{V}$. The diagram contributing to Γ_H above is thus the sunset Goldstone self-energy



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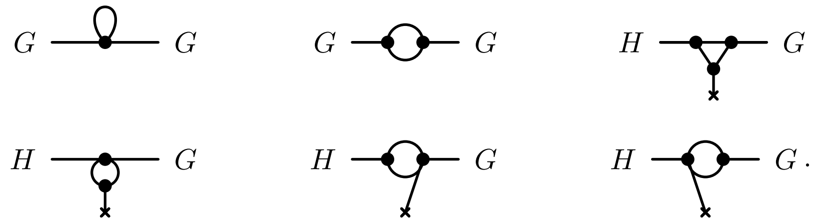
which has the incorrect kinematics and lacks the required group theory $(N-1)$ and Bose symmetry $(1/2)$ factors as well. In fact, a perturbative evaluation of this diagram gives $\Gamma_H=0$ due to the threshold at $p^2=m_H^2$! What has been shown here is that no matter the form of the exact spectral functions, there cannot be a non-perturbative enhancement of this graph large enough to give the correct Γ_H for $N>3$. The neglected vertex corrections give a leading $\mathcal{O}\left(\hbar\right)$ contribution to Γ_H which must be included.

Now consider the three loop truncation of the symmetry improved 3PIEA. Since one loop vertex corrections appear at this order one expects that Γ_H should be correct at least to $\mathcal{O}(\hbar)$. Since the previous result was incorrect by group theory factors already at $\mathcal{O}(\hbar)$ one is motivated in seeking only the $\mathcal{O}(\hbar)$ decay width at first, delaying the requirement for numerical computation as much as possible. Thus one can make use of only the one loop terms in the Higgs equation of motion, which are



The crossed vertices represent the injection of a zero momentum Goldstone boson with amplitude v. Furthermore, by iterating the equations of motion one may replace all propagators and vertices by their perturbative values to $\mathcal{O}(\hbar)$. This will leave out contributions of higher order decay processes such as $H \to GGGG$. Evaluating the decay width predicted by the SI-3PIEA more accurately requires a numerical solution of the equations of motion and so is left to future work.

The contributions of the various terms to the imaginary part of Σ_H can be determined using Cutkosky cutting rules [36]. In particular, the Hartree-Fock diagram and the first bubble vertex correction diagram (left diagram, bottom row above) have no cuts such that all cut lines can be put on shell. Also, cuts through intermediate states with both Goldstone and Higgs lines contribute to the process $H \to HG$, which is prohibited due to the unbroken $\mathrm{O}(N-1)$ symmetry and anyway vanishes due to the zero phase space at threshold. This means one can drop the sunset diagram and the last bubble vertex correction (right diagram, bottom row). Similarly cuts through two intermediate Higgs lines can be dropped since $H \to HH$ is impossible on shell. This means one can drop the contributions to the triangle and remaining bubble diagram where the leftmost vertex is V_N rather than $\bar{V}$. The relevant contributions can now be displayed explicitly:

$$\begin{aligned}
-\Sigma_H &\supset \bar{V} v \\
&\supset v \left[i\hbar \left(-\frac{\lambda v}{3} \right)^3 \int_\ell \frac{1}{(\ell - p)^2 - m_G^2 + i\epsilon} \frac{1}{\ell^2 - m_G^2 + i\epsilon} \frac{1}{\ell^2 - m_H^2 + i\epsilon} \right. \\
&\quad \left. + \frac{i\hbar\lambda}{6} (N+1) \left(-\frac{\lambda v}{3} \right) \int_\ell \frac{1}{(\ell - p)^2 - m_G^2 + i\epsilon} \frac{1}{\ell^2 - m_G^2 + i\epsilon} \right],
\end{aligned} \tag{4.150}$$

where the first and second term are the triangle and bubble graphs respectively. Now cut the Goldstone lines by replacing each cut propagator

$$\left(p^2 - m_G^2 + i\epsilon\right)^{-1} \to -2\pi i \delta\left(p^2 - m_G^2\right), \tag{4.151}$$

to give $-2i\mathrm{Im}\Sigma_H$ (because the cutting rules give the *discontinuity* of the diagram, which is $2i$ times the imaginary part), yielding

$$\begin{aligned}-2i\mathrm{Im}\Sigma_H \supset -i\hbar v\left[\left(-\frac{\lambda v}{3}\right)^3 \frac{1}{-m_H^2}+\frac{\lambda}{6}(N+1)\left(-\frac{\lambda v}{3}\right)\right]\int_\ell 2\pi\delta\left((\ell-p)^2\right)2\pi\delta\left(\ell^2\right)\\ = i\hbar\frac{\lambda^2 v^2}{3^2 2}(N-1)\int_\ell 2\pi\delta\left((\ell-p)^2\right)2\pi\delta\left(\ell^2\right),\end{aligned} \tag{4.152}$$

The integral can be evaluated by elementary techniques, giving

$$\begin{aligned}\int_\ell 2\pi\delta\left((\ell-p)^2\right)2\pi\delta\left(\ell^2\right) &= \frac{1}{4\pi^2}\int \mathrm{d}^4\ell\delta\left(\ell^2-2\ell\cdot p+p^2\right)\delta\left(\ell^2\right)\\ &= \frac{1}{4\pi^2}\int \mathrm{d}\ell_0\mathrm{d}l 4\pi l^2\delta\left(-2\ell_0 m_H+m_H^2\right)\delta\left(\ell_0^2-l^2\right)\\ &= \frac{1}{\pi}\int \mathrm{d}l l^2\frac{1}{2m_H}\frac{\delta\left(\frac{m_H}{2}-l\right)}{2\frac{m_H}{2}}\\ &= \frac{1}{8\pi},\end{aligned} \tag{4.153}$$

and finally

$$-\mathrm{Im}\Sigma_H(m_H) = \frac{N-1}{2}\frac{\hbar}{16\pi}\left(\frac{\lambda v}{3}\right)^2+\mathcal{O}\left(\hbar^2\right). \tag{4.154}$$

This exactly matches the expected Γ_H, including group theory and Bose symmetry factors. The full non-perturbative solution will give corrections to this accounting for loop corrections as well as cascade decay processes $H \to GG \to (GG)^2 \to \cdots$. While the full evaluation of Γ_H is left to future work as discussed, it has been shown that the one loop vertex corrections are required to get the correct Γ_H at leading order.

4.11 Summary

The symmetry improvement formalism of Pilaftsis and Teresi is able to enforce the preservation of global symmetries in two particle irreducible effective actions, allowing among other things the accurate description of phase transitions in strongly coupled theories with spontaneously broken global symmetries. It restores Goldstone's theorem and produces physically reasonable absorptive parts in propagators [1] and has been shown to restore the second order phase transition of the O (4) linear sigma model in the Hartree-Fock approximation [116]. It has been used to study pion strings evolving in the thermal bath of a heavy ion collision [128] and has been demonstrated to improve the evaluation of the effective potential of the standard model by taming the infrared divergences of the Higgs sector, treated as an O (4) scalar field theory with gauge interactions turned off [129, 130].

However, the development of a first principles non-perturbative kinetic theory for the gauge theories of real physical interest requires the use of n-particle irreducible effective actions with $n \geq 3$ [16, 68, 69]. A step in this direction has been achieved by extending the symmetry improvement formalism to the 3PIEA for a scalar field theory with spontaneous breaking of a global $\mathrm{O}(N)$ symmetry. In the SI-3PIEA formalism an extra Ward identity involving the vertex function must be imposed. Since the constraints are singular this requires a careful consideration of the variational procedure, namely one must be careful to impose constraints in a way that satisfies d'Alembert's principle. Once this is done the theory can be renormalised in a more or less standard way, though the counter-terms differ in value from the unimproved case. The finite equations of motion and counter-terms for the Hartree-Fock truncation, two loop truncation, and three loop truncation of the effective action were found.

Several important qualitative results have been found already in this investigation. First, symmetry improvement breaks the equivalence hierarchy of nPIEA. Second, the numerical solution of the Hartree-Fock truncation gave mixed results: Goldstone's theorem was satisfied, but the order of the phase transition was incorrectly predicted to be weakly first order (though there was still a large quantitative improvement over the unimproved 2PIEA case). Third, the two loop truncation incorrectly predicts the Higgs decay width as a consequence of the optical theorem, though the three loop truncation gives the correct value, at least to $\mathcal{O}(\hbar)$. These results could be considered strong circumstantial evidence that one should not apply symmetry improvement to nPIEA at a truncation to less than n loops. One could test this conjecture further by, for example, computing the symmetry improved 4PIEA. The prediction is that unsatisfactory results of some kind will be found for any truncation of this to < 4 loops.

The results presented here are limited in several ways due to the lack of numerical solutions for the full SI-3PI equations of motion. The results presented are for the Hartree-Fock truncation, which completely misses vertex corrections and therefore most of the physical content of the 3PI scheme. In other words, the results presented are essentially those for the SI-2PI effective action with the Higgs propagator equation of motion replaced by a truncated version of the vertex Ward identity. There is therefore no basis, given the results presented here, to conclude that the SI-3PI is problematic or not once the vertex corrections are taken into account. Further limitations consider the renormalisation procedure. The renormalisation of the theory at two and three loops was performed in vacuum. The only finite temperature computation performed here was for the Hartree-Fock approximation. The extension of the two or three loop truncations to finite temperature, or an extension to non-equilibrium situations, will require a much heavier numerical effort than has yet been attempted and so falls beyond the scope of the present thesis. Likewise, it would be interesting to compare the self-consistent Higgs decay rate in the symmetry improved 2PI and 3PI formalisms. Similarly, analytical results for the renormalisation of the three loop truncation were only presented in $1+2$ dimensions, since the renormalisation was not analytically tractable in $1+3$ dimensions. Because these investigations involve a numerical effort substantially greater than anything else in this thesis they are left to future work.

The general renormalisation theory presented here, based on counter-terms, is difficult to use in practice. It would be interesting to see if symmetry improvement could work along with the counter-term-free functional renormalisation group approach [131]. Such an approach may not be easier to set up in the first instance, but once developed would likely be easier to extend to higher loop order and n than the current method. Of course it will be interesting to see if this work can be extended to gauge symmetries and, eventually, the standard model of particle physics. If successful, such an effort could serve to open a new window to the non-perturbative physics of these theories in high temperature, high density and strong coupling regimes. In an attempt to address some of problems with the SI-nPIEA, a novel method of "soft" symmetry improvement is studied in the next chapter. Following that, as a stepping stone towards non-equilibrium applications, the theory of the linear response of a system to external perturbations is studied in the SI-2PIEA formalism in Chapter 6.

Chapter 5

Soft Symmetry Improvement

5.1 Synopsis

Chapter 4 investigated the symmetry improvement formalism for nPIEA, an attempt to improve the symmetry properties of solutions of non-perturbative equations of motion by directly imposing the appropriate Ward identities as constraints. As discussed there are some successes attributable to this method. However, there are also pathologies. In particular: truncations of SI-nPIEA may not admit solutions, too low order truncations of SI-nPIEA ($\mathcal{O}\left(\hbar^l\right)$ with $l < n$) are not fully self-consistent due to over-imposed constraints, and, as will be shown in Chapter 6, solutions of SI-nPI equations of motion likely do not exist beyond equilibrium (certainly not in the linear response approximation) because violations of $> n$th order WIs feed back into the solutions for the $\leq n$th order correlation functions. All of this suggests that symmetry improvement *over-imposes* the WIs. This motivates the investigation of methods which only enforce WIs approximately rather than exactly. The idea is that the extra freedom may allow solutions to exist while violations of the WIs may be acceptably small phenomenologically depending on the application one has in mind. This chapter investigates a novel method called *soft symmetry improvement* (SSI) with this aim.

The idea of soft symmetry improvement is to relax the WI constraint of the symmetry improvement method. Instead, the WI is enforced *softly* in the sense of least squares: a solution is found which minimises at the same time violations of the WI and the violation of the unimproved equations of motion. The relative cost of WI violations is controlled by a new *stiffness* parameter ξ, such that $\xi \to 0\,(\infty)$ reduces to the SI(unimproved)-2PIEA respectively. The motivating hope behind the investigation is that for some range of finite values of ξ the extra freedom to allow small violations of the WIs leads to a formalism which is phenomenologically useful in the sense that physical quantities can be computed to some desired accuracy fixed by the particular application in mind. It is shown in this chapter that this program cannot be fully realised, at least in the infinite volume / low temperature limit, due to the infrared sensitivity of the SSI-2PIEA: the limit is a subtle one and the only consistent limiting procedures reduce to either the unimproved or SI-2PIEA, or to a novel limit which has pathological properties of its own.

This chapter formulates the SSI-2PIEA, then renormalises and solves the resulting equations of motion in the Hartree-Fock approximation in thermal equilibrium. The infinite volume / low temperature limit ($V\beta \to \infty$) is examined carefully and three consistent limiting procedures are found. Two reduce to previous work (the unimproved and SI-2PIEAs respectively) but the third is new. The properties of this new limit are examined in equilibrium at finite and zero temperature. All of this work is new and forms the basis of a paper by the author [5].

5.2 Soft Symmetry Improvement of 2PIEA

The simplest way to arrive at the definition of the SSI-2PIEA is to start with the standard 2PIEA $\Gamma[\varphi, \Delta]$ (suppressing indices and spacetime arguments where these just clutter) and the trivial identity

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi, \Delta]\right) = \int \mathcal{D}\phi\delta(\phi - \varphi)\exp\left(\frac{i}{\hbar}\Gamma[\phi, \Delta]\right). \tag{5.1}$$

The usual symmetry improved action $\Gamma^{\mathrm{SI}}[\varphi, \Delta]$ is then defined by inserting a delta function

$$\exp\left(\frac{i}{\hbar}\Gamma^{\mathrm{SI}}[\varphi, \Delta]\right) = N\int \mathcal{D}\phi\delta(\phi - \varphi)\exp\left(\frac{i}{\hbar}\Gamma[\phi, \Delta]\right)\delta(\mathcal{W}[\phi, \Delta]) \tag{5.2}$$

where $\mathcal{W}[\phi, \Delta] = 0$ is the Ward identity and the normalisation factor N is chosen so that $\Gamma^{\mathrm{SI}}[\varphi, \Delta]$ numerically equals $\Gamma[\varphi, \Delta]$ when the arguments satisfy the Ward identity[1][2]. $\Gamma^{\mathrm{SI}}[\varphi, \Delta]$ is defined only for field configurations satisfying the Ward identity, and equals the usual effective action on those configurations. Thus $\Gamma^{\mathrm{SI}}[\varphi, \Delta]$ is nothing but the SI-2PIEA as discussed in chapter 4 arrived at in a new way.

It seems reasonable that the problems encountered with PT symmetry improvement are due to the singular nature of the constraint surface, as embodied by the delta function above. Due to these problems a *soft symmetry improved* (SSI) effective action $\Gamma^{\mathrm{SSI}}_{\xi}[\varphi, \Delta]$ can be introduced where the Ward identity is no longer strictly enforced. Small violations $\mathcal{W} \neq 0$ are allowed but punished in the functional integral. A new free parameter controls how strictly the constraint is enforced. The hope is that the added freedom allows consistent solutions with non-trivial dynamics (e.g. linear response to external sources), while the stiffness can be tuned to make violations of the Ward identity acceptably small in practice. To achieve this replace the delta function by a smoothed version $\delta(\mathcal{W}) \to \delta_{\xi}(\mathcal{W})$ defined as follows

[1] Formally $N = [\delta(0)]^{-1}$ though it is not necessary to worry about rigorously defining this here.

[2] Note that if one wants invariance under redefinitions of $\mathcal{W}$ then one also needs to insert a factor of $\mathrm{Det}\left(\frac{\delta\mathcal{W}}{\delta\phi}\right)$. This can be handled straightforwardly in the functional formalism through the introduction of Faddeev-Popov ghost fields[18, 36, 73]. However, this is not necessary here because there is no strong reason to consider transformations of $\mathcal{W}$.

$$\begin{aligned}
\exp\left(\frac{i}{\hbar}\Gamma^{\mathrm{SSI}}_{\xi}[\varphi,\Delta]\right) &= N_0\int\mathcal{D}\phi\,\delta(\phi-\varphi)\exp\left(\frac{i}{\hbar}\Gamma[\phi,\Delta]\right)\delta_{\xi}(\mathcal{W}[\phi,\Delta]) \\
&= N_1\int\mathcal{D}[\phi,\lambda_{\phi},\lambda_W]\exp\left(\frac{i}{\hbar}\left[\lambda_{\phi}(\phi-\varphi)+\Gamma[\phi,\Delta]+\lambda_W\mathcal{W}-\frac{1}{2}\xi\lambda_W^2\right]\right) \\
&= N_2\int\mathcal{D}[\phi,\lambda_{\phi}]\exp\left(\frac{i}{\hbar}\left[\lambda_{\phi}(\phi-\varphi)+\Gamma[\phi,\Delta]+\frac{1}{2\xi}\mathcal{W}^2\right]\right) \\
&= \exp\left(\frac{i}{\hbar}\left[\Gamma[\varphi,\Delta]+\frac{1}{2\xi}\mathcal{W}^2[\varphi,\Delta]\right]\right). \qquad (5.3)
\end{aligned}$$

The first line is a formal expression that is defined by the next line. The Fourier representation of the delta functions are used to replace $\delta(\phi-\varphi)\to\int\mathcal{D}\lambda_{\phi}\exp\frac{i}{\hbar}\lambda_{\phi}(\phi-\varphi)$ etc. The $\frac{1}{2}\xi\lambda_W^2$ term is the one responsible for smoothing the delta function, with the limit $\xi\to 0$ corresponding to a stiffening of the constraint. In the third line the integral over λ_W, which is Gaussian, is performed. Finally, the integral over λ_{ϕ} yields a delta function which kills the ϕ integral, resulting in

$$\Gamma^{\mathrm{SSI}}_{\xi}[\varphi,\Delta]=\Gamma[\varphi,\Delta]+\frac{1}{2\xi}\mathcal{W}^2[\varphi,\Delta]. \qquad (5.4)$$

In a slight generalisation of this derivation one can use the smoothing term $-\frac{1}{2}\xi\lambda_W R^{-1}\lambda_W$ where R^{-1} is an arbitrary positive definite symmetric kernel which may depend on φ and Δ, which gives

$$\Gamma^{\mathrm{SSI}}_{\xi R}[\varphi,\Delta]=\Gamma[\varphi,\Delta]+\frac{1}{2\xi}\mathcal{W}R\mathcal{W}-\frac{i\hbar}{2}\mathrm{Tr}\ln R. \qquad (5.5)$$

This is the most general form of the definition of the SSI-2PIEA. The simpler form $\Gamma^{\mathrm{SSI}}_{\xi}[\varphi,\Delta]$ corresponds to a trivial kernel (now with indices explicit)

$$R^{AB}_{ab}(x,y)=\delta^{AB}\delta_{ab}\delta(x-y), \qquad (5.6)$$

which is used exclusively in the following, though the freedom to choose a non-trivial R in the definition of the SSI effective action may be useful in certain circumstances. The end result is simply that $\mathcal{W}=0$ is enforced in the sense of (possibly weighted if R is non-trivial) least-squared error, rather than as a strict constraint.

Taking as the SSI equations of motion $\delta\Gamma^{\mathrm{SSI}}_{\xi}=0$, it is straightforward to derive the equations of motion, now including indices:

$$\begin{aligned}
\frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi_a} &= -\frac{1}{\xi}\mathcal{W}^A_c[\varphi,\Delta]\frac{\delta}{\delta\varphi_a}\mathcal{W}^A_c[\varphi,\Delta] \\
&= -\frac{1}{\xi}\left(\Delta^{-1}_{cf}T^A_{fg}\varphi_g\right)\Delta^{-1}_{cd}T^A_{da}, \qquad (5.7) \\
\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_{ab}} &= -\frac{1}{\xi}\mathcal{W}^A_c[\varphi,\Delta]\frac{\delta}{\delta\Delta_{ab}}\mathcal{W}^A_c[\varphi,\Delta] \\
&= \frac{1}{\xi}\left(\Delta^{-1}_{cf}T^A_{fg}\varphi_g\right)\Delta^{-1}_{ca}\left(\Delta^{-1}_{bd}T^A_{de}\varphi_e\right). \qquad (5.8)
\end{aligned}$$

Now the spontaneous symmetry breaking ansatz (3.30), reproduced here

$$\varphi_a = v\delta_{aN}, \tag{5.9}$$

$$\Delta^{-1}_{ab} = \begin{cases} \Delta^{-1}_G & a = b \neq N, \\ \Delta^{-1}_H & a = b = N, \end{cases} \tag{5.10}$$

can be used, yielding

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi_g(x)} = 0,\ (g \neq N), \tag{5.11}$$

$$\begin{aligned} \frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi_N(x)} &= \frac{1}{\xi}2(N-1)v\int_{yz}\Delta^{-1}_G(y,z)\Delta^{-1}_G(y,x) \\ &= \frac{1}{\xi}2(N-1)vm^4_G, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_G(x,y)} &= -\frac{1}{\xi}2v^2\int_{wrz}\Delta^{-1}_G(w,r)\Delta^{-1}_G(w,x)\Delta^{-1}_G(y,z) \\ &= \frac{1}{\xi}2v^2m^6_G, \end{aligned} \tag{5.13}$$

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_H} = 0. \tag{5.14}$$

Note that if one takes $\xi \to 0$ proportionally to vm^4_G one obtains for the non-trivial right hand sides above $2(N-1)vm^4_G/\xi \to$ constant and $2v^2m^6_G/\xi \to$ (const.) $\times\, vm^2_G \to 0$ and one recovers the usual SI-2PIEA scheme in the limit (c.f. (4.11)-(4.13)). In fact it is shown through a careful treatment of the infinite volume limit in section 5.4.3 that this limit is not quite correct, but there does exist a scaling limit with $\xi \to 0$ which reduces to the SI-2PIEA. This confirms the intuition that $\xi \to 0$ approaches hard symmetry improvement and that $\Gamma^{\mathrm{SSI}}_\xi[\varphi,\Delta] \to \Gamma^{\mathrm{SI}}[\varphi,\Delta]$ which really is just the standard symmetry improved effective action. In the next sections these equations of motion are renormalised and solved in the Hartree-Fock approximation.

5.3 Renormalisation of the Hartree-Fock truncation

Since it turns out that the SSI method is sensitive to the $\mathrm{V}\beta \to \infty$ limit the theory will be formulated in Euclidean spacetime (i.e. the Matsubara formalism) in a box of volume $\mathrm{V} = L^3$ with periodic boundary conditions of period L in the space directions and β in the time $\tau = it$ direction. This mirrors the treatment of finite temperature scalar fields in section 1.4. The Euclidean continuation leads to $\partial_\mu\partial^\mu \to -\nabla^2$, $\int_x \to -i\int_{x_E}$ and the conventions

$$f(x_E) = \frac{1}{V\beta}\sum_{n,\boldsymbol{k}} \mathrm{e}^{i(\omega_n\tau + \boldsymbol{k}\cdot\boldsymbol{x})} f(n,\boldsymbol{k}), \tag{5.15}$$

$$f(n,\boldsymbol{q}) = \int_{x_E} \mathrm{e}^{-i(\omega_m\tau + \boldsymbol{q}\cdot\boldsymbol{x})} f(x_E), \tag{5.16}$$

for Fourier transforms. This leads to the Fourier transforms of the propagators

$$\Delta_{G/H}(x,y) = \frac{1}{V\beta}\sum_{n,\boldsymbol{k}} \mathrm{e}^{i(\omega_n(\tau_x-\tau_y) + \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}))} \Delta_{G/H}(n,\boldsymbol{k}), \tag{5.17}$$

$$\Delta^{-1}_{G/H}(x,y) = \frac{1}{V\beta}\sum_{n,\boldsymbol{k}} \mathrm{e}^{i(\omega_n(\tau_x-\tau_y) + \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}))} \Delta^{-1}_{G/H}(n,\boldsymbol{k}), \tag{5.18}$$

and

$$\Delta_{G/H}(n,\boldsymbol{k}) = \int_{x-y} \mathrm{e}^{-i(\omega_n(\tau_x-\tau_y) + \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}))} \Delta_{G/H}(x,y), \tag{5.19}$$

etc., where the Matsubara frequencies are $\omega_n = 2\pi n/\beta$ and the wave vectors $\boldsymbol{k}$ are discretised on a lattice of spacing $2\pi/L$. The four dimensional Euclidean shorthand $k_E = (\omega_n, \boldsymbol{k})$ is often useful.

It is straightforward to modify the real time 2PIEA of chapter 2 to the Euclidean formalism. The results are collected here for easy reference. The defining functional integral of the field theory is the partition function

$$Z[J,K] = \int \mathcal{D}[\phi] \exp\left(-S_E[\phi] - J_a\phi_a - \frac{1}{2}\phi_a K_{ab}\phi_b\right), \tag{5.20}$$

where

$$S_E[\phi] = \int_x \frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2}m^2\phi_a\phi_a + \frac{1}{4!}\lambda(\phi_a\phi_a)^2, \tag{5.21}$$

is the Euclidean action. Then $W[J,K] = -\ln Z[J,K]$ is the connected generating functional and

$$\Gamma[\varphi,\Delta] = W - J\frac{\delta W}{\delta J} - K\frac{\delta W}{\delta K}, \tag{5.22}$$

is the standard 2PIEA once J and K are eliminated in terms of φ and Δ using

$$\frac{\delta W}{\delta J_a} = \langle\phi_a\rangle = \varphi_a, \tag{5.23}$$

$$\frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle\phi_a\phi_b\rangle = \frac{1}{2}(\Delta_{ab} + \varphi_a\varphi_b). \tag{5.24}$$

As before, the Legendre transform can be evaluated by the saddle point method, which results in

$$\Gamma = S_E[\varphi] + \frac{1}{2}\mathrm{Tr}\ln\left(\Delta^{-1}\right) + \frac{1}{2}\mathrm{Tr}\left(\Delta_0^{-1}\Delta - 1\right) + \Gamma_2, \tag{5.25}$$

where Γ_2 is the set of two particle irreducible graphs and $\Delta_0^{-1} = \delta^2 S_E / \delta\phi\delta\phi$ is the unperturbed propagator

$$\Delta_{0ab}^{-1} = \left(-\nabla^2 + m^2 + \frac{1}{6}\lambda\varphi^2\right)\delta_{ab} + \frac{1}{3}\lambda\varphi_a\varphi_b. \tag{5.26}$$

To two loop order

$$\Gamma_2 = \frac{1}{4!}\lambda\Delta_{aa}\Delta_{bb} + \frac{1}{12}\lambda\Delta_{ab}\Delta_{ab} - \frac{1}{36}\lambda^2\varphi_b\Delta_{ac}\Delta_{ac}\Delta_{bd}\varphi_d - \frac{1}{18}\lambda^2\varphi_b\Delta_{ac}\Delta_{ad}\Delta_{bc}\varphi_d + \cdots . \tag{5.27}$$

To form $\Gamma_\xi^{\mathrm{SSI}}$ one adds the soft-symmetry improvement term $-\frac{1}{2\xi}\mathcal{W}^2$ where the Ward identity (3.15) is

$$\mathcal{W}_a^A = \Delta_{ab}^{-1} T_{bc}^A \varphi_c. \tag{5.28}$$

(Note that $\mathcal{W}$ is pure imaginary due to the i in T^A so $-\mathcal{W}^2$ is positive definite.) Using the spontaneous symmetry breaking ansatz and inserting the ten renormalisation constants Z, Z_Δ, $\delta m_{0,1}^2$, $\delta\lambda_0$, $\delta\lambda_{1,2}^{A,B}$, $\delta\lambda$ (c.f. section 3.4) and a new one Z_ξ for ξ gives the renormalised SSI effective action:

$$\begin{aligned}
\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta] = &\int_x \left(Z_\Delta^{-1}\frac{m^2+\delta m_0^2}{2}v^2 + \frac{\lambda+\delta\lambda_0}{4!}v^4\right) \\
&+\frac{1}{2}(N-1)\,\mathrm{Tr}\ln\left(Z^{-1}Z_\Delta^{-1}\Delta_G^{-1}\right) + \frac{1}{2}\mathrm{Tr}\ln\left(Z^{-1}Z_\Delta^{-1}\Delta_H^{-1}\right) \\
&+\frac{1}{2}(N-1)\,\mathrm{Tr}\left[\left(-ZZ_\Delta\nabla^2 + m^2 + \delta m_1^2 + Z_\Delta\frac{\lambda+\delta\lambda_1^A}{6}v^2\right)\Delta_G\right] \\
&+\frac{1}{2}\mathrm{Tr}\left[\left(-ZZ_\Delta\nabla^2 + m^2 + \delta m_1^2 + Z_\Delta\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2\right)\Delta_H\right] \\
&+\Gamma_2 + Z_\xi^{-1}\frac{1}{\xi}(N-1)\,v^2 Z^{-1}Z_\Delta^{-2}\int_{xyz}\Delta_G^{-1}(x,y)\,\Delta_G^{-1}(x,z)
\end{aligned} \tag{5.29}$$

with

$$\begin{aligned}
\Gamma_2 = &\frac{1}{4!}Z_\Delta^2\left(\lambda+\delta\lambda_2^A\right)\Delta_{aa}\Delta_{bb} + \frac{1}{12}Z_\Delta^2\left(\lambda+\delta\lambda_2^B\right)\Delta_{ab}\Delta_{ab} \\
&-\frac{1}{36}Z_\Delta^3(\lambda+\delta\lambda)^2\varphi_b\Delta_{ac}\Delta_{ac}\Delta_{bd}\varphi_d - \frac{1}{18}Z_\Delta^3(\lambda+\delta\lambda)^2\varphi_b\Delta_{ac}\Delta_{ad}\Delta_{bc}\varphi_d + \cdots
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
= &\frac{1}{4!}Z_\Delta^2(N-1)\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right]\Delta_G\Delta_G \\
&+\frac{1}{4!}Z_\Delta^2\left(\lambda+\delta\lambda_2^A\right)2(N-1)\Delta_G\Delta_H + \frac{1}{4!}Z_\Delta^2\left[3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B\right]\Delta_H\Delta_H \\
&-\frac{1}{36}(N-1)Z_\Delta^3(\lambda+\delta\lambda)^2 v^2\Delta_G\Delta_G\Delta_H - \frac{1}{12}Z_\Delta^3(\lambda+\delta\lambda)^2 v^2\Delta_H\Delta_H\Delta_H + \cdots .
\end{aligned} \tag{5.31}$$

This can be simplified using the mode expansions for $\Delta_{G/H}$ and doing the integrals, giving

$$\Gamma_{\xi}^{\mathrm{SSI}}\left[\varphi,\Delta\right] = \mathrm{V}\beta\left(Z_{\Delta}^{-1}\frac{m^2+\delta m_0^2}{2}v^2 + \frac{\lambda+\delta\lambda_0}{4!}v^4\right) + \frac{1}{2}N\mathrm{V}\beta\ln\left(Z^{-1}Z_{\Delta}^{-1}\right)$$
$$+\frac{1}{2}\left(N-1\right)\sum_{n,\boldsymbol{k}}\ln\frac{1}{\Delta_G\left(n,\boldsymbol{k}\right)} + \frac{1}{2}\sum_{n,\boldsymbol{k}}\ln\frac{1}{\Delta_H\left(n,\boldsymbol{k}\right)}$$
$$+\frac{1}{2}\left(N-1\right)\sum_{n,\boldsymbol{k}}\left(ZZ_{\Delta}k_E^2+m^2+\delta m_1^2+Z_{\Delta}\frac{\lambda+\delta\lambda_1^A}{6}v^2\right)\Delta_G\left(n,\boldsymbol{k}\right)$$
$$+\frac{1}{2}\sum_{n,\boldsymbol{k}}\left(ZZ_{\Delta}k_E^2+m^2+\delta m_1^2+Z_{\Delta}\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2\right)\Delta_H\left(n,\boldsymbol{k}\right)$$
$$+\Gamma_2+Z_{\xi}^{-1}\frac{1}{\xi}\left(N-1\right)v^2Z^{-1}Z_{\Delta}^{-2}\mathrm{V}\beta\left[\Delta_G^{-1}\left(0,\mathbf{0}\right)\right]^2, \tag{5.32}$$

noting that the integrals in the SSI term give

$$\int_{xyz}\Delta_G^{-1}\left(x,y\right)\Delta_G^{-1}\left(x,z\right) = \mathrm{V}\beta\left[\Delta_G^{-1}\left(0,\mathbf{0}\right)\right]^2. \tag{5.33}$$

Now one can make a basic consistency check by examining the tree level equations of motion, which are (setting renormalisation constants to their trivial values)

$$0 = \mathrm{V}\beta v\left\{\left(m^2+\frac{\lambda}{6}v^2\right)+\frac{2\left(N-1\right)}{\xi}\left[\Delta_G^{-1}\left(0,\mathbf{0}\right)\right]^2\right\}, \tag{5.34}$$
$$\frac{1}{\Delta_G\left(n,\boldsymbol{k}\right)} = k_E^2+m^2+\frac{\lambda}{6}v^2,\ n,\boldsymbol{k}\neq 0, \tag{5.35}$$
$$\Delta_G^{-1}\left(0,\mathbf{0}\right) = m^2+\frac{\lambda}{6}v^2-\frac{4\mathrm{V}\beta}{\xi}v^2\left[\Delta_G^{-1}\left(0,\mathbf{0}\right)\right]^3, \tag{5.36}$$
$$\frac{1}{\Delta_H\left(n,\boldsymbol{k}\right)} = k_E^2+m^2+\frac{\lambda}{2}v^2. \tag{5.37}$$

Indeed these have $v^2=-6m^2/\lambda$, $\Delta_G^{-1}\left(n,\boldsymbol{k}\right)=k_E^2$ and $\Delta_H^{-1}\left(n,\boldsymbol{k}\right)=k_E^2+m_H^2=k_E^2+\frac{\lambda}{3}v^2$ as a consistent solution as expected. Since these equations are self-consistent there is a valid concern about possible spurious solutions. This can be investigated by solving the v and $\Delta_G^{-1}\left(0,\mathbf{0}\right)$ equations together on the assumption that $v^2\neq 0,-6m^2/\lambda$. Using the v equation to reduce the degree of the $\Delta_G^{-1}\left(0,\mathbf{0}\right)$ to first order gives the potential spurious solution

$$-\frac{\xi}{2\left(N-1\right)} = \left(m^2+\frac{\lambda}{6}v^2\right)/\left[1-\frac{2\mathrm{V}\beta}{N-1}v^2\left(m^2+\frac{\lambda}{6}v^2\right)\right]^2, \tag{5.38}$$
$$\Delta_G^{-1}\left(0,\mathbf{0}\right) = \left(m^2+\frac{\lambda}{6}v^2\right)/\left[1-\frac{2\mathrm{V}\beta}{N-1}v^2\left(m^2+\frac{\lambda}{6}v^2\right)\right]. \tag{5.39}$$

However, the condition that there are no tachyons (i.e. excitations with $m^2<0$ indicating the instability of the vacuum) requires $\Delta_G^{-1}\left(0,\mathbf{0}\right)\geq 0$ which implies

$$0\leq m^2+\frac{\lambda}{6}v^2<\frac{N-1}{2\mathrm{V}\beta v^2}, \tag{5.40}$$

which then implies that the right hand side of the first equation is positive, but then the left hand side $\propto -\xi$ is negative leading to a contradiction. Thus the only spurious solutions are tachyonic and so easily dismissable.

At the Hartree-Fock level

$$\begin{aligned}\Gamma_2 = &\frac{1}{4!} Z_\Delta^2 (N-1) \left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} \sum_{n,\boldsymbol{k}} \Delta_G(n,\boldsymbol{k}) \sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) \\ &+ \frac{1}{4!} Z_\Delta^2 \left(\lambda + \delta\lambda_2^A\right) 2(N-1) \frac{1}{\mathrm{V}\beta} \sum_{n,\boldsymbol{k}} \Delta_G(n,\boldsymbol{k}) \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q}) \\ &+ \frac{1}{4!} Z_\Delta^2 \left[3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} \sum_{n,\boldsymbol{k}} \Delta_H(n,\boldsymbol{k}) \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q}) .\end{aligned} \tag{5.41}$$

Further, since ξ is so far arbitrary, there is no loss of generality in setting $Z_\xi = Z_\Delta^{-1}$. The resulting equations of motion are for the vev:

$$\begin{aligned}0 = &\mathrm{V}\beta \left(Z_\Delta^{-1} \frac{m^2 + \delta m_0^2}{2} 2v + \frac{\lambda + \delta\lambda_0}{4!} 4v^3 \right) + \frac{1}{2}(N-1) \left(Z_\Delta \frac{\lambda + \delta\lambda_1^A}{6} 2v \right) \sum_{n,\boldsymbol{k}} \Delta_G(n,\boldsymbol{k}) \\ &+ \frac{1}{2} \left(Z_\Delta \frac{3\lambda + \delta\lambda_1^A + 2\delta\lambda_1^B}{6} 2v \right) \sum_{n,\boldsymbol{k}} \Delta_H(n,\boldsymbol{k}) + \frac{1}{\xi}(N-1) 2v Z^{-1} Z_\Delta^{-1} \mathrm{V}\beta \left[\Delta_G^{-1}(0,\mathbf{0})\right]^2 ,\end{aligned} \tag{5.42}$$

for the Goldstone propagator:

$$\begin{aligned}\frac{1}{\Delta_G(n,\boldsymbol{k})} = &Z Z_\Delta k_E^2 + m^2 + \delta m_1^2 + Z_\Delta \frac{\lambda + \delta\lambda_1^A}{6} v^2 \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} 2 \sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left(\lambda + \delta\lambda_2^A\right) 2 \frac{1}{\mathrm{V}\beta} \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q}) \\ &+ 2\delta_{n0}\delta_{\boldsymbol{k}\mathbf{0}} \frac{1}{\xi} v^2 Z^{-1} Z_\Delta^{-1} \mathrm{V}\beta (-2) \left[\Delta_G^{-1}(0,\mathbf{0})\right]^3 ,\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}\frac{1}{\Delta_H(n,\boldsymbol{k})} = &Z Z_\Delta k_E^2 + m^2 + \delta m_1^2 + Z_\Delta \frac{3\lambda + \delta\lambda_1^A + 2\delta\lambda_1^B}{6} v^2 \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left(\lambda + \delta\lambda_2^A\right) 2(N-1) \frac{1}{\mathrm{V}\beta} \sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left[3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} 2 \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q})\end{aligned} \tag{5.44}$$

for the Higgs propagator.

Since the self-energies are momentum independent (except for the $\delta_{n0}\delta_{\boldsymbol{k}\mathbf{0}}$ term in Δ_G) the propagators can be written as

$$\Delta_G(n,\boldsymbol{k}) = \begin{cases} \Delta_G(0,\mathbf{0}) & n=\boldsymbol{k}=0 \\ \frac{1}{k_E^2+m_G^2} & n,\boldsymbol{k}\neq 0 \end{cases}, \tag{5.45}$$

$$\Delta_H(n,\boldsymbol{k}) = \frac{1}{k_E^2+m_H^2}. \tag{5.46}$$

Substituting these in the propagator equations of motion gives $ZZ_\Delta = 1$ and

$$\begin{aligned} m_G^2 = {}& m^2 + \delta m_1^2 + Z_\Delta \frac{\lambda+\delta\lambda_1^A}{6} v^2 \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} 2 \sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left(\lambda+\delta\lambda_2^A\right) 2\frac{1}{\mathrm{V}\beta} \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q}), \end{aligned} \tag{5.47}$$

from the non-zero Goldstone modes,

$$\begin{aligned} \frac{1}{\Delta_G(0,\mathbf{0})} = {}& m^2 + \delta m_1^2 + Z_\Delta \frac{\lambda+\delta\lambda_1^A}{6} v^2 \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} 2 \sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left(\lambda+\delta\lambda_2^A\right) 2\frac{1}{\mathrm{V}\beta} \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q}) \\ &+ 2\frac{1}{\xi} v^2 \mathrm{V}\beta(-2)\left[\Delta_G^{-1}(0,\mathbf{0})\right]^3 \\ = {}& m_G^2 - 4\frac{1}{\xi} v^2 \mathrm{V}\beta \left[\Delta_G^{-1}(0,\mathbf{0})\right]^3, \end{aligned} \tag{5.48}$$

for the zero mode and

$$\begin{aligned} m_H^2 = {}& m^2 + \delta m_1^2 + Z_\Delta \frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6} v^2 \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left(\lambda+\delta\lambda_2^A\right) 2(N-1) \frac{1}{\mathrm{V}\beta} \sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) \\ &+ 2\frac{1}{4!} Z_\Delta^2 \left[3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B\right] \frac{1}{\mathrm{V}\beta} 2 \sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q}) \end{aligned} \tag{5.49}$$

for the Higgs mass.

Now there are two cases which must be distinguished. In the first, $m_G^2 = 0$ and the zero mode equation has the solutions

$$\Delta_G^{-1}(0,\mathbf{0}) = \begin{cases} 0, \\ \pm i\sqrt{\frac{\xi}{4v^2 \mathrm{V}\beta}}, \end{cases} \tag{5.50}$$

the latter two of which are clearly unphysical. However the first solution is just what one would have if $\Delta_G(n,\boldsymbol{k}) = k_E^{-2}$ as usual for a massless particle (i.e. the zero mode need no longer be treated separately). Then $\sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q})$ and $\sum_{m,\boldsymbol{q}} \Delta_H(m,\boldsymbol{q})$ are just the familiar Hartree-Fock tadpole sums, which in the infinite volume limit are

$$\sum_{n,\boldsymbol{k}} \Delta_G(n,\boldsymbol{k}) = \beta\mathrm{V}\left(\mathcal{T}_G^{\infty} + \mathcal{T}_G^{\text{fin}} + \mathcal{T}_G^{\text{th}}\right), \tag{5.51}$$

$$\sum_{n,\boldsymbol{k}} \Delta_H(n,\boldsymbol{k}) = \beta\mathrm{V}\left(\mathcal{T}_H^{\infty} + \mathcal{T}_H^{\text{fin}} + \mathcal{T}_H^{\text{th}}\right), \tag{5.52}$$

where

$$\mathcal{T}_{G/H}^{\infty} = c_0\Lambda^2 + c_1 m_{G/H}^2 \ln\frac{\Lambda^2}{\mu^2}, \tag{5.53}$$

$$\mathcal{T}_{G/H}^{\text{fin}} = \frac{m_{G/H}^2}{16\pi^2}\ln\frac{m_{G/H}^2}{\mu^2}, \tag{5.54}$$

are the vacuum contribution and $\mathcal{T}_{G/H}^{\text{th}}$ are the Bose-Einstein integrals (3.78)

$$\mathcal{T}_{G/H}^{\text{th}} = \int_{\boldsymbol{k}} \frac{1}{\omega_{\boldsymbol{k}}}\frac{1}{\mathrm{e}^{\beta\omega_{\boldsymbol{k}}}-1}, \quad \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m_{G/H}^2}. \tag{5.55}$$

μ, Λ and the dimensionless constants c_0 and c_1 are regularisation scheme dependent. In $\overline{\text{MS}}$ in $4-2\eta$ dimensions, $c_0 = 0$, $c_1 = -1/16\pi^2$ and $\Lambda^2 = 4\pi\mu^2\exp\left(\frac{1}{\eta} - \gamma + 1\right)$.

If, on the other hand, $m_G^2 \neq 0$ then Δ_G no longer has the usual form and the Goldstone tadpole must be handled differently. In this case it can be rewritten as

$$\begin{aligned}
\sum_{m,\boldsymbol{q}} \Delta_G(m,\boldsymbol{q}) &= \sum_{m,\boldsymbol{q}\neq 0} \Delta_G(m,\boldsymbol{q}) + \Delta_G(0,\mathbf{0}) \\
&= \sum_{m,\boldsymbol{q}\neq 0} \Delta_G(m,\boldsymbol{q}) + \frac{1}{m_G^2} + \Delta_G(0,\mathbf{0}) - \frac{1}{m_G^2} \\
&= \sum_{m,\boldsymbol{q}} \tilde{\Delta}_G(m,\boldsymbol{q}) + \Delta_G(0,\mathbf{0}) - \frac{1}{m_G^2}
\end{aligned} \tag{5.56}$$

where $\tilde{\Delta}_G$ is an auxiliary propagator defined to have the usual form

$$\tilde{\Delta}_G(n,\boldsymbol{k}) = \frac{1}{k_E^2 + m_G^2}. \tag{5.57}$$

Then $\sum_{m,\boldsymbol{q}} \tilde{\Delta}_G(m,\boldsymbol{q})$ is just the familiar Hartree-Fock tadpole sum for a particle of mass m_G. The terms $\Delta_G(0,\mathbf{0}) - \frac{1}{m_G^2}$ in the Goldstone tadpole account for the shift in the zero mode propagator from its usual value. Focusing on the zero mode equation, let

$$\Delta_G\left(0,\mathbf{0}\right)=\frac{1}{m_G^2\epsilon}, \tag{5.58}$$

which defines ϵ. Then,

$$\epsilon=1-\frac{4v^2\mathrm{V}\beta m_G^4}{\xi}\epsilon^3=1-\frac{4\epsilon^3}{27\hat{\xi}}, \tag{5.59}$$

where

$$\hat{\xi}=\frac{\xi}{27v^2\mathrm{V}\beta m_G^4} \tag{5.60}$$

and the numeric factor is chosen for later convenience. The real solution of this cubic equation is

$$\epsilon=\frac{3}{2}\sqrt[3]{\hat{\xi}\left(\sqrt{\hat{\xi}+1}-1\right)}\left(\sqrt[3]{\frac{\hat{\xi}}{\left(\sqrt{\hat{\xi}+1}-1\right)^2}}-1\right), \tag{5.61}$$

which is monotonically increasing from 0 to 1 as $\hat{\xi}$ goes from 0 to $+\infty$ and behaves asymptotically as

$$\epsilon\sim\begin{cases}\frac{3\hat{\xi}^{1/3}}{2^{2/3}}+\mathcal{O}\left(\hat{\xi}^{2/3}\right), & \hat{\xi}\to 0,\\ 1-\frac{4}{27\hat{\xi}}+\mathcal{O}\left(\hat{\xi}^{-2}\right), & \hat{\xi}\to\infty.\end{cases} \tag{5.62}$$

This is shown in Figure 5.1.

The remaining equations are renormalised by demanding that kinematically distinct divergences vanish. Details of this laborious and unenlightening calculation are included in the supplemental MATHEMATICA [99] notebook (Appendix C). The resulting renormalisation constants are

$$\begin{aligned}
\delta m_0^2 &= \delta m_1^2=-\frac{\hbar\lambda}{3}\left(c_0\Lambda^2+c_1m^2\ln\frac{\Lambda^2}{\mu^2}\right)\left(\frac{\delta\lambda_1^A}{\delta\lambda_1^B}-1\right), && (5.63)\\
\delta\lambda_0 &= \delta\lambda_1^A+2\delta\lambda_1^B, && (5.64)\\
\delta\lambda_1^A &= \delta\lambda_2^A=\left(1+\frac{3\left(N+2\right)}{6+\left(N+2\right)\hbar\lambda c_1\ln\frac{\Lambda^2}{\mu^2}}\right)\delta\lambda_1^B, && (5.65)\\
\delta\lambda_1^B &= \delta\lambda_2^B=\lambda\left(-1+\frac{3}{3+\hbar\lambda c_1\ln\frac{\Lambda^2}{\mu^2}}\right), && (5.66)\\
Z &= Z_\Delta=Z_\xi=1, && (5.67)
\end{aligned}$$

and the resulting finite equations of motion are

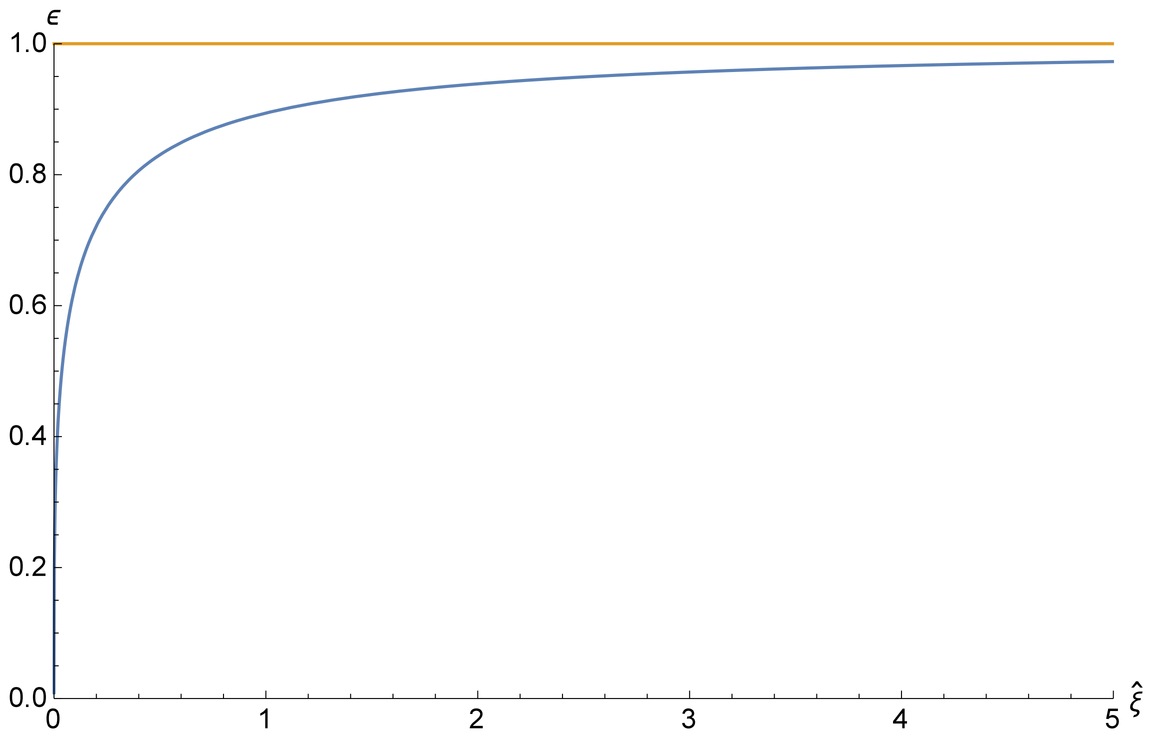


Figure 5.1: Plot of ϵ (blue) versus $\hat{\xi}$ solving the Goldstone boson zero mode equation (5.61). The line at $\epsilon = 1$ is to guide the eye.

$$v = 0, \text{ or} \tag{5.68}$$

$$0 = m^2 + \frac{\lambda}{6}v^2 + \frac{1}{2}(N-1)\frac{\lambda}{3}\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\text{th}} + \frac{1}{\text{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon} - 1\right)\right)$$
$$+ \frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right) + \frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2, \tag{5.69}$$

for the vev, and

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{1}{6}(N+1)\lambda\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\text{th}} + \frac{1}{\text{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon} - 1\right)\right)$$
$$+ \frac{1}{6}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right), \tag{5.70}$$

and

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{1}{6}\lambda(N-1)\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\text{th}} + \frac{1}{\text{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon} - 1\right)\right)$$
$$+ \frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right), \tag{5.71}$$

for the Goldstone and Higgs masses, respectively.

5.4 Solution in the infinite volume / low temperature limit

5.4.1 Symmetric phase

At high temperatures there should be a symmetric phase solution to the equations of motion. Therefore examine the $v \to 0$ limit of the equations of motion. As $v \to 0$ at fixed ξ, $\epsilon \to 1 - \frac{4v^2 \mathrm{V}\beta m_G^4}{\xi}$ so long as m_G does not go to infinity faster than $1/\sqrt{v}$. Then

$$\frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right) \to \frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{1-\frac{4v^2\mathrm{V}\beta m_G^4}{\xi}}-1\right) \to \frac{4v^2 m_G^2}{\xi} \to 0, \tag{5.72}$$

and the equations of motion (5.69)-(5.71) reduce to

$$m_G^2 = m_H^2 = m^2 + \frac{1}{6}(N+2)\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right), \tag{5.73}$$

which is a symmetric (i.e. equal mass) phase as expected. Indeed, the gap equation is unmodified by the presence of soft symmetry improvement in the symmetric phase. This phase terminates at the critical point $m_G^2 = m_H^2 = 0$ which gives the critical temperature

$$T_\star = \sqrt{\frac{12\bar{v}^2}{N+2}}, \tag{5.74}$$

where $m^2 = -\lambda\bar{v}^2/6$. That $T_\star$ is independent of ξ follows because $\mathcal{W} = 0$ trivially in the symmetric phase, and is consistent with the previously noted result that the same critical point is found in all schemes (c.f. chapter (4)). There is no subtlety involved in the $\mathrm{V}\beta \to \infty$ limit in this case.

5.4.2 Broken phase with $m_G^2 \neq 0$

Attempting to describe the broken phase with the SSI equations of motion is rather more complicated than the symmetric phase. Decreasing temperature at *fixed* ξ gives $\hat{\xi} \to 0$ and

$$\frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right) \to \frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{2^{2/3}}{3\left(\frac{\xi}{27v^2\mathrm{V}\beta m_G^4}\right)^{1/3}}-1\right) \to \left(\frac{4v^2}{\xi m_G^2(\mathrm{V}\beta)^2}\right)^{1/3}, \tag{5.75}$$

so the equations of motion (5.69)-(5.71) become

$$\begin{aligned}0 = m^2 &+ \frac{\lambda}{6}v^2 + \frac{1}{2}(N-1)\frac{\lambda}{3}\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\mathrm{th}} + \left(\frac{4v^2}{\xi m_G^2(\mathrm{V}\beta)^2}\right)^{1/3}\right)\\ &+ \frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right) + \frac{1}{\xi}(N-1)\left(\frac{\xi m_G^2}{\sqrt{2}v^2\mathrm{V}\beta}\right)^{2/3},\end{aligned} \tag{5.76}$$

for the vev, and

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{1}{6}(N+1)\lambda\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\mathrm{th}} + \left(\frac{4v^2}{\xi m_G^2 (\mathrm{V}\beta)^2}\right)^{1/3}\right)$$
$$+ \frac{1}{6}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right), \tag{5.77}$$

and

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{1}{6}\lambda(N-1)\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\mathrm{th}} + \left(\frac{4v^2}{\xi m_G^2 (\mathrm{V}\beta)^2}\right)^{1/3}\right)$$
$$+ \frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right). \tag{5.78}$$

Note that all of the soft symmetry improvement terms vanish in the limit $\mathrm{V}\beta \to \infty$. Thus the SSI-2PIEA reduces to the unimproved 2PIEA if $\mathrm{V}\beta \to \infty$ at fixed ξ. It is necessary to allow ξ to vary as the $\mathrm{V}\beta \to \infty$ limit is taken to obtain a non-trivial correction to the unimproved 2PIEA. This is the first sign that the limit is non-trivial.

This section examines the simplest scheme to find a non-trivial limit, letting ξ vary with $\mathrm{V}\beta$ as $\xi = (\mathrm{V}\beta)^\alpha \zeta$ where ζ is a constant (with mass dimension $[\zeta] = 2 + 4\alpha$). If $\alpha \geq 1$ the SSI terms vanish in the limit. If $\alpha < 1$

$$\epsilon \to \left(\frac{(\mathrm{V}\beta)^\alpha \zeta}{4v^2 \mathrm{V}\beta m_G^4}\right)^{1/3}, \tag{5.79}$$

and the symmetry improvement terms are

$$\frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon} - 1\right) \quad\to\quad \frac{1}{m_G^2}\left(\left(\frac{4v^2 m_G^4}{\zeta}\right)^{1/3}(\mathrm{V}\beta)^{(-\alpha-2)/3} - \frac{1}{\mathrm{V}\beta}\right), \tag{5.80}$$
$$\frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2 \quad\to\quad \frac{1}{\zeta}(N-1)\left(\frac{\zeta m_G^2}{\sqrt{2}v^2}\right)^{2/3}(\mathrm{V}\beta)^{(-2-\alpha)/3}. \tag{5.81}$$

The only non-trivial possibility is $\alpha = -2$, for which

$$\epsilon \quad\to\quad \frac{1}{\mathrm{V}\beta}\left(\frac{\zeta}{4v^2 m_G^4}\right)^{1/3} \to 0, \tag{5.82}$$
$$\frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon} - 1\right) \quad\to\quad \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}, \tag{5.83}$$
$$\frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2 \quad\to\quad \frac{1}{\zeta}(N-1)\left(\frac{\zeta m_G^2}{\sqrt{2}v^2}\right)^{2/3}. \tag{5.84}$$

The equations of motion are then

$$0 = m^2 + \frac{\lambda}{6}v^2 + \frac{1}{2}(N-1)\frac{\lambda}{3}\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\mathrm{th}} + \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}\right)$$
$$+ \frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right) + \frac{1}{\zeta}(N-1)\left(\frac{\zeta m_G^2}{\sqrt{2}v^2}\right)^{2/3}, \tag{5.85}$$

for the vev, and

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{1}{6}(N+1)\lambda\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\mathrm{th}} + \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}\right)$$
$$+ \frac{1}{6}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right), \tag{5.86}$$

and

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{1}{6}\lambda(N-1)\left(\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\mathrm{th}} + \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}\right)$$
$$+ \frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\mathrm{th}}\right), \tag{5.87}$$

for the Goldstone and Higgs masses, respectively. Importantly, note that the mass appearing in Goldstone's theorem is ϵm_G^2 (from the definition of ϵ: $\Delta_G^{-1}(0,\mathbf{0}) = \epsilon m_G^2$), which obeys $\epsilon m_G^2 \to 0$ as $\mathrm{V}\beta \to \infty$ thanks to the scaling chosen for ξ. Thus this scheme satisfies Goldstone's theorem even if $m_G^2 \neq 0$. What $m_G^2 \neq 0$ indicates here is not actually a violation of Goldstone's theorem, but a *non*-communication of the masslessness of the Goldstone zero mode to the other modes.

Defining the dimensionless variables

$$x = v^2/\mu^2, \qquad y = m_G^2/\mu^2, \qquad z = m_H^2/\mu^2, \tag{5.88}$$
$$\bar{x} = \bar{v}^2/\mu^2, \qquad \bar{X} = -6m^2/\lambda\mu^2, \qquad \hat{\zeta} = \zeta\mu^6, \tag{5.89}$$
$$T_{G/H} = \mu^{-2}\mathcal{T}_{G/H}^{\mathrm{th}}, \tag{5.90}$$

(note the distinction between the Lagrangian parameter $\bar{X}$ and the zero temperature value of the mean field $\bar{x}$, which happen to be equal at tree level and in the usual renormalisation scheme for the Hartree-Fock approximation) this system becomes

$$0 = \frac{\lambda}{6}\left(x - \bar{X}\right) + \frac{1}{6}\left(N-1\right)\lambda\left(\frac{1}{16\pi^2}y\ln y + T_G + \left(\frac{4x}{\hat{\zeta}y}\right)^{1/3}\right) + \frac{1}{2}\lambda\left(\frac{1}{16\pi^2}z\ln z + T_H\right)$$
$$+ \frac{1}{\hat{\zeta}}\left(N-1\right)\left(\frac{\hat{\zeta}y}{\sqrt{2}x}\right)^{2/3}, \tag{5.91}$$
$$y = \frac{\lambda}{6}\left(x - \bar{X}\right) + \frac{1}{6}\left(N+1\right)\lambda\left(\frac{1}{16\pi^2}y\ln y + T_G + \left(\frac{4x}{\hat{\zeta}y}\right)^{1/3}\right) + \frac{1}{6}\lambda\left(\frac{1}{16\pi^2}z\ln z + T_H\right), \tag{5.92}$$
$$z = \frac{\lambda}{3}x - \frac{1}{\hat{\zeta}}\left(N-1\right)\left(\frac{\hat{\zeta}y}{\sqrt{2}x}\right)^{2/3}. \tag{5.93}$$

Looking for a zero temperature solution gives the system

$$0 = \frac{\lambda}{6}\left(\bar{x} - \bar{X}\right) + \frac{1}{6}\left(N-1\right)\lambda\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + \left(\frac{4\bar{x}}{\hat{\zeta}\bar{y}}\right)^{1/3}\right) + \frac{1}{2}\lambda\frac{1}{16\pi^2}\bar{z}\ln\bar{z} + \left(N-1\right)\left(\frac{\bar{y}^2}{2\hat{\zeta}\bar{x}^2}\right)^{1/3}, \tag{5.94}$$
$$\bar{y} = \frac{\lambda}{6}\left(\bar{x} - \bar{X}\right) + \frac{1}{6}\left(N+1\right)\lambda\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + \left(\frac{4\bar{x}}{\hat{\zeta}\bar{y}}\right)^{1/3}\right) + \frac{1}{6}\lambda\frac{1}{16\pi^2}\bar{z}\ln\bar{z}, \tag{5.95}$$
$$\bar{z} = \frac{\lambda\bar{x}}{3} - \left(N-1\right)\left(\frac{\bar{y}^2}{2\hat{\zeta}\bar{x}^2}\right)^{1/3}. \tag{5.96}$$

First, ignoring the SSI terms one finds the usual unimproved 2PI solution $\bar{x} = \bar{X}$, $\bar{y} = 0$, $\bar{z} = \lambda\bar{x}/3 = 1$. Now examine the large N limit of these equations, taking as the scaling limit $g = \lambda N =$ constant and $\bar{x}, \bar{X} \sim N^1$, $\bar{y}, \bar{z} \sim N^0$ and $\hat{\zeta} \sim N^a$ with a to be determined. To leading order

$$0 = \frac{g}{6N}\left(\bar{x} - \bar{X}\right) + \frac{g}{6}\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + N^{(1-a)/3}\left(\frac{4\bar{x}/N}{\left(\hat{\zeta}/N^a\right)\bar{y}}\right)^{1/3}\right)$$
$$+ N^{(1-a)/3}\left(\frac{\bar{y}^2}{2\left(\hat{\zeta}/N^a\right)\left(\bar{x}/N\right)^2}\right)^{1/3}, \tag{5.97}$$
$$\bar{y} = \frac{g}{6N}\left(\bar{x} - \bar{X}\right) + \frac{g}{6}\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + N^{(1-a)/3}\left(\frac{4\bar{x}/N}{\left(\hat{\zeta}/N^a\right)\bar{y}}\right)^{1/3}\right), \tag{5.98}$$
$$\bar{z} = \frac{g\bar{x}}{3N} - N^{(1-a)/3}\left(\frac{\bar{y}^2}{2\left(\hat{\zeta}/N^a\right)\left(\bar{x}/N\right)^2}\right)^{1/3}. \tag{5.99}$$

Scaling limits exist if $a \geq 1$. Note that the SSI term in $\Gamma_\xi^{\mathrm{SSI}}$ goes as $\xi^{-1}Nv^2 \sim N^{3-a}$ so that one needs $a \geq 2$ for a scaling limit for $\Gamma_\xi^{\mathrm{SSI}}$ to exist. $a = 1$ can also be ruled out by considering the equations of motion, for in this case the leading approximation is

$$0 = \frac{g}{6N}\left(\bar{x} - \bar{X}\right) + \frac{g}{6}\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + \left(\frac{4\bar{x}/N}{\left(\hat{\zeta}/N\right)\bar{y}}\right)^{1/3}\right) + \left(\frac{\bar{y}^2}{2\left(\hat{\zeta}/N\right)\left(\bar{x}/N\right)^2}\right)^{1/3}, \tag{5.100}$$

$$\bar{y} = \frac{g}{6N}\left(\bar{x} - \bar{X}\right) + \frac{g}{6}\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + \left(\frac{4\bar{x}/N}{\left(\hat{\zeta}/N\right)\bar{y}}\right)^{1/3}\right), \tag{5.101}$$

$$\bar{z} = \frac{g\bar{x}}{3N} - \left(\frac{\bar{y}^2}{2\left(\hat{\zeta}/N\right)\left(\bar{x}/N\right)^2}\right)^{1/3}. \tag{5.102}$$

Using the first equation to simplify the second,

$$\bar{y} = -\left(\frac{\bar{y}^2}{2\left(\hat{\zeta}/N\right)\left(\bar{x}/N\right)^2}\right)^{1/3}, \tag{5.103}$$

which has the solution

$$\bar{y} = -\frac{1}{2\left(\hat{\zeta}/N\right)\left(\bar{x}/N\right)^2} < 0, \tag{5.104}$$

with an unphysical tachyonic Goldstone $m_G^2 < 0$. This is not *necessarily* a problem because the zero mode $\Delta_G(0, \mathbf{0})$ is always positive and, in finite volume with β and L sufficiently small (i.e. $\beta, L < 2\pi/|m_G|$), each mode $\Delta_G(n, \boldsymbol{k}) = 1/\left(\omega_n^2 + \boldsymbol{k}^2 + m_G^2\right)$ with $n, \boldsymbol{k} \neq \mathbf{0}$ is still positive. Physically, confinement energy is stabilising the tachyon. However, a second condition is that the imaginary part of the first equation of motion vanishes, yielding

$$0 = -\frac{1}{16\pi^2}|\bar{y}|\left(2k+1\right)\pi - \sin\left(\frac{\left(2k+1\right)\pi}{3}\right)\left(\frac{4\bar{x}/N}{\left(\hat{\zeta}/N\right)|\bar{y}|}\right)^{1/3}, \tag{5.105}$$

where the branch chosen is $\bar{y} = |\bar{y}|\exp\left(i\pi\left(2k+1\right)\right)$ where k is an integer. Using the solution for $\bar{y}$ this becomes

$$k = -\frac{1}{2} - 32\pi\left(\hat{\zeta}/N\right)\left(\bar{x}/N\right)^3\sin\left(\frac{\left(2k+1\right)\pi}{3}\right), \tag{5.106}$$

which only has solutions of the form $k = 3j$ if

$$j = -\frac{1}{6} - \frac{16\pi}{\sqrt{3}}\left(\hat{\zeta}/N\right)\left(\bar{x}/N\right)^3, \tag{5.107}$$

is an integer. The existence of solutions only for certain discrete values of $\hat{\zeta}\bar{x}^3$ is troubling and highly counter-intuitive (note especially that the relationship between $\hat{\zeta}$ and $\bar{x}$ for a given j is independent of g, so that no matter how g is varied at fixed m^2 and $\hat{\zeta}$, $\bar{x}$ is fixed even though one expects $\bar{x} \sim N/g$).

If $a > 1$ the SSI terms in the equation of motion are of higher order and the leading large N approximation is just the standard one, i.e.

$$0 = \frac{g}{6N}\left(\bar{x} - \bar{X}\right) + \frac{g}{6}\frac{1}{16\pi^2}\bar{y}\ln\bar{y}, \tag{5.108}$$

$$\bar{y} = \frac{g}{6N}\left(\bar{x} - \bar{X}\right) + \frac{g}{6}\frac{1}{16\pi^2}\bar{y}\ln\bar{y}, \tag{5.109}$$

$$\bar{z} = \frac{g\bar{x}}{3N}, \tag{5.110}$$

which has the solution $\bar{x} = \bar{X}$, $\bar{y} = 0$ and $\bar{z} = \lambda\bar{x}/3$ as expected. Now note that if $1 < a < 4$ the SSI terms goes as a fractional power of N between N^0 and N^{-1} which cannot balance any of the terms coming from diagrams, which all go as integer powers of N^{-1}. This implies that if $a > 1$ it must be of the form $a = 4 + 3k$ where $k = 0, 1, 2, \cdots$. The SSI terms then scale as $N^{-(1+k)}$ in the equation of motion and N^{-1-3k} in $\Gamma_\xi^{\mathrm{SSI}}$. Thus the SSI equations of motion possess a satisfactory leading large N limit, but only if the scaling is such that the SSI terms are of higher order. This is the first sign that the SSI terms are problematic.

Now consider the case where symmetry improvement is only weakly imposed, i.e. the SSI terms are a small perturbation. Intuitively this can be achieved by taking $\hat{\zeta}$ sufficiently large. It is thus natural to solve the equations of motion (5.94)-(5.96) perturbatively in powers of $\hat{\zeta}^{-1/3}$ as $\hat{\zeta} \to \infty$. Writing $\bar{x} = \bar{x}_0 + \hat{\zeta}^{-1/3}\bar{x}_1 + \hat{\zeta}^{-2/3}\bar{x}_2 + \cdots$ and so on, the leading equations of motion are just the unimproved 2PI ones

$$0 = \frac{\lambda}{6}\left(\bar{x}_0 - \bar{X}\right) + \frac{1}{6}\left(N-1\right)\lambda\frac{1}{16\pi^2}\bar{y}_0\ln\bar{y}_0 + \frac{1}{2}\lambda\frac{1}{16\pi^2}\bar{z}_0\ln\bar{z}_0, \tag{5.111}$$

$$\bar{y}_0 = \frac{\lambda}{6}\left(\bar{x}_0 - \bar{X}\right) + \frac{1}{6}\left(N+1\right)\lambda\frac{1}{16\pi^2}\bar{y}_0\ln\bar{y}_0 + \frac{1}{6}\lambda\frac{1}{16\pi^2}\bar{z}_0\ln\bar{z}_0, \tag{5.112}$$

$$\bar{z}_0 = \frac{\lambda\bar{x}_0}{3}. \tag{5.113}$$

The first order perturbation obeys a system of equations which can be arranged as the matrix equation

$$\begin{pmatrix} \frac{\lambda}{6} & \frac{(N-1)\lambda}{96\pi^2}\left(1+\ln\bar{y}_0\right) & \frac{\lambda}{32\pi^2}\left(1+\ln\bar{z}_0\right) \\ \frac{\lambda}{6} & \frac{(N+1)\lambda}{96\pi^2}\left(1+\ln\bar{y}_0\right) - 1 & \frac{\lambda}{96\pi^2}\left(1+\ln\bar{z}_0\right) \\ \frac{\lambda}{3} & 0 & -1 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \\ \bar{z}_1 \end{pmatrix} = \left(N-1\right)\left(\frac{\bar{y}_0^2}{2\bar{x}_0^2}\right)^{1/3}\begin{pmatrix} -\frac{\lambda\bar{x}_0}{3\bar{y}_0} - 1 \\ -\frac{N+1}{N-1}\frac{\lambda\bar{x}_0}{3\bar{y}_0} \\ 1 \end{pmatrix}. \tag{5.114}$$

Note that this equation is singular in the limit $\bar{y}_0 \to 0$. The solution for $\bar{y}_1$ in this limit is

$$\bar{y}_1 \to -\frac{32\pi^2}{\ln\bar{y}_0}\left(\frac{\bar{x}_0}{2\bar{y}_0}\right)^{1/3} \to \infty. \tag{5.115}$$

There is no sense in which the SSI terms are a small perturbation, no matter the value of $\hat{\zeta}$. This can also be seen from a direct examination of the full equations of motion. In the limit $\bar{y} \to 0$ the $\left(\frac{4\bar{x}}{\hat{\zeta}\bar{y}}\right)^{1/3}$ terms always dominate for any finite value of $\hat{\zeta}$. The

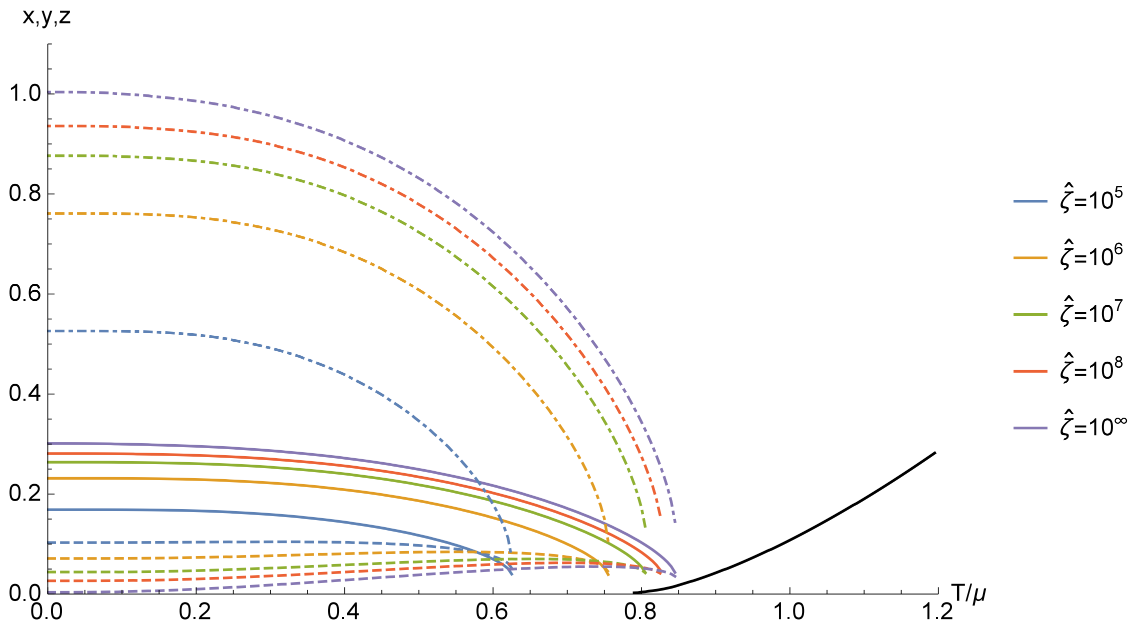


Figure 5.2: Solutions of (5.91)-(5.93), the SSI equations of motion in the Hartree-Fock approximation. Shown are broken phase x (solid), y (dashed) and z (dot-dashed) and symmetric phase $y = z$ (solid black) versus temperature for $\lambda = 10$, $N = 4$, $\bar{X} = 0.3$ and several values of $\hat{\zeta}$ from 10^5 to ∞ (unimproved). The critical temperature is $T_\star/\mu \approx 0.775$.

result is that the SSI solution must always have $\bar{y} \neq 0$, even at zero temperature. For the same reason a perturbation analysis near the critical temperature also fails and, in fact, real valued solutions do not exist in a ($\hat{\zeta}$ dependent) range of temperatures beneath the critical temperature. Further, m_G^2 appears to increase as the SSI terms are more strongly imposed. Physically, the unimproved 2PI equations of motion "would like to have" a non-zero Goldstone mass. When the mass of the zero mode is forced to vanish the SSI-2PIEA adjusts by *increasing* the mass of the other modes. This can be verified by examining numerical solutions.

Numerical solutions of the (5.91)-(5.93) are shown in Figure 5.2 for $\lambda = 10$, $N = 4$, $\bar{X} = 0.3$ and several values of $\hat{\zeta}$ from 10^4 to ∞. The critical temperature for these values is $T_\star/\mu \approx 0.775$. For very large $\hat{\zeta}$ the solution is near the unimproved solution. However, as $\hat{\zeta}$ is decreased, x and z decrease and y *increases* (this is consistent with the perturbation y_1 being positive). Broken phase solutions cease to exist above the upper spinodal temperature $T_{\rm us}\left(\hat{\zeta}\right)$ which depends on $\hat{\zeta}$. Note that $T_{\rm us}\left(\hat{\zeta}\right)$ drops below $T_\star$ for all $\hat{\zeta} < \hat{\zeta}_c$ where $\hat{\zeta}_c$ is somewhere between 10^6 and 10^7. This means that, for $\hat{\zeta} < \hat{\zeta}_c$ there is *no solution* between $T_{\rm us}\left(\hat{\zeta}\right)$ and $T_\star$. Further, as $\hat{\zeta} \to 0$, $T_{\rm us}\left(\hat{\zeta}\right) \to 0$. This behaviour can be seen in Figure 5.3. At a critical value $\hat{\zeta} = \hat{\zeta}_\star$ one has $T_{\rm us}\left(\hat{\zeta}_\star\right) = 0$ and real solutions cease to exist for all $\hat{\zeta} \leq \hat{\zeta}_\star$.

The mathematical origin of this loss of solution can be understood by considering the zero temperature equations of motion, reproduced below.

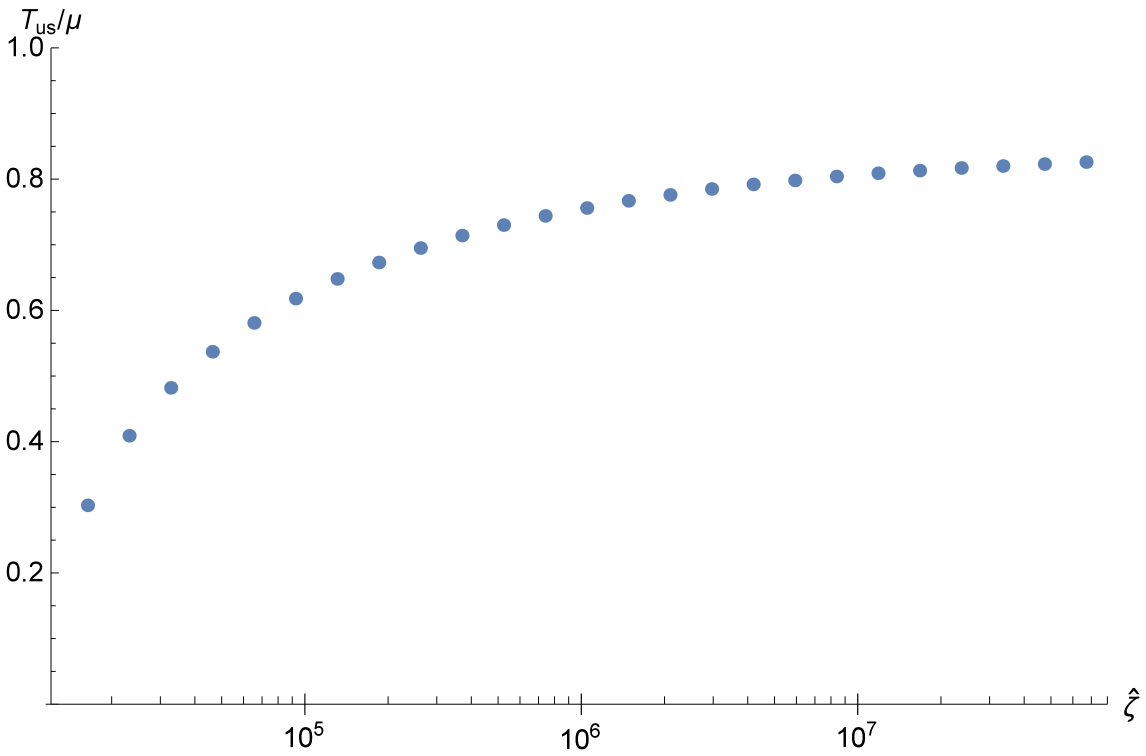


Figure 5.3: The upper spinodal temperature T_{us} versus $\hat{\zeta}$ for broken phase solutions of the SSI equations of motion in the Hartree-Fock approximation for $\lambda = 10$, $N = 4$, and $\bar{X} = 0.3$. The critical temperature is $T_\star/\mu \approx 0.775$.

$$0 = \frac{\lambda}{6}\left(\bar{x} - \bar{X}\right) + \frac{1}{6}(N-1)\,\lambda\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + \left(\frac{4\bar{x}}{\hat{\zeta}\bar{y}}\right)^{1/3}\right) + \frac{1}{2}\lambda\frac{1}{16\pi^2}\bar{z}\ln\bar{z} + (N-1)\left(\frac{\bar{y}^2}{2\hat{\zeta}\bar{x}^2}\right)^{1/3}, \tag{5.116}$$

$$\bar{y} = \frac{\lambda}{6}\left(\bar{x} - \bar{X}\right) + \frac{1}{6}(N+1)\,\lambda\left(\frac{1}{16\pi^2}\bar{y}\ln\bar{y} + \left(\frac{4\bar{x}}{\hat{\zeta}\bar{y}}\right)^{1/3}\right) + \frac{1}{6}\lambda\frac{1}{16\pi^2}\bar{z}\ln\bar{z}, \tag{5.117}$$

$$\bar{z} = \frac{\lambda\bar{x}}{3} - (N-1)\left(\frac{\bar{y}^2}{2\hat{\zeta}\bar{x}^2}\right)^{1/3}. \tag{5.118}$$

Use (5.118) to eliminate $\bar{z}$ in the first two equations and consider the real and imaginary parts of the right hand sides of these equations as functions of $\bar{x}$ and $\bar{y}$. The relevant regions of the $\bar{x} - \bar{y}$ plane are shown in Figure 5.4. The blue regions satisfy $\Re\left((5.116)\right) > 0$, yellow regions satisfy $\Im\left((5.116)\right) \neq 0$ and the green regions satisfy $\Re\left((5.117)\right) > \bar{y}$. Note that, for $\bar{x}, \bar{y} > 0$, $\Im\left((5.116)\right) \neq 0$ is equivalent to $\bar{z} > 0$, as is $\Im\left((5.117)\right) \neq 0$ which does not give anything new.

Valid solutions of the equations of motion are on the boundary of the blue and green regions simultaneously *and* outside of the yellow region. As $\hat{\zeta}$ is decreased it can be seen that the blue region "closes in" towards the origin, the green region grows upwards, and the yellow region grows to the right. Solutions cease to exist for $\hat{\zeta} = \hat{\zeta}_\star \approx 12200$ where all three regions intersect at a common point. For all $\hat{\zeta} < \hat{\zeta}_\star$ there are no solutions (intersection

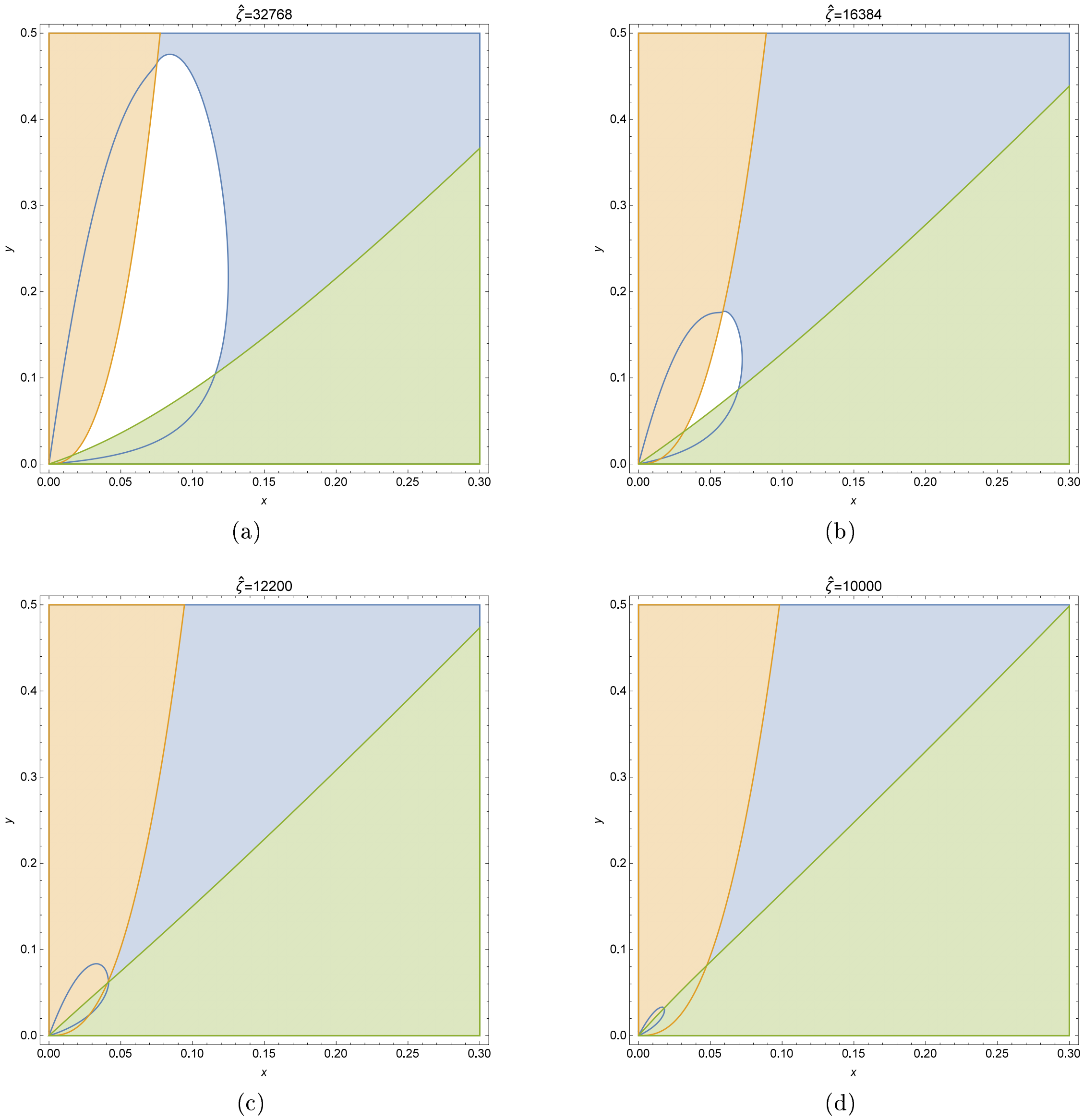


Figure 5.4: Real and imaginary parts of the right hand sides of (5.116) and (5.117) as functions of $\bar{x}$ and $\bar{y}$ for $\lambda = 10$, $N = 4$, $\bar{X} = 0.3$ and $\hat{\zeta} = 2^{15}$ (upper left), 2^{14} (upper right), 12200 (lower left) and 10^4 (lower right). The blue regions satisfy $\Re\left((5.116)\right) > 0$, the yellow regions satisfy $\Im\left((5.116)\right) \neq 0$ and the green regions satisfy $\Re\left((5.117)\right) > \bar{y}$.

points between the blue and green curves) which are also real (outside the yellow region). If the temperature is non-zero, the thermal contributions increase the real parts of the right hand sides which, in comparison with Figure 5.4, hastens the onset of loss of solutions, which is therefore achieved at a greater value $\hat{\zeta}$. Conversely, the temperature at which solutions are lost for a given $\hat{\zeta}$ increases as a function of $\hat{\zeta}$, which, of course, matches the behaviour seen in Figures 5.2 and 5.3.

5.4.3 Broken phase with $m_G^2 \to 0$

In order to find a broken phase solution without the pathological properties of the previous section one can try to find solutions with $m_G^2 \to 0$ in the V$\beta \to \infty$ limit. To achieve this, take the scalings

$$\xi = (\mathrm{V}\beta)^{\alpha}\,\zeta = (\mathrm{V}\beta)^{\alpha}\,\mu^{2+4\alpha}\hat{\zeta}, \tag{5.119}$$

$$m_G^2 = (\mathrm{V}\beta)^{-\gamma}\,\mu^{2-4\gamma}y, \tag{5.120}$$

where $\gamma > 0$. The definitions of the other dimensionless variables (x, z, etc.) are as before. Then

$$\epsilon \sim \begin{cases} \left(\frac{(\mathrm{V}\beta)^{\alpha+2\gamma-1}}{(\mu^{-4})^{\alpha+2\gamma-1}}\right)^{1/3}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}, & \alpha+2\gamma-1<0, \\ 1-\frac{(\mu^{-4})^{\alpha+2\gamma-1}}{(\mathrm{V}\beta)^{\alpha+2\gamma-1}}\frac{4xy^2}{\hat{\zeta}}, & \alpha+2\gamma-1>0. \end{cases} \tag{5.121}$$

One can take the equations of motion (5.69)-(5.71) with the prescription $\frac{1}{\mathrm{V}\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right)\to 0$ because, as discussed in section 5.3, the Goldstone tadpole reduces to the unmodified form in the massless case. The result is the equation of motion

$$0 = m^2+\frac{\lambda}{6}v^2+\frac{1}{2}(N-1)\frac{\lambda}{3}\frac{T^2}{12}+\frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2}+\mathcal{T}_H^{\mathrm{th}}\right)+\frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2, \tag{5.122}$$

for the vev, and

$$0 = m^2+\frac{\lambda}{6}v^2+\frac{1}{6}(N+1)\lambda\frac{T^2}{12}+\frac{1}{6}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2}+\mathcal{T}_H^{\mathrm{th}}\right), \tag{5.123}$$

and

$$m_H^2 = m^2+\frac{\lambda}{2}v^2+\frac{1}{6}\lambda(N-1)\frac{T^2}{12}+\frac{1}{2}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2}+\mathcal{T}_H^{\mathrm{th}}\right). \tag{5.124}$$

Note that, remarkably, the last two equations are nothing but the SI-2PIEA equations of motion (4.15)-(4.16), having used that $\mathcal{T}_G^{\mathrm{fin}} = T^2/12$ for $m_G^2 = 0$. The only thing new is the modification of the vev equation by the term $\frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2$. To examine this further one must consider the three cases $\alpha+2\gamma-1 \gtreqless 0$ which govern the possible scaling behaviours of this term.

In the $\alpha+2\gamma-1>0$ case, $\epsilon\to 1$ and

$$\frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2 \to \mu^2\frac{(\mu^{-4})^{\alpha+2\gamma}}{(\mathrm{V}\beta)^{\alpha+2\gamma}}\frac{(N-1)\,2y^2}{\hat{\zeta}}\to 0. \tag{5.125}$$

If on the other hand $\alpha+2\gamma-1=0$, ϵ is a constant as $\mathrm{V}\beta\to\infty$ and

$$\frac{1}{\xi}(N-1)\,2\left[m_G^2\epsilon\right]^2 \to \mu^2\frac{(\mu^{-4})^{\alpha+2\gamma}}{(\mathrm{V}\beta)^{\alpha+2\gamma}}\frac{1}{\hat{\zeta}}(N-1)\,2y^2\epsilon^2 = \mu^2\frac{(\mu^{-4})}{(\mathrm{V}\beta)}\frac{1}{\hat{\zeta}}(N-1)\,2y^2\epsilon^2\to 0. \tag{5.126}$$

In both of these cases the vev equation is unmodified by SSI and cannot hold at the same time as the other two equations of motion. To see this solve the SI-2PI equations (c.f. section 4.3) to get

$$m_H^2 = -2m^2 - \frac{1}{3}\lambda\,(N+2)\,\frac{T^2}{12}, \tag{5.127}$$

and

$$\frac{\lambda}{6}v^2 = -m^2 - \frac{1}{6}\,(N+1)\,\lambda\frac{T^2}{12} - \frac{1}{6}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right). \tag{5.128}$$

Now use these in the vev equation to get

$$\frac{T^2}{12} = \frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}, \tag{5.129}$$

which only holds at $T=0$ (for $\mu = \bar{m}_H$) and $T = T_\star$. There is no solution at any other temperature.

The remaining case is $\alpha + 2\gamma - 1 < 0$. For this case the SSI term becomes

$$\frac{1}{\xi}\,(N-1)\,2\left[m_G^2\epsilon\right]^2 \to \mu^2\left(\frac{\left(\mu^{-4}\right)^{\alpha+2\gamma+2}}{(\mathrm{V}\beta)^{\alpha+2\gamma+2}}\right)^{1/3}(N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3}. \tag{5.130}$$

The only consistent solution possible is $\alpha + 2\gamma + 2 = 0$ (which automatically satisfies the condition $\alpha + 2\gamma - 1 < 0$). In this case the vev equation of motion reduces to (in terms of dimensionless variables now)

$$0 = -\frac{\lambda}{6}\bar{X} + \frac{\lambda}{6}x + \frac{1}{6}\,(N-1)\,\lambda\frac{T^2/\mu^2}{12} + \frac{1}{2}\lambda\left(\frac{z}{16\pi^2}\ln z + T_H\right) + (N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3}. \tag{5.131}$$

Subtracting the m_H^2 equation from the vev equation gives

$$(N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3} = \frac{\lambda}{3}x - z, \tag{5.132}$$

which can be easily solved for y, giving

$$y^2 = 2\hat{\zeta}x^2\left(\frac{\frac{\lambda}{3}x - z}{N-1}\right)^3. \tag{5.133}$$

Note that $m_G^2 = 0$ regardless of the value of y. The only constraint is that $0 \le y < \infty$ which requires $\lambda x/3 \ge z$. This can be verified using the solution of the SI-2PI equations of motion

$$z = 1 - \frac{T^2}{T_\star^2}, \tag{5.134}$$

$$x = \bar{X} - \left((N+1)\,\frac{T^2/\mu^2}{12} + T_H\right), \tag{5.135}$$

(recalling $\bar{z} = 1$ and $\bar{X} = 3/\lambda$ are the zero temperature solutions and $T_\star^2 = 12\bar{X}\mu^2/\left(N+2\right)$) so that

$$\frac{\lambda}{3}x - z = \frac{\lambda}{3}\left[\bar{X} - \left((N+1)\frac{T^2/\mu^2}{12} + T_H\right)\right] - \left(1 - \frac{T^2}{T_\star^2}\right)$$
$$= (N+2)\frac{T^2/\mu^2}{12\bar{X}} - (N+1)\frac{T^2/\mu^2}{12\bar{X}} - \frac{\lambda}{3}T_H = \frac{1}{\bar{X}}\left(\frac{T^2/\mu^2}{12} - T_H\right) \geq 0, \quad (5.136)$$

since the thermal integral T_H is maximised for massless particles. Thus $0 \leq y^2 < \infty$ and y can always be chosen in $0 \leq y < \infty$. Thus $\xi \sim (\mathrm{V}\beta)^{-2\gamma-2}$ and $m_G^2 \sim (\mathrm{V}\beta)^{-\gamma}$ with $\gamma > 0$ is identified as the unique limiting procedure that gives back the old SI-2PIEA from the SSI-2PIEA.

One can see that this procedure is the unique way of connecting the SI and SSI methods by directly matching the SSI term in $\Gamma_\xi^{\mathrm{SSI}}$ with the Lagrange multiplier term in Γ^{SI}. Recall that the constraint term in the SI-2PIEA is (c.f. (4.3))

$$\mathcal{C} = \frac{i}{2}\ell_A^a \mathcal{W}_a^A. \quad (5.137)$$

Recall also that the constraint is singular. One must proceed by violating the constraint by an amount $\sim \eta$ then taking a limit $\eta \to 0$ such that $\ell\eta$ is a constant. Previously this procedure was carried out at the level of the equations of motion. Now it is convenient to implement this at the level of the action by shifting the constraint term to

$$\frac{i}{2}\ell_A^a\left(\mathcal{W}_a^A - i\mathcal{F}_a^A\right), \quad (5.138)$$

where $\mathcal{F}_a^A \sim \eta$ is the regulator written in $\mathrm{O}(N)$-covariant form. Setting the SI constraint term equal to the SSI term gives

$$\frac{i}{2}\ell_A^a\left(\mathcal{W}_a^A - i\mathcal{F}_a^A\right) = -\frac{1}{2\xi}\mathcal{W}_a^A\mathcal{W}_a^A. \quad (5.139)$$

Substituting $\mathcal{W}_a^A$ gives

$$2\frac{i}{2}\mathrm{V}\beta\ell_{cN}^a\left(iP_{ca}^\perp\left[\Delta_G^{-1}(0,\mathbf{0})\right]v - i\mathcal{F}_a^{cN}\right) = -\frac{1}{2\xi}\left(-2(N-1)v^2\mathrm{V}\beta\left[\Delta_G^{-1}(0,\mathbf{0})\right]^2\right), \quad (5.140)$$

having used $\int_y \Delta_G^{-1}(x,y) = \Delta_G^{-1}(0,\mathbf{0})$. Without loss of generality one can set

$$\ell_{cN}^a = P_{ac}^\perp\left(\frac{1}{N-1}\ell_{dN}^d\right), \quad (5.141)$$
$$\mathcal{F}_a^{cN} = P_{ac}^\perp\mathcal{F}, \quad (5.142)$$

and find

$$-\ell_{cN}^c\left(\Delta_G^{-1}(0,\mathbf{0})v - \mathcal{F}\right) = \frac{1}{\xi}(N-1)v^2\left[\Delta_G^{-1}(0,\mathbf{0})\right]^2. \quad (5.143)$$

Now recall that the usual form of the SI regulator is $\Delta_G^{-1}(0,\mathbf{0})v = m_G^2 v = \eta m^3$ where m is some arbitrary mass scale (it is convenient to take $m = \mu$). This identifies $\mathcal{F} = \eta m^3$. The $\eta \to 0$ limit is taken so that $\eta\ell_{cN}^c = \ell_0 v$ is a constant. Using this and $\Delta_G^{-1}(0,\mathbf{0}) = \epsilon m_G^2$

gives

$$-\frac{\ell_0 v}{\eta}\left(\epsilon m_G^2 v-\eta\mu^3\right)=\frac{1}{\xi}\left(N-1\right)v^2\left[\epsilon m_G^2\right]^2. \tag{5.144}$$

It is now convenient to take $\eta=(\mathrm{V}\beta)^{-\delta}\,\mu^{-4\delta}$ with $\delta>0$. Taking also the usual scalings for ξ and m_G^2 one finds

$$-\left(\mathrm{V}\beta\right)^{\delta-\gamma}\mu^{4(\delta-\gamma)}\epsilon\ell_0 xy+\ell_0\sqrt{x}=\frac{\mu^{-4\alpha-8\gamma}}{(\mathrm{V}\beta)^{\alpha+2\gamma}}\epsilon^2\left(N-1\right)\frac{xy^2}{\hat{\zeta}}. \tag{5.145}$$

If $\alpha+2\gamma-1>0$ asymptotic balance is impossible (dominant terms can be matched, but not subdominant terms). Likewise, balance cannot be achieved for $\alpha+2\gamma-1=0$. If $\alpha+2\gamma-1<0$, however,

$$\begin{aligned}-\left(\mathrm{V}\beta\right)^{\delta-\gamma}\mu^{4(\delta-\gamma)}\left(\frac{(\mathrm{V}\beta)^{\alpha+2\gamma-1}}{(\mu^{-4})^{\alpha+2\gamma-1}}\right)^{1/3}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}&\ell_0 xy+\ell_0\sqrt{x}\\ &=\frac{\mu^{-4\alpha-8\gamma}}{(\mathrm{V}\beta)^{\alpha+2\gamma}}\left(\frac{(\mathrm{V}\beta)^{\alpha+2\gamma-1}}{(\mu^{-4})^{\alpha+2\gamma-1}}\right)^{2/3}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{2/3}\left(N-1\right)\frac{xy^2}{\hat{\zeta}}.\end{aligned} \tag{5.146}$$

Matching powers of $(\mathrm{V}\beta)$ on both sides gives

$$0=3\delta-1+\alpha-\gamma, \tag{5.147}$$

$$0=\alpha+2\gamma+2, \tag{5.148}$$

which has the solution

$$\alpha=-2\delta, \tag{5.149}$$

$$\gamma=\delta-1. \tag{5.150}$$

$\gamma>0$ requires $\delta>1$. Substituting this into the SI=SSI equation gives

$$-\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}\ell_0 xy+\ell_0\sqrt{x}=\left(\frac{\hat{\zeta}}{4xy^2}\right)^{2/3}\left(N-1\right)\frac{xy^2}{\hat{\zeta}}, \tag{5.151}$$

which can be solved for $\hat{\zeta}$, giving

$$\hat{\zeta}^{1/3}=\left(\frac{1}{2\sqrt{xy}}\right)^{1/3}\left(1\pm\sqrt{1-\left(N-1\right)\frac{y}{\ell_0}}\right). \tag{5.152}$$

This is the desired connection between the SSI stiffness parameter $\hat{\zeta}$ and the SI Lagrange multiplier ℓ_0.

5.5 Summary

This chapter introduced, for the first time to the author's knowledge, a new method of symmetry improvement called *soft symmetry improvement* or SSI. As opposed to the *hard* symmetry improvement (SI) of Chapter 4, Ward identities are imposed weakly in the sense of least squared error. The relative strength of the soft constraint is determined by a new stiffness parameter ξ which can be tuned from zero (equivalent to hard SI) to ∞ (equivalent to no constraint), with the hope that the added freedom allows the SSI method to circumvent some of the problems of the SI method for some range of finite ξ. Here the SSI method was studied for the 2PIEA in the Hartree-Fock approximation, though the generalisation to the nPIEA and higher order approximations is conceptually straightforward (leading only to greater technical complication).

Unfortunately the goals of this program cannot be fully realised in the most interesting case of the infinite volume limit $\beta V \to \infty$. Due to an infrared sensitivity of the SSI method, ξ must be scaled with βV in the limit in order to obtain non-trivial behaviour. Three consistent limiting behaviours were found. Two of these are equivalent to the unimproved 2PIEA and SI-2PIEA, respectively, both of which have been studied previously. The remaining case is a novel one and was studied in detail in section 5.4.2.

In the novel limit the Goldstone self-energy is momentum dependent even in the Hartree-Fock approximation. The zero momentum Goldstone mode is massless as required by Goldstone's theorem, but the remaining modes are massive with a constant $m_G^2 \neq 0$. Perhaps counter-intuitively, m_G^2 *increases* as the symmetry improvement constraint is more strongly imposed. A strong first order phase transition is found for all values of the (scaled) stiffness parameter $\hat{\zeta}$. Further, there is a critical value $\hat{\zeta}_c$ such that for $\hat{\zeta} < \hat{\zeta}_c$ solutions cease to exist for a range of temperatures below the critical temperature. As $\hat{\zeta} \to 0$ this range increases and at some finite value $\hat{\zeta}_\star$ the solution ceases to exist even at zero temperature. This loss of solution was confirmed both analytically and numerically.

This behaviour highlights the difficulty faced by the symmetry improvement program. Ward identities are by nature infrared properties of a field theory and imposing them amounts to modifying a theory at large distances, while trying not to modify anything at short distance scales. However, due to the use of self-consistent propagators and vertices, nPIEAs inherently couple IR and UV behaviour. This can be seen in the loss of solutions of the SI-2PIEA in the two loop truncation as discussed in section 4.4: it cannot be guaranteed for the SI-2PIEA in a given truncation that all UV regulators lead to the same IR physics. In the SSI method in the new limit, the imposition of the Ward identity (even weakly) is incompatible with the existence of massless non-zero modes of the Goldstone propagator in the $\beta V \to 0$ limit. This again leads to loss of solutions. Further, due to the unavoidable scaling of an IR quantity (the SSI term) in the $\beta V \to \infty$ limit, the unimproved, SI and new limits become decoupled. There is no version of the theory which smoothly interpolates between the unimproved and symmetry improved cases in the $\beta V \to \infty$ limit. To put it colloquially, infinite volume is too big a space to "softly" impose anything on.

It would be interesting to examine whether at finite βV any physically reasonable results could be obtained. This could be studied by, for example, putting the universe in a box the size of the Hubble scale. However, this would required redoing the renormalisation procedure in finite volume and re-evaluating the thermal sums which are no longer trivial. Thus, this investigation is deferred to future work.

Chapter 6

Approaching non-equilibrium: Linear Response Theory for Symmetry Improved Effective Actions

6.1 Synopsis

Chapter 2 introduced nPIEAs as a useful general tool for calculations in QFT which can be extended beyond equilibrium and beyond perturbation theory. However, chapter 3 demonstrated that one of the major disadvantages of nPIEAs is that finite order truncations of the effective actions generically do not respect the symmetry properties one expects from the exact theory. A particularly promising remedy for this problem is the symmetry improvement formalism introduced by Pilaftsis and Teresi [1] for 2PIEAs and reviewed in chapter 4. Chapter 4 also extended the formalism to the symmetry improvement of 3PIEA and investigated the properties of this scheme in equilibrium. Chapter 5 introduced the idea of softly imposed symmetry improvement as an attempt to avoid some of the problems of the symmetry improvement formalism, though this was also considered only in equilibrium. This chapter focuses on the extension to non-equilibrium problems, closely following the paper by the author [4].

So far symmetry improvement has only been applied to non-gauged scalar field theories in equilibrium. It restores Goldstone's theorem and produces physically reasonable absorptive parts in propagators [1] and has been shown to restore the second order phase transition of the $\mathrm{O}(4)$ linear sigma model in the Hartree-Fock approximation [116]. It has been used to study pion strings evolving in the thermal bath of a heavy ion collision [128] (though note that the SI-2PIEA was only used to calculate an equilibrium finite temperature effective potential; this work did not constitute a true non-equilibrium calculation using symmetry improved effective actions). Symmetry improvement has also been demonstrated to improve the evaluation of the effective potential of the standard model by taming the infrared divergences of the Higgs sector, treated as an $\mathrm{O}(4)$ scalar field theory with gauge interactions turned off [129, 130].

There is a strong motivation to extend symmetry improvement beyond equilibrium since one of the major reasons for using nPIEAs in the first place is their ability to handle non-equilibrium situations. nPIEAs give an entirely mechanical way to set up the generic initial value problem as a *closed* system of integro-differential equations directly for the mean fields and low order correlation functions, which are simply related to the handful of physical quantities (densities, conserved currents, etc.) one is most often interested in. Apart from a truncation to some finite loop order these equations need not be subject to any further approximation. Hence, apart from the symmetry issue and issues involved in the renormalisation process, nPIEAs give potentially the most general and accurate framework available for the computation of real time properties in quantum field theory. Thus an extension of the symmetry improvement technique to non-equilibrium situations is well motivated. The ultimate goal of this program would be a tractable and manifestly gauge invariant set of equations of motion for highly excited Yang-Mills-Higgs theories with chiral fermion matter based on the self-consistently complete 4PIEA. In the meantime this chapter is restricted to an analysis of the symmetry improved 2PIEA for scalar fields in the linear response regime.

The linear response approximation is investigated rather than a generic non-equilibrium situation for several reasons. First, the linear response approximation is simply far more tractable than the general non-equilibrium situation as the response functions only depend on the equilibrium properties of the theory. Second, linear response is widely applicable in the real world: many systems are "close enough" to equilibrium for practical purposes. Third, the linear response approximation is a nice laboratory to isolate the novel features of symmetry improvement constraints in non-equilibrium settings. Finally, it is expected that any physically reasonable formalism will have a well formed linear response approximation, though this depends on the assumption that the exact behaviour is an analytic (or at least not too singular) function of the external perturbation within some neighbourhood of zero perturbation. This is true of all quantum mechanical systems (so long as the Hamiltonian remains bounded below), but for field theories the infinite number of degrees of freedom may complicate the situation.

The outline of the remainder of this chapter is as follows. In section 6.2 linear response theory is reviewed in connection with 2PIEAs. In section 6.3 a mechanical analogy is used, very similar to the one used to justify the d'Alembert formalism in chapter 4, to illustrate the linear response procedure for constrained systems. In section 6.4 the consequences of the constraints for the linear response functions of the SI-2PIEA are derived, noting that a careful treatment of the constraint procedure requires that not just the WI, but also its derivatives, must vanish. Finally the results are discussed and lessons drawn in section 6.5.

6.2 Linear response theory and nPIEA

Linear response theory studies the effect of small externally applied perturbations on a system initially in equilibrium (see, e.g., [11, 63, 132] for informative discussions). Consider a quantum system which is subjected to an external driving potential $-J(t)\,\hat{B}(t)$ where $J(t)$ is a c-number function of time representing the strength of the driving and $\hat{B}(t)$ is the interaction Hamiltonian (the reason for the name will become apparent). If the initial state of the system is described by a density matrix ρ_0 at time t_0, with $J(t) = 0$ for $t \leq t_0$, then at time $t > t_0$ the expectation of an operator $\hat{A}$ (in the interaction picture with respect to the external perturbation) is:

$$\begin{aligned}
\left\langle \hat{A}(t) \right\rangle &= \mathrm{Tr}\left\{ \rho(t) \hat{A}(t) \right\} \\
&= \mathrm{Tr}\left\{ U(t, t_0) \rho_0 U(t, t_0)^\dagger \hat{A}(t) \right\} \\
&= \mathrm{Tr}\left\{ \mathrm{T}\left[\mathrm{e}^{i \int_{t_0}^{t} J(\tau)\hat{B}(\tau)\mathrm{d}\tau} \right] \rho_0 \tilde{\mathrm{T}}\left[\mathrm{e}^{-i \int_{t_0}^{t} J(\tau)\hat{B}(\tau)\mathrm{d}\tau} \right] \hat{A}(t) \right\} \\
&= \left\langle \tilde{\hat{A}}(t) \right\rangle + i \int_{t_0}^{t} \left\langle \left[\hat{A}(t), \hat{B}(\tau) \right] \right\rangle J(\tau)\,\mathrm{d}\tau + \mathcal{O}\left(J^2\right), \qquad (6.1)
\end{aligned}$$

where $\left\langle \tilde{\hat{A}}(t) \right\rangle$ denotes the expected value in the absence of perturbation. This leads to the definition of the response function

$$\chi^{AB}(t-\tau) = i \left\langle \left[\hat{A}(t), \hat{B}(\tau) \right] \right\rangle \Theta(t-\tau), \qquad (6.2)$$

(which only depends on the time difference due to the equilibrium assumption about ρ_0) such that

$$\delta A(t) \equiv \left\langle \hat{A}(t) \right\rangle - \left\langle \tilde{\hat{A}}(t) \right\rangle = \int_{t_0}^{t} \chi^{AB}(t-\tau) J(\tau)\,\mathrm{d}\tau + \mathcal{O}\left(J^2\right). \qquad (6.3)$$

(The limits can be pushed to $\pm\infty$ thanks to the step function in χ^{AB}, and the equation becomes trivial in the Fourier domain.) The goal of linear response theory is to compute $\chi^{AB}(t-\tau)$ for perturbations $\hat{B}$ and observables $\hat{A}$ of interest. The condition for validity of the approximation is that the quadratic term, which is

$$-\int_{t_0}^{t} \int_{t_0}^{\tau_1} J(\tau_1) J(\tau_2) \left\langle \left[\left[\hat{A}(t), \hat{B}(\tau_1) \right], \hat{B}(\tau_2) \right] \right\rangle \mathrm{d}\tau_2 \mathrm{d}\tau_1, \qquad (6.4)$$

is much smaller than the linear term, which occurs for sufficiently small sources J and times $t - t_0$.

Specialising to the scalar $\mathrm{O}(N)$ field theory of the previous few chapters, one must consider the 2PIEA $\Gamma[\varphi, \Delta]$ which can be connected to the linear response theory by expanding $\varphi \to \tilde{\varphi} + \delta\varphi$ and $\Delta \to \tilde{\Delta} + \delta\Delta$ about their source-free equilibrium values $\tilde{\varphi}$ and $\tilde{\Delta}$ determined by $(\delta\Gamma/\delta\varphi)_{\varphi=\tilde{\varphi},\Delta=\tilde{\Delta}} = (\delta\Gamma/\delta\Delta)_{\varphi=\tilde{\varphi},\Delta=\tilde{\Delta}} = 0$ and matching terms order by order in the sources, treating the responses $\delta\varphi$ and $\delta\Delta$ as first order, as typical of a perturbation theory analysis. At lowest order one finds the usual 2PI equations of motion with no sources for the equilibrium solutions and at first order one finds equations of motion for the perturbations:

$$\frac{\delta^2\Gamma}{\delta\varphi_b\delta\varphi_a}\delta\varphi_b + \frac{\delta^2\Gamma}{\delta\Delta_{bc}\delta\varphi_a}\delta\Delta_{bc} = -J_a - K_{ab}\tilde{\varphi}_b, \qquad (6.5)$$

$$\frac{\delta^2\Gamma}{\delta\varphi_c\delta\Delta_{ab}}\delta\varphi_c + \frac{\delta^2\Gamma}{\delta\Delta_{cd}\delta\Delta_{ab}}\delta\Delta_{cd} = -\frac{1}{2}i\hbar K_{ab}, \qquad (6.6)$$

where all derivatives on the left hand sides are evaluated at the equilibrium values. It is possible to eliminate the fluctuations from these equations by introducing the linear response functions $\chi^{\phi J}_{ab}$, $\chi^{\phi K}_{abc}$, $\chi^{\Delta J}_{abc}$ and $\chi^{\Delta K}_{abcd}$:

$$\delta\varphi_a = \chi^{\phi J}_{ab} J_b + \frac{1}{2}\chi^{\phi K}_{abc} K_{bc}, \tag{6.7}$$

$$\delta\Delta_{ab} = \chi^{\Delta J}_{abc} J_c + \frac{1}{2}\chi^{\Delta K}_{abcd} K_{cd}, \tag{6.8}$$

and demanding that the resulting equations hold for any value of the sources J, K. Doing this leads to the system

$$\frac{\delta^2\Gamma}{\delta\varphi_b\delta\varphi_a}\chi^{\phi J}_{bd} + \frac{\delta^2\Gamma}{\delta\Delta_{bc}\delta\varphi_a}\chi^{\Delta J}_{bcd} = -\delta_{ad}, \tag{6.9}$$

$$\frac{\delta^2\Gamma}{\delta\varphi_c\delta\Delta_{ab}}\chi^{\phi J}_{ce} + \frac{\delta^2\Gamma}{\delta\Delta_{cd}\delta\Delta_{ab}}\chi^{\Delta J}_{cde} = 0, \tag{6.10}$$

$$\frac{\delta^2\Gamma}{\delta\varphi_b\delta\varphi_a}\chi^{\phi K}_{bde} + \frac{\delta^2\Gamma}{\delta\Delta_{bc}\delta\varphi_a}\chi^{\Delta K}_{bcde} = -\left(\delta_{ad}\tilde{\varphi}_e + \delta_{ae}\tilde{\varphi}_d\right), \tag{6.11}$$

$$\frac{\delta^2\Gamma}{\delta\varphi_c\delta\Delta_{ab}}\chi^{\phi K}_{cef} + \frac{\delta^2\Gamma}{\delta\Delta_{cd}\delta\Delta_{ab}}\chi^{\Delta K}_{cdef} = -\frac{1}{2}i\hbar\left(\delta_{ae}\delta_{bf} + \delta_{af}\delta_{be}\right), \tag{6.12}$$

where note that in the last two equations one must symmetrise the right hand sides before removing the source K (since by the symmetry of K only the symmetric part contributes). These equations determine the linear response functions entirely in terms of the equilibrium properties of the theory (in particular, the second derivatives of the effective action evaluated at the equilibrium solution). Note that the last equation can be recast as a Bethe-Salpeter equation for the $\chi^{\Delta K}$ by using $\Delta^{-1} = \Delta_0^{-1} - \Sigma + K$ to write

$$\begin{aligned}
-\Delta^{-1}_{ab} + \Delta^{-1}_{0ab}[\varphi] - \Sigma_{ab}[\varphi,\Delta] &= -\Delta^{-1}_{ab} + \left(\Delta^{-1}_{0ab}[\varphi] - \Delta^{-1}_{0ab}[\tilde{\varphi}]\right) \\
&\quad + \left(\Delta^{-1}_{0ab}[\tilde{\varphi}] - \Sigma_{ab}\left[\tilde{\varphi},\tilde{\Delta}\right]\right) + \left(\Sigma_{ab}\left[\tilde{\varphi},\tilde{\Delta}\right] - \Sigma_{ab}[\varphi,\Delta]\right) \\
&= -\delta\Delta^{-1}_{ab} + \frac{\delta\Delta^{-1}_{0ab}}{\delta\varphi_c}\delta\varphi_c - \left(\frac{\delta\Sigma_{ab}}{\delta\varphi_c}\delta\varphi_c + \frac{\delta\Sigma_{ab}}{\delta\Delta_{cd}}\delta\Delta_{cd}\right),
\end{aligned} \tag{6.13}$$

then using the identity

$$\delta\Delta^{-1}_{ab} = -\tilde{\Delta}^{-1}_{ac}\delta\Delta_{cd}\tilde{\Delta}^{-1}_{db} + \mathcal{O}\left(J^2, K^2, JK\right), \tag{6.14}$$

and the definitions of the linear response functions followed by some rearrangement to give

$$\begin{aligned}
\chi^{\Delta K}_{abef} &= -\tilde{\Delta}_{ac}\left(\delta_{ce}\delta_{df} + \delta_{cf}\delta_{de}\right)\tilde{\Delta}_{db} - \tilde{\Delta}_{ag}\left(\frac{\delta\Delta^{-1}_{0gh}}{\delta\varphi_c} - \frac{\delta\Sigma_{gh}}{\delta\varphi_c}\right)\tilde{\Delta}_{hb}\chi^{\phi K}_{cef} \\
&\quad + \left(\tilde{\Delta}_{ag}\frac{\delta\Sigma_{gh}}{\delta\Delta_{cd}}\tilde{\Delta}_{hb}\right)\chi^{\Delta K}_{cdef}.
\end{aligned} \tag{6.15}$$

This is an equation which determines the four point kernel $\chi^{\Delta K}$ iteratively, i.e. a Bethe-Salpeter equation with the last quantity in braces being the Bethe-Salpeter kernel.

6.3 Mechanical analogy to illustrate constrained linear response theory

Here follows a brief discussion of a very simple mechanical system which illustrates several of the unusual features of the constraint procedure and linear response formulation which will be used. The chief unusual feature is that the variables being perturbed are constrained by symmetry improvement, so a *constrained* linear response formalism is required. This mechanical example shows how this can be done and, in particular, why (a) the Lagrange multiplier diverges, (b) constraints must be imposed in the linear response approximation to begin with, and (c) secondary constraints arise. Consider a unit mass classical particle constrained to move without friction on a circular hoop of radius r in the $x-y$ plane. Its Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \lambda W + j_x x + j_y y, \tag{6.16}$$

$$W = \frac{1}{4}\left(x^2 + y^2 - r^2\right)^2, \tag{6.17}$$

where λ is the Lagrange multiplier and the form of the constraint $W = 0$ is chosen to mimic the singular constraint procedure. The equations of motion are

$$\begin{aligned}\ddot{x} &= -\lambda \partial_x W + j_x \\ &= -\lambda\left(x^2 + y^2 - r^2\right) x + j_x,\end{aligned} \tag{6.18}$$

$$\begin{aligned}\ddot{y} &= -\lambda \partial_y W + j_y \\ &= -\lambda\left(x^2 + y^2 - r^2\right) y + j_y.\end{aligned} \tag{6.19}$$

Now consider the source free case $j_x = j_y = 0$. The constraint terms vanish unless $\lambda \to \infty$ as $x^2 + y^2 - r^2 \to 0$. Set $x^2 + y^2 - r^2 = \eta$ and $\lambda\eta = \omega^2$ and take the limit such that ω^2 is a constant (recall "d'Alembert's principle" c.f. section 4.6). Note that $W = \eta^2/4$. Then the equations of motion become

$$\ddot{x} = -\omega^2 x, \tag{6.20}$$

$$\ddot{y} = -\omega^2 y, \tag{6.21}$$

with the solutions

$$x = r\cos\left[\omega\left(t - t_0\right)\right], \tag{6.22}$$

$$y = r\sin\left[\omega\left(t - t_0\right)\right], \tag{6.23}$$

where t_0 and ω are determined by the initial conditions. For simplicity the equilibrium solution is chosen as $x = r$ and $y = 0$, which determines $\omega = 0$ and $t_0 = 0$ (without loss of generality).

Now turn on the sources j_x and j_y and investigate the linear response by setting $x \to \tilde{x} + \delta x$, $y \to \tilde{y} + \delta y$, $\lambda \to \tilde{\lambda} + \delta\lambda$ where the tilde variables are the source free solutions. The variation of the constraint is

$$\delta W = \eta\left(\tilde{x}\delta x + \tilde{y}\delta y\right) \to 0, \tag{6.24}$$

regardless of the behaviour of δx and δy, so long as they are non-singular in the $\eta \to 0$ limit. However, the first order equations of motion become

$$\begin{aligned}
\delta\ddot{x} &= -\delta\lambda\eta\tilde{x} - 2\tilde{\lambda}\left(\tilde{x}\delta x + \tilde{y}\delta y\right)\tilde{x} - \tilde{\lambda}\eta\delta x + j_x \\
&= F_x^{\mathrm{rad}} - \omega^2\delta x + j_x = -\omega^2\delta x + j_x^{\perp}, \qquad (6.25)\\
\delta\ddot{y} &= -\delta\lambda\eta\tilde{y} - 2\tilde{\lambda}\left(\tilde{x}\delta x + \tilde{y}\delta y\right)\tilde{y} - \tilde{\lambda}\eta\delta y + j_y \\
&= F_y^{\mathrm{rad}} - \omega^2\delta y + j_y = -\omega^2\delta y + j_y^{\perp}, \qquad (6.26)
\end{aligned}$$

where the radial force is defined as $\mathbf{F}^{\mathrm{rad}} = -\left[\delta\lambda\eta + 2\tilde{\lambda}\left(\tilde{x}\delta x + \tilde{y}\delta y\right)\right](\tilde{x}, \tilde{y})$, whose physical function is to balance the applied force normal to the constraint surface, resulting in the net transverse source $\mathbf{j}^{\perp}$.

Now notice the terms proportional to $\tilde{\lambda}\left(\tilde{x}\delta x + \tilde{y}\delta y\right)$ in the equations of motion. In order for these terms to be well behaved in the limit $\tilde{\lambda} \to \infty$ one must have $\tilde{x}\delta x + \tilde{y}\delta y \to 0$, i.e. the response remains within the constraint surface (to first order). Thus the vanishing of these terms in addition to the vanishing of δW is required to fully enforce that the response be tangential to the constraint surface. Also note that by examining the $\delta\lambda$ terms in the equation of motion one can identify which component of the applied force acts normal to the constraint surface (and hence produces no physical response).

Applying the equilibrium solution one find $\mathbf{F}^{\mathrm{rad}} = -\left[\delta\lambda\eta + 2\tilde{\lambda}r\delta x\right](r, 0)$. For this to be well behaved as $\tilde{\lambda} \to \infty$ requires $\delta x = 0$, which also determines $j_x^{\perp} = 0$ via the δx equation of motion. The δy equation of motion is

$$\delta\ddot{y} = j_y, \tag{6.27}$$

with the solution (taking into account the initial conditions $\delta y_0 = \delta\dot{y}_0 = 0$):

$$\delta y = \int_0^t \int_0^\tau j_y\left(\tau'\right)\mathrm{d}\tau'\mathrm{d}\tau = \int_0^t (t-\tau)\, j_y(\tau)\,\mathrm{d}\tau. \tag{6.28}$$

Now compare this to the exact solution. Substituting the ansatz $x = r\cos\theta(t)$, $y = r\sin\theta(t)$, the Lagrangian and equation of motion become

$$L = \frac{1}{2}r^2\dot{\theta}^2 + j_x r\cos\theta + j_y r\sin\theta, \tag{6.29}$$

$$\ddot{\theta} = -\frac{j_x}{r}\sin\theta + \frac{j_y}{r}\cos\theta = \frac{j_\theta}{r}, \tag{6.30}$$

where $j_\theta = -j_x\sin\theta + j_y\cos\theta$ is the tangential component of the force. The solution satisfying the initial conditions $\theta_0 = \dot{\theta}_0 = 0$ is

$$\theta = \int_0^t (t-\tau)\frac{j_\theta(\tau)}{r}\mathrm{d}\tau. \tag{6.31}$$

To linear order in j_θ the x and y components are just

$$x = r\cos\theta = 0, \tag{6.32}$$

$$y = r\sin\theta = \int_0^t (t-\tau)\, j_\theta(\tau)\, \mathrm{d}\tau, \tag{6.33}$$

which, on putting $j_\theta = j_y + \mathcal{O}\left(j_{x,y}^2\right)$, gives

$$y = \int_0^t (t-\tau)\, j_y(\tau)\, \mathrm{d}\tau, \tag{6.34}$$

which is just the solution obtained previously.

If in contrast one never imposed the constraints on the linear response solution one would obtain the equations of motion

$$\delta\ddot{x} = j_x, \tag{6.35}$$

$$\delta\ddot{y} = j_y, \tag{6.36}$$

with the solutions

$$\delta x = \int_0^t (t-\tau)\, j_x(\tau)\, \mathrm{d}\tau, \tag{6.37}$$

$$\delta y = \int_0^t (t-\tau)\, j_y(\tau)\, \mathrm{d}\tau. \tag{6.38}$$

The δy component is correct but δx is in error already at linear order. In fact, the solution should not even depend on j_x until second order.

6.4 Symmetry improvement and linear response

6.4.1 The simple constraint scheme

In the linear response approximation J_a and K_{ab} no longer vanish, and the SI-2PI equations of motion must be solved to first order in the sources without any assumption of homogeneity. Now it makes a difference whether one uses the simple constraint (4.3)

$$\mathcal{C} = \frac{i}{2}\ell_A^c \mathcal{W}_c^A, \tag{6.39}$$

or projected constraint (4.4)

$$\mathcal{C}' = \frac{i}{2}\ell_A^c P_{cd}^\perp \mathcal{W}_d^A \tag{6.40}$$

for symmetry improvement. In this section only the simple constraint is considered. The projected constraint is considered in the next section. The SI-2PI equations of motion were given in chapter 4, though chapter 4 (along with all of the published SI literature to date) neglects the WI (3.14), reproduced here:

$$\mathcal{W}^A = -J_a T_{ab}^A \varphi_b. \tag{6.41}$$

The consistency of the linear response theory requires that this identity also be enforced if $J_a \neq 0$. Therefore a new constraint $\mathcal{C}'' = i\ell_A \mathcal{W}^A$ must be added to Γ and the symmetry improved equations of motion are altered to

$$\mathcal{W}^A = -J_a T^A_{ab}\varphi_b = 0, \tag{6.42}$$

$$\mathcal{W}^A_c = \Delta^{-1}_{ca} T^A_{ab}\varphi_b - J_a T^A_{ac} = 0, \tag{6.43}$$

$$\frac{\delta\Gamma}{\delta\varphi_d(z)} = -iJ_a(z)\,\ell_A T^A_{ad} + \frac{i}{2}\int_x \ell^c_A(x)\,\Delta^{-1}_{ca}(x,z)\,T^A_{ad} - J_d(z) - \int_w K_{de}(z,w)\,\varphi_e(w), \tag{6.44}$$

$$\frac{\delta\Gamma}{\delta\Delta_{de}(z,w)} = -\frac{i}{2}\int_x \ell^c_A(x)\,\Delta^{-1}_{cd}(x,z)\int_y \Delta^{-1}_{ea}(w,y)\,T^A_{ab}\varphi_b(y) - \frac{1}{2}i\hbar K_{de}(z,w). \tag{6.45}$$

The only effect of the new WI on the previously established equations of motion is the addition of a term $\propto J$ on the right hand side of the φ equation of motion and a linear constraint saying that φ must be proportional to J if $J \neq 0$. This WI has no consequences if $J_a = 0$ so none of the previous results are affected. Also note that the new Lagrange multipliers ℓ_A are finite because the term on the right hand side of the φ equation of motion does not vanish identically.

To find the linear response expand all quantities $\varphi \to \tilde{\varphi} + \delta\varphi$, $\Delta \to \tilde{\Delta} + \delta\Delta$ and $\ell \to \tilde{\ell} + \delta\ell$ and match terms order by order, considering the $\delta\varphi$ etc. as first order. Working first on the WIs gives the equations:

$$0 = J_a T^A_{ab}\tilde{\varphi}_b \tag{6.46}$$

$$0 = \tilde{\Delta}^{-1}_{ca} T^A_{ab}\tilde{\varphi}_b, \tag{6.47}$$

$$0 = \delta\Delta^{-1}_{ca} T^A_{ab}\tilde{\varphi}_b + \tilde{\Delta}^{-1}_{ca} T^A_{ab}\delta\varphi_b - J_a T^A_{ac}. \tag{6.48}$$

The first equation requires that the source components in the Goldstone directions vanish, that is $J_a = (0, \cdots, 0, J_N)$. The generation of Goldstone field fluctuations is outside the scope of linear response theory. The second equation is simply the equilibrium constraint as expected. The third equation is new.

To understand (6.48) start by taking the variation of Δ^{-1} from its definition in terms of the 1PIEA (2.31) (recall that $\Delta^{-1} = \delta^2\Gamma^{(1)}/\delta\varphi\delta\varphi$). One can write

$$\delta\Delta^{-1}_{ca} = \frac{\delta^3\Gamma^{(1)}}{\delta\varphi_d\delta\varphi_c\delta\varphi_a}\delta\varphi_d + \mathcal{O}\left(\delta\varphi^2\right), \tag{6.49}$$

and note that $\delta^3\Gamma^{(1)}/\delta\varphi_d\delta\varphi_c\delta\varphi_a = V_{dca}$ is the three point vertex function. These relations no longer hold identically for the 2PI correlation functions, but they do hold numerically for the *exact* solutions of the untruncated 2PI equations of motion. This can be extended to the 2PIEA by writing

$$\delta\Delta^{-1}_{ca} = V_{dca}\delta\varphi_d + \mathcal{K}_{ca}, \tag{6.50}$$

where $\mathcal{K}_{ca}$ encapsulates the additional variations of Δ in the truncated 2PIEA formalism. Also recall there is a WI for the three point vertex function (3.16),

$$\mathcal{W}^A_{dc} = V_{dca} T^A_{ab} \varphi_b + \Delta^{-1}_{ca} T^A_{ad} + \Delta^{-1}_{da} T^A_{ac}, \tag{6.51}$$

which is unaffected by the presence of sources. As with $\mathcal{W}^A_c$, $\mathcal{W}^A_{dc} = 0$ in the exact theory, but not automatically in truncations. Combining these relations with (6.48) gives:

$$\begin{aligned} 0 &= \delta\varphi_d \left(\mathcal{W}^A_{dc} - \tilde{\Delta}^{-1}_{ca} T^A_{ad} - \tilde{\Delta}^{-1}_{da} T^A_{ac} \right) + \mathcal{K}_{ca} T^A_{ab} \tilde{\varphi}_b + \tilde{\Delta}^{-1}_{ca} T^A_{ab} \delta\varphi_b - J_a T^A_{ac} \\ &= \delta\varphi_d \left(\mathcal{W}^A_{dc} - \tilde{\Delta}^{-1}_{da} T^A_{ac} \right) + \mathcal{K}_{ca} T^A_{ab} \tilde{\varphi}_b - J_a T^A_{ac}, \end{aligned} \tag{6.52}$$

which can be rearranged as

$$\left(\delta\varphi_d \tilde{\Delta}^{-1}_{da} + J_a \right) T^A_{ac} = \delta\varphi_d \mathcal{W}^A_{dc} + \mathcal{K}_{ca} T^A_{ab} \tilde{\varphi}_b. \tag{6.53}$$

If the right hand side vanishes one has (using the symmetry of $\tilde{\Delta}^{-1}$)

$$\tilde{\Delta}^{-1}_{ad} \delta\varphi_d = -J_a, \tag{6.54}$$

which is the correct equation of motion for fluctuations. The failure of these equations to be satisfied is measured by the terms on the right hand side of (6.53), which have two distinct meanings. The first, $\delta\varphi_d \mathcal{W}^A_{dc}$, measures the failure of the *vertex* WI to be satisfied by 2PI approximations, while the second term, $\mathcal{K}_{ca} T^A_{ab} \tilde{\varphi}_b$, measures the failure of the variations in the 2PI functions φ and Δ to be linked according to the 1PI relations. Generically one cannot expect these terms to cancel as they are logically independent (i.e., they do not cancel as the result of any basic identity). The vanishing of the right hand side of (6.53) is an additional constraint on the linear response functions, which are fully determined by the 2PI equations of motion. Therefore the existence of a physically reasonable linear response approximation is a "fine tuned" property of particular truncations.

The "anomalous" terms on the right hand side can be understood in greater detail by expanding $\mathcal{K}_{ca}$ using the identity[1] $\delta\Delta^{-1}_{ca} = -\Delta^{-1}_{cd} \delta\Delta_{de} \Delta^{-1}_{ea} + \mathcal{O}\left(\delta\Delta^2\right)$ and the expressions for $\delta\varphi$ and $\delta\Delta$ in terms of the linear response functions (6.7)-(6.8) to obtain

$$\begin{aligned} \mathcal{K}_{ca} &= -\Delta^{-1}_{cd} \left(\chi^{\Delta J}_{def} J_f + \frac{1}{2} \chi^{\Delta K}_{defg} K_{fg} \right) \Delta^{-1}_{ea} - V_{dca} \left(\chi^{\phi J}_{de} J_e + \frac{1}{2} \chi^{\phi K}_{def} K_{ef} \right) \\ &= \left(-\Delta^{-1}_{cd} \chi^{\Delta J}_{def} \Delta^{-1}_{ea} - V_{dca} \chi^{\phi J}_{df} \right) J_f + \frac{1}{2} \left(-\Delta^{-1}_{cd} \chi^{\Delta K}_{defg} \Delta^{-1}_{ea} - V_{dca} \chi^{\phi K}_{dfg} \right) K_{fg}. \end{aligned} \tag{6.55}$$

[1] Note that one must be careful using this identity because $\delta\Delta$ can develop an IR divergence. $\delta\Delta^{-1}$ is at least "infrared safe."

The anomalous right hand side becomes

$$
\begin{aligned}
\mathcal{F}^A_c &\equiv \delta\varphi_d \mathcal{W}^A_{dc} + \mathcal{K}_{ca} T^A_{ab}\tilde{\varphi}_b \\
&= \left(\chi^{\phi J}_{df} J_f + \frac{1}{2}\chi^{\phi K}_{dfg} K_{fg}\right)\left(V_{dca}T^A_{ab}\tilde{\varphi}_b + \tilde{\Delta}^{-1}_{ca}T^A_{ad} + \tilde{\Delta}^{-1}_{da}T^A_{ac}\right) \\
&\quad + \left[\left(-\tilde{\Delta}^{-1}_{cd}\chi^{\Delta J}_{def}\tilde{\Delta}^{-1}_{ea} - V_{dca}\chi^{\phi J}_{df}\right) J_f + \frac{1}{2}\left(-\tilde{\Delta}^{-1}_{cd}\chi^{\Delta K}_{defg}\tilde{\Delta}^{-1}_{ea} - V_{dca}\chi^{\phi K}_{dfg}\right) K_{fg}\right] T^A_{ab}\tilde{\varphi}_b.
\end{aligned}
\tag{6.56}
$$

Demanding that this vanishes independent of the sources gives the pair of conditions

$$0 = \chi^{\phi J}_{df}\left(\tilde{\Delta}^{-1}_{ca}T^A_{ad} + \tilde{\Delta}^{-1}_{da}T^A_{ac}\right) - \tilde{\Delta}^{-1}_{cd}\chi^{\Delta J}_{def}\tilde{\Delta}^{-1}_{ea}T^A_{ab}\tilde{\varphi}_b, \tag{6.57}$$

$$0 = \chi^{\phi K}_{dfg}\left(\tilde{\Delta}^{-1}_{ca}T^A_{ad} + \tilde{\Delta}^{-1}_{da}T^A_{ac}\right) - \tilde{\Delta}^{-1}_{cd}\chi^{\Delta K}_{defg}\tilde{\Delta}^{-1}_{ea}T^A_{ab}\tilde{\varphi}_b, \tag{6.58}$$

which must be satisfied by the linear response functions in order for the linear response approximation to correctly describe the propagation of field fluctuations.

The equations of motion following from (6.53) can be worked out by substituting particular values for A and c and working out the components. The nontrivial components are

$$\tilde{\Delta}^{-1}_G\delta\varphi_g + J_g = \delta\varphi_g\left(V_{GHG}v + \tilde{\Delta}^{-1}_G - \tilde{\Delta}^{-1}_H\right) + \mathcal{K}_{Ng}v, \qquad A = (g,N)\,,\ c = N, \tag{6.59}$$

$$0 = \mathcal{K}_{cg}v, \qquad A = (g,N)\,,\ c \neq g, N, \tag{6.60}$$

$$\tilde{\Delta}^{-1}_H\delta\varphi_N + J_N = -\delta\varphi_N\left(V_{HGG}v + \tilde{\Delta}^{-1}_G - \tilde{\Delta}^{-1}_H\right) - \mathcal{K}_{gg}v, \qquad A = (g,N)\,,\ c = g, \tag{6.61}$$

$$\tilde{\Delta}^{-1}_G\delta\varphi_g + J_g = 0, \qquad A = \left(g, g'\right),\ g, g' \neq N,\ c = g', c \neq g. \tag{6.62}$$

(Note that $\mathcal{K}_{gg}$ is not summed on g.) These equations are not completely independent but neither are they trivial. In particular, the consistency of the first and fourth equation requires

$$\delta\varphi_g\left(V_{GHG}v + \tilde{\Delta}^{-1}_G - \tilde{\Delta}^{-1}_H\right) + \mathcal{K}_{Ng}v = 0, \tag{6.63}$$

which is not guaranteed by the SI-2PI equations of motion. The corresponding conditions on the response functions are

$$0 = \chi^{\phi J}_{kf}\left(\tilde{\Delta}^{-1}_G - \tilde{\Delta}^{-1}_H\right) - \tilde{\Delta}^{-1}_H\chi^{\Delta J}_{Nkf}\tilde{\Delta}^{-1}_G v, \tag{6.64}$$

$$0 = \chi^{\phi K}_{kfg}\left(\tilde{\Delta}^{-1}_G - \tilde{\Delta}^{-1}_H\right) - \tilde{\Delta}^{-1}_H\chi^{\Delta K}_{Nkfg}\tilde{\Delta}^{-1}_G v. \tag{6.65}$$

There are further constraints lurking in the right hand sides of the SI-2PI equations of motion. Since the constraint procedure involves a limit $\tilde{\ell} \to \infty$ there is a danger that terms on the right hand sides can diverge. Consider the expansion of the right hand side of the φ equation of motion to first order:

$$\frac{i}{2}\int_x \delta\ell^c_A(x)\,\tilde{\Delta}^{-1}_{ca}(x,z)\,T^A_{ad}+\frac{i}{2}\int_x \tilde{\ell}^c_A(x)\,\delta\Delta^{-1}_{ca}(x,z)\,T^A_{ad}$$
$$-J_d(z)-\int_w K_{de}(z,w)\,\tilde{\varphi}_e(w)\,. \quad (6.66)$$

Using (4.14) (recall $\ell^a_{bN}=-\ell^a_{Nb}=P^\perp_{ab}\left(\frac{1}{N-1}\ell^c_{cN}\right)$) the term proportional to $\tilde{\ell}$ can be written

$$-\frac{1}{N-1}\tilde{\ell}^e_{eN}\left(P^\perp_{cg}\int_x \delta\Delta^{-1}_{cg}(x,z)\,\delta_{Nd}-P^\perp_{cd}\int_x \delta\Delta^{-1}_{cN}(x,z)\right), \quad (6.67)$$

which, for the $\tilde{\ell}^e_{eN}\to\infty$ limit to exist, requires that the term in braces vanishes, i.e.

$$0=\delta_{Nd}\int_x P^\perp_{cg}\delta\Delta^{-1}_{cg}(x,z)-P^\perp_{cd}\int_x \delta\Delta^{-1}_{cN}(x,z)\,. \quad (6.68)$$

A similar analysis for the Δ equation of motion gives

$$0=vm^2_G\int_x\left(P^\perp_{ea}\delta\Delta^{-1}_{ad}(x,z)+\delta\Delta^{-1}_{ea}(w,x)\,P^\perp_{ad}\right)$$
$$-P^\perp_{db}\delta_{eN}m^2_G\int_y \tilde{\Delta}^{-1}_H(w,y)\,\delta\varphi_b(y)+P^\perp_{de}m^2_G\int_y \tilde{\Delta}^{-1}_G(w,y)\,\delta\varphi_N(y)\,. \quad (6.69)$$

The constraints (6.68) and (6.69) are called the *secondary* constraints of the scheme and, by contrast, equation (6.48) the *primary* constraint[2]. The secondary constraints must be enforced so that no divergences appear in the $\tilde{\ell}\to\infty$ limit of the equations of motion. Note that one can take $m^2_G\to 0$ without any problems in (6.69), so that constraint is satisfied identically. Also, (6.68) can be simplified following the same procedure as (6.53), yielding two further conditions on the linear response functions

$$0=\int_{xwv}\left[-\delta_{Nd}P^\perp_{ab}\Delta^{-1}_G(x,w)\,\Delta^{-1}_G(v,z)+P^\perp_{ad}\delta_{bN}\Delta^{-1}_G(x,w)\,\Delta^{-1}_H(v,z)\right]\chi^{\Delta J}_{abc}(w,v,u)\,, \quad (6.70)$$

$$0=\int_{xwv}\left[-\delta_{Nd}P^\perp_{ab}\Delta^{-1}_G(x,w)\,\Delta^{-1}_G(v,z)+P^\perp_{ad}\delta_{bN}\Delta^{-1}_G(x,w)\,\Delta^{-1}_H(v,z)\right]\chi^{\Delta K}_{abcd}(w,v,u,y)\,. \quad (6.71)$$

With one exception it is not known whether any truncation satisfies all of these conditions. The exception is the truncation of the *equations of motion*[3] to lowest order in $\hbar$,

[2] These should not to be confused with the terminology from Dirac's constrained quantisation method [133].

[3] The truncation of Γ to $\mathcal{O}(\hbar^0)$ leaves Δ undetermined.

for which

$$
\begin{aligned}
\delta\Delta_{ca}^{-1} &= \delta\Delta_{0ca}^{-1} \\
&= \left(-\frac{\lambda}{3}\tilde{\varphi}_d\delta\varphi^d\delta_{ac} - \frac{\lambda}{3}\delta\varphi_a\tilde{\varphi}_c - \frac{\lambda}{3}\tilde{\varphi}_a\delta\varphi_c\right)\delta\left(x-y\right) \\
&= -\frac{\lambda}{3}v\left(\delta\varphi_N\delta_{ac} + \delta\varphi_a\delta_{cN} + \delta_{aN}\delta\varphi_c\right)\delta\left(x-y\right). \qquad (6.72)
\end{aligned}
$$

Substituting this into the WI gives

$$
0 = -\frac{\lambda v^2}{3}\left(\delta\varphi_N\left(x\right)\delta_{ac} + \delta\varphi_a\left(x\right)\delta_{cN}\right)T^A_{aN} + \int_y \tilde{\Delta}^{-1}_{ca}\left(x,y\right)T^A_{ab}\delta\varphi_b\left(y\right) - J_a T^A_{ac}, \qquad (6.73)
$$

and working out the components gives the set of equations

$$
\int_y \left[\tilde{\Delta}^{-1}_G\left(x,y\right) - \frac{\lambda v^2}{3}\delta\left(x-y\right)\right]\delta\varphi_N\left(y\right) = -J_N, \qquad (6.74)
$$

$$
\int_y \left[\tilde{\Delta}^{-1}_H\left(x,y\right) + \frac{\lambda v^2}{3}\delta\left(x-y\right)\right]\delta\varphi_g\left(y\right) = -J_g, \qquad (6.75)
$$

$$
\int_y \tilde{\Delta}^{-1}_G\left(x,y\right)\delta\varphi_g\left(y\right) = -J_g. \qquad (6.76)
$$

This system is consistent if $m_H^2 = m_G^2 + \lambda v^2/3 + \mathcal{O}\left(\hbar\right)$ which, of course, is true. Thus, for the 1PIEA the WI propagates Higgs and Goldstone fluctuations with the correct masses and source terms. This is expected because the WI was constructed to be satisfied by the 1PI correlation functions. One further has that $\delta\varphi_g = 0$ as a consequence of $J_g = 0$. Furthermore, from the $d = N$ component of the secondary constraint one has

$$
\begin{aligned}
0 &= P^\perp_{cg}\int_x \delta\Delta^{-1}_{cg}\left(x,z\right) \\
&= -\frac{\lambda}{3}vP^\perp_{cg}\int_x \left(\delta\varphi_N\delta_{gc} + \delta\varphi_g\delta_{cN} + \delta_{gN}\delta\varphi_c\right)\delta\left(x-z\right) \\
&= -\frac{\lambda}{3}v\left(N-1\right)\delta\varphi_N\left(z\right), \qquad (6.77)
\end{aligned}
$$

which implies that $\delta\varphi_N = 0$, which in turn implies $\delta\Delta^{-1} = 0$, which in turn requires $K = 0$ which is a consequence of the 2PI equation of motion. Thus the classical approximation is consistent, but trivial. There is no longer any reason for this to go through if $\delta\Delta$ is independent of $\delta\varphi$, which is the case in higher order truncations.

6.4.2 The projected constraint scheme

Now consider using the projected constraint

$$
\mathcal{C}' = \frac{i}{2}\ell^c_A P^\perp_{cd}\mathcal{W}^A_d, \qquad (6.78)
$$

instead of the simple constraint used above for the symmetry improvement. The projection operator forces only Goldstone modes to be involved in the constraint. Note that the constraints which have been projected out are valid WIs which are obeyed by the 1PIEA. This alternate scheme consists of picking a *subset* of the WIs to enforce, chosen in the only $\mathrm{O}(N)$-covariant way available. The new constraint is

$$0 = P^{\perp}_{cd}\left(\Delta^{-1}_{da} T^{A}_{ab}\varphi_b + T^{A}_{da} J_a\right). \tag{6.79}$$

At lowest order this becomes

$$0 = -v m_G^2 T^{A}_{cN}, \tag{6.80}$$

which is the same as before. However, at first order there are new terms due to the variation of $P^{\perp}$. The first and second derivatives of $P^{\perp}$ evaluated at $\tilde{\varphi}$ are needed, and are

$$\frac{\delta P^{\perp}_{dc}(x)}{\delta\varphi_e(z)}[\tilde{\varphi}] = -\frac{1}{v}\left(\delta_{de}\delta_{cN} + \delta_{dN}\delta_{ce} - 2\delta_{dN}\delta_{cN}\delta_{eN}\right)\delta(x-z), \tag{6.81}$$

$$\frac{\delta^2 P^{\perp}_{dc}(x)}{\delta\varphi_f(w)\,\delta\varphi_e(z)}[\tilde{\varphi}] = \frac{1}{v^2} F_{dcfe}\,\delta(z-w)\,\delta(x-z), \tag{6.82}$$

where

$$\begin{aligned} F_{dcfe} = {} & 2\delta_{cN}\delta_{dN}\tilde{P}^{\perp}_{ef} - \left(\tilde{P}^{\perp}_{ce}\tilde{P}^{\perp}_{df} + \tilde{P}^{\perp}_{de}\tilde{P}^{\perp}_{cf}\right) \\ & + \left(\tilde{P}^{\perp}_{ce}\delta_{dN}\delta_{fN} + \tilde{P}^{\perp}_{de}\delta_{cN}\delta_{fN} + \tilde{P}^{\perp}_{cf}\delta_{dN}\delta_{eN} + \tilde{P}^{\perp}_{df}\delta_{cN}\delta_{eN}\right), \end{aligned} \tag{6.83}$$

which is symmetric in cd and in ef. In the linear response approximation one requires

$$\delta P^{\perp}_{cd}(x) \to \int_z \frac{\delta P^{\perp}_{dc}(x)}{\delta\varphi_e(z)}[\tilde{\varphi}]\,\delta\varphi_e(z) = -\frac{1}{v}\left(\delta_{cN}\delta\varphi_d(x) + \delta_{dN}\delta\varphi_c(x) - 2\delta_{dN}\delta_{cN}\delta\varphi_N(x)\right), \tag{6.84}$$

$$\delta\left[\frac{\delta P^{\perp}_{cd}(x)}{\delta\varphi_e(z)}\right] \to \int_w \frac{\delta^2 P^{\perp}_{dc}(x)}{\delta\varphi_f(w)\,\delta\varphi_e(z)}[\tilde{\varphi}]\,\delta\varphi_f(w) = \frac{1}{v^2}F_{dcfe}\,\delta(x-z)\,\delta\varphi_f(x). \tag{6.85}$$

The first order constraint is

$$0 = P^{\perp}_{cd}\left(\delta\Delta^{-1}_{da}T^{A}_{ab}\tilde{\varphi}_b + \tilde{\Delta}^{-1}_{da}T^{A}_{ab}\delta\varphi_b + T^{A}_{da}J_a\right) + \delta P^{\perp}_{cd}\left(\tilde{\Delta}^{-1}_{da}T^{A}_{ab}\tilde{\varphi}_b\right). \tag{6.86}$$

The second term vanishes due to the zeroth order WI and the first term is just (6.48) with only the $c \neq N$ equations picked out. This gives the set of wave equations:

$$0 = \mathcal{K}_{cg} v, \qquad A = (g, N),\ c \neq g, N, \tag{6.87}$$

$$\tilde{\Delta}^{-1}_{H}\delta\varphi_N + J_N = -\delta\varphi_N\left(V_{HGG}v + \tilde{\Delta}^{-1}_{G} - \tilde{\Delta}^{-1}_{H}\right) - \mathcal{K}_{gg}v, \qquad A = (g, N),\ c = g, \tag{6.88}$$

$$\tilde{\Delta}^{-1}_{G}\delta\varphi_g + J_g = 0, \qquad A = (g, g'),\ g, g' \neq N,\ c = g', c \neq g. \tag{6.89}$$

In order to find the secondary constraints one writes the equations of motion

$$\frac{\delta\Gamma}{\delta\varphi_d(z)} = -iJ_a(z)\,\ell_A T^A_{ad} + \frac{i}{2}\int_x \ell^f_A(x)\,\frac{\delta P^\perp_{fc}(x)}{\delta\varphi_d(z)}\left(\int_y \Delta^{-1}_{ca}(x,y)\,T^A_{ab}\varphi_b(y) + T^A_{ca}J_a(x)\right)$$
$$+\frac{i}{2}\int_x \ell^f_A(x)\,P^\perp_{fc}(x)\,\Delta^{-1}_{ca}(x,z)\,T^A_{ad} - J_d(z) - \int_w K_{de}(z,w)\,\varphi_e(w)\,, \qquad (6.90)$$
$$\frac{\delta\Gamma}{\delta\Delta_{de}(z,w)} = -\frac{i}{2}\int_x \ell^f_A(x)\,P^\perp_{fc}(x)\,\Delta^{-1}_{cd}(x,z)\int_y \Delta^{-1}_{ea}(w,y)\,T^A_{ab}\varphi_b(y) - \frac{1}{2}i\hbar K_{de}(z,w)\,. \qquad (6.91)$$

The secondary constraints are that the variation of the terms multiplying ℓ^f_A vanish, since if they did not divergences would arise as $\tilde{\ell}\to\infty$. Starting work on the right hand side of (6.91) by demanding

$$0 = -\frac{i}{2}\int_x \tilde{\ell}^f_A\,\delta\left[P^\perp_{fc}(x)\,\Delta^{-1}_{cd}(x,z)\int_y \Delta^{-1}_{ea}(w,y)\,T^A_{ab}\varphi_b(y)\right]$$
$$= -i\frac{1}{N-1}\tilde{\ell}^h_{hN}\tilde{P}^\perp_{gf}\int_x \delta\left[P^\perp_{fc}(x)\,\Delta^{-1}_{cd}(x,z)\int_y \Delta^{-1}_{ea}(w,y)\,T^{gN}_{ab}\varphi_b(y)\right], \qquad (6.92)$$

gives the constraint

$$0 = -i\tilde{P}^\perp_{gf}\int_x \delta\left[P^\perp_{fc}(x)\,\Delta^{-1}_{cd}(x,z)\int_y \Delta^{-1}_{ea}(w,y)\,T^{gN}_{ab}\varphi_b(y)\right]$$
$$= \tilde{P}^\perp_{ef}\int_x \delta P^\perp_{fc}(x)\,\tilde{\Delta}^{-1}_{cd}(x,z)\int_y \tilde{\Delta}^{-1}_G(w,y)\,v + \tilde{P}^\perp_{ec}\int_x \delta\Delta^{-1}_{cd}(x,z)\int_y \tilde{\Delta}^{-1}_G(w,y)\,v$$
$$+\tilde{P}^\perp_{ad}\int_x \tilde{\Delta}^{-1}_G(x,z)\int_y \delta\Delta^{-1}_{ea}(w,y)\,v - \delta_{eN}\tilde{P}^\perp_{gd}\int_x \tilde{\Delta}^{-1}_G(x,z)\int_y \tilde{\Delta}^{-1}_H(w,y)\,\delta\varphi_g(y)$$
$$+\tilde{P}^\perp_{ed}\int_x \tilde{\Delta}^{-1}_G(x,z)\int_y \tilde{\Delta}^{-1}_G(w,y)\,\delta\varphi_N(y)\,, \qquad (6.93)$$

and one again finds that every term is proportional to $m^2_G\to 0$ so the constraint is satisfied automatically.

Now working on the right hand side of (6.90) gives the secondary constraint

$$0 = i\tilde{P}^\perp_{gf}\int_x \delta\left[\frac{\delta P^\perp_{fc}(x)}{\delta\varphi_d(z)}\left(\int_y \Delta^{-1}_{ca}(x,y)\,T^{gN}_{ab}\varphi_b(y) + T^{gN}_{ca}J_a(x)\right) + P^\perp_{fc}(x)\,\Delta^{-1}_{ca}(x,z)\,T^{gN}_{ad}\right]$$
$$= i\tilde{P}^\perp_{gf}\int_x \delta\left[\frac{\delta P^\perp_{fc}(x)}{\delta\varphi_d(z)}\right]\int_y \tilde{\Delta}^{-1}_{ca}(x,y)\,T^{gN}_{ab}\tilde{\varphi}_b(y) + i\tilde{P}^\perp_{gf}\int_x \frac{\delta P^\perp_{fc}(x)}{\delta\varphi_d(z)}[\tilde{\varphi}]\int_y \delta\Delta^{-1}_{ca}(x,y)\,T^{gN}_{ab}\tilde{\varphi}_b(y)$$
$$+ i\tilde{P}^\perp_{gf}\int_x \frac{\delta P^\perp_{fc}(x)}{\delta\varphi_d(z)}[\tilde{\varphi}]\int_y \tilde{\Delta}^{-1}_{ca}(x,y)\,T^{gN}_{ab}\delta\varphi_b(y) + i\tilde{P}^\perp_{gf}\int_x \frac{\delta P^\perp_{fc}(x)}{\delta\varphi_d(z)}[\tilde{\varphi}]\,T^{gN}_{ca}J_a(x)$$
$$+ i\tilde{P}^\perp_{gf}\int_x \delta P^\perp_{fc}(x)\,\tilde{\Delta}^{-1}_{ca}(x,z)\,T^{gN}_{ad} + i\tilde{P}^\perp_{gf}\int_x \tilde{P}^\perp_{fc}(x)\,\delta\Delta^{-1}_{ca}(x,z)\,T^{gN}_{ad}\,. \qquad (6.94)$$

(Note that the J_a term in the third line is present because in the first line the $\delta[\cdots]$ truly means "linear piece of $[\cdots]$", not "variation of $[\cdots]$.") Plugging in the expressions for $\delta P^\perp$,

$\delta P^{\perp}/\delta\varphi$ and $\delta\left[\delta P^{\perp}/\delta\varphi\right]$ and simplifying gives

$$
\begin{aligned}
0 = \frac{1}{v} F_{fchd}\delta\varphi_h(z)\, m_G^2 P_{fc}^{\perp} + \int_y \delta\Delta_{Na}^{-1}(z,y)\, P_{ad}^{\perp} - \frac{1}{v}\int_y \tilde{\Delta}_H^{-1}(z,y)\, P_{db}^{\perp}\delta\varphi_b(y) \\
- \frac{1}{v} P_{da}^{\perp} J_a(z) - \frac{1}{v}\int_x \delta\varphi_g(x)\, \tilde{\Delta}_H^{-1}(x,z)\, P_{gd}^{\perp} + i\int_x \delta\Delta_{ga}^{-1}(x,z)\, T_{ad}^{gN}.
\end{aligned} \tag{6.95}
$$

The first term vanishes as $m_G^2 \to 0$. The remaining terms become, for $d = N$,

$$
0 = (N-1)\int_{xw} V_{HGG}(w,x,z)\,\delta\varphi_N(w) + \int_x P_{ga}^{\perp}\mathcal{K}_{ga}(x,z), \tag{6.96}
$$

and for $d \neq N$

$$
\int_y \tilde{\Delta}_H^{-1}(z,y)\,\delta\varphi_d(y) + \frac{1}{2}J_d(z) = v\int_{yw} V_{GHG}(w,z,y)\,\delta\varphi_d(w) + v\int_y \mathcal{K}_{Nd}(z,y). \tag{6.97}
$$

These equations are again apparently incompatible with the usual SI-2PI equations of motion.

6.5 Discussion and summary

This chapter has shown that the imposition of symmetry improvement constraints is incompatible with the linear response approximation, except possibly in finely tuned truncations which satisfy a number of constraints above and beyond the usual equations of motion. Since the original symmetry improvement scheme of Pilaftsis and Teresi was not formulated $\mathrm{O}(N)$-covariantly there are actually two natural generalisations: a scheme used previously ([2] and chapter (4)) based on the projected constraint and a one based on the simpler constraint without a projection operator. Both schemes are equivalent in equilibrium and both lead, in different ways, to pathologies in the linear response approximation.

There is no simple modification of the constraint which could possibly fix these problems since one can understand them as a consequence of treating φ and Δ as independent variables in the 2PIEA and the inability of symmetry improved 2PIEA to enforce also the three point *vertex* WI. There are two error terms in the equations of motion for fluctuations. The first is due to the violation of the *vertex* WI by the equilibrium solutions of the SI-2PI equations of motion. The second is due to the independence of Δ and φ in the 2PIEA. One could try to eliminate the second error term by constraining the variation $\delta\Delta$ to be related to $\delta\varphi$ in an appropriate way. This would no longer be working strictly within the 2PIEA formalism. Rather, it would define a hybrid 2PI-1PI scheme where one computes the equilibrium properties using the symmetry improved 2PIEA, then defines a resummed 1PIEA by eliminating Δ from Γ in an appropriate way. This is similar to the usual resummed 1PIEA method, only now symmetry improvement is applied self-consistently at the 2PI level.

The first error term is more formidable. It comes down to the failure of the master 1PI WI in nPIEA truncations. The master WI encodes relations between *all* correlation functions in the exact theory. However, a symmetry improved nPIEA scheme only has the power to enforce constraints between the lowest n of them. Violations of WIs involving higher order correlation functions are inevitable in any scheme with fixed finite n. This is not a major issue in equilibrium because one can solve for the n-point correlation functions in a self-consistently complete n-loop truncation, and so long as one does not care about the behaviour of higher order correlation functions their problems remain invisible. However, once external sources are turned on and the system departs from equilibrium, variations of the correlation functions appear and these can be related to higher order correlation functions. Violations of the higher order Ward identities then feed back into the equations of motion for the fluctuations, leading to an inconsistent system overall. This leads to the general conclusion: it is not possible to formulate a linear response theory for SI-nPIEAs by requiring (as in the 1PIEA) the vanishing of WI violations and the linking of correlation functions of different order (e.g. φ and Δ). Instead, one must rely upon a network of cancellations among the *violations* of the usual 1PI relations by the corresponding nPI quantities. Such a situation is delicate and highly truncation dependent, so that, so far, it is not know that any truncation beyond the *classical* theory actually obeys all of the required conditions.

It would be interesting to examine whether alternatives to symmetry improvement can be extended to non-equilibrium situations. This motivates the investigation of the linear response of the soft symmetry improvement formalism as a potential workaround for the issues found here. There are two reasons that the results of this chapter cannot be directly transferred to the SSI formalism of Chapter 5. The first is that the WI is not exactly imposed in the SSI formalism, so that the treatment given above no longer applies. Instead, one must explicitly compute the left hand sides of (6.9)-(6.12) and solve for the response functions. This results in a very unwieldy set of equations. The second reason is that, due to the infrared sensitivity of the formalism, the SSI investigation of Chapter 5 was carried out in a finite volume in Euclidean space. However, the linear response equations are formulated in an infinite volume and real time. Thus, an analytical continuation from imaginary to real time must be performed. In principle this is straightforward (see, e.g. [11]), although in practice it is complicated by the unusual behaviour of the zero mode in the SSI formalism. As a result, though an investigation of the linear response theory of soft symmetry improved effective actions is clearly motivated, it is deferred to future work.

Chapter 7

Discussion

7.1 Summary of the thesis

This thesis has examined the theory of global symmetries in the n-particle irreducible effective action formalism for quantum field theory with the goal of improving the symmetry properties of solutions obtained from practical approximation schemes. Chapter 1 motivated this investigation by showing that the inability to perform reliable calculations in regimes where there are strongly interacting particles or many particles far from equilibrium has caused gaps to form between theory and experiment in regimes that are otherwise well understood. These gaps are not just a practical problem, but they hinder efforts to learn new physics. This was illustrated with the case study of electroweak baryogenesis (EWBG), a theoretical proposal for the origin of the matter-antimatter asymmetry of the universe. In EWBG, a strong first order phase transition of the Higgs field occurs in the early universe. During the phase transition, bubbles of “broken” phase form and expand in a background of “symmetric” phase. As the bubbles expand, bubble wall friction drives a flow in the symmetric phase plasma which separates left and right handed particles. This asymmetry between left and right handed particles is converted into a matter-antimatter asymmetry by a tunnelling process (“sphalerons”) which is intrinsically non-perturbative, and it is this resulting asymmetry that persists to the present day. Every step of this complex process involves non-perturbative or non-equilibrium physics which is difficult to treat reliably with presently available methods. It is hard to even estimate the uncertainties of the predictions accurately. As a result, it is difficult to say whether a given model can be excluded on the grounds of an incorrect baryogenesis prediction. These challenges motivate the improvement of nPIEA techniques discussed in the remainder of the thesis. Chapter 1 then concluded with a review of basic quantum field theory.

Chapter 2 then reviewed the nPIEA formalism. The nPIEA were derived for $n = 1$, 2 and 3 for a generic quartically coupled scalar field theory. Then, in novel work (published by the author as [3]), the properties of the 2PIEA viewed as a resummation scheme were examined. For this purpose a toy model – effectively the quartic scalar field theory in zero dimensions – was used which enabled comparisons with exact results, perturbation theory, Borel, Borel-Padé resummation and a novel hybrid 2PI-Padé resummation scheme. It was shown that the 2PIEA yielded predictions which were competitive with or exceeding standard resummation methods. This result could be understood in terms of the ability of the 2PIEA, due to its self-consistency, to capture intrinsically non-perturbative properties of the theory, in particular the vacuum instability for negative values of the coupling constant.

Chapter 3 began the study of symmetries in the nPIEAs formalism. The generic scalar field theory was specialised to an $\mathrm{O}(N)$ symmetric scalar field theory which formed the basis of the rest of the thesis. The parameters of the theory were chosen to exhibit spontaneous symmetry breaking from $\mathrm{O}(N) \to \mathrm{O}(N-1)$, which results in $N-1$ massless Goldstone bosons and one massive "Higgs" boson. At high temperatures the symmetry is restored and there are N equally massive bosons. On the basis of universality arguments and lattice computations the phase transition is known to be second order in four dimensions. The Ward identities governing the symmetry were derived in the 1PIEA and 2PIEA formalisms and it was shown that solutions of the truncated 2PI equations of motion generically violate the 1PI Ward identities. In particular, the Goldstone bosons are massive in the Hartree-Fock approximation. As a result the phase transition is predicted to be strongly first order. More subtle violations persist in higher order approximation schemes. These results were contrasted with the large N method, which satisfied the Ward identities at leading order in $1/N$ but violates them again at higher orders, and the external propagator method. The external propagator has been used extensively in the literature on 2PIEA and does obey the lowest order Ward identity (i.e. Goldstone's theorem), but is not self-consistent.

Then chapter 4 studied the symmetry improvement method (SI) first proposed by Pilaftsis and Teresi [1] for the 2PIEA. Symmetry improvement imposes the 1PI Ward identities directly onto the solutions of nPI equations of motion through the use of Lagrange multipliers. The constraint turns out to be singular and a delicate limiting procedure is required to make sense of the theory. It is known that solutions are not guaranteed to exist in arbitrary truncations of the symmetry improved effective action [6]. Pilaftsis and Teresi [1] considered the method for the 2PIEA of a scalar $\mathrm{O}(2)$ theory. In novel work (published by the author as [2]), chapter 4 extended their formulation to the symmetry improvement of the 3PIEA for an $\mathrm{O}(N)$ theory. In the process two forms of the constraint procedure and an ambiguity in the limiting process were discovered. The choice of constraint turns out to make no difference in equilibrium, and the ambiguity in the limiting procedure was fixed by introducing the *d'Alembert* formalism, which chooses uniquely the simplest possibility. The SI-3PIEA was then renormalised in the Hartree-Fock and two loop truncations in four dimensions and in the three loop truncation in three dimensions. The Hartree-Fock equations of motion were solved and compared to the unimproved and SI-2PI Hartree-Fock solutions. The result was that in the SI-2PI solutions the phase transition is second order and Goldstone's theorem is satisfied (in agreement with previous literature), but in the SI-3PI solutions the phase transition is first order even though Goldstone's theorem is satisfied. This showed for the first time that masslessness of the Goldstone bosons in the formalism does not imply that the phase transition is computed correctly. This is because the Hartree-Fock truncation is not fully self-consistent for the 3PIEA. However, the first order transition in the SI-3PI case is much weaker than in the unimproved 2PI case, so the symmetry improvement does help matters even so. Further checks of the formalism were performed: it was shown that the SI-3PIEA obeys the Coleman-Mermin-Wagner theorem and that the absorptive part of the propagator is consistent with unitarity to $\mathcal{O}(\hbar)$, but only if the full three loop truncation is kept.

Chapter 5 introduced a novel method of soft symmetry improvement (SSI) for nPIEA. SSI differs from SI in that the Ward identities are imposed *softly* in the sense of (weighted) least squared error rather than as strong constraints on the solutions. A new *stiffness* parameter controls the strength of the constraint. The motivation for this method is that the singular nature of the constraint in SI leads to pathological behaviour such as the non-existence of solutions in certain truncations and the breakdown of the theory out of equilibrium (which is examined more closely in Chapter 6). The hope was that by relaxing the constraint some solutions with reasonable physical properties could be found interpolating between the unimproved and symmetry improved limits for finite values of the stiffness parameter. The SSI-nPIEA is sensitive to the infrared (large distance) boundary conditions of the problem. In order to regulate the IR behaviour the SSI-2PIEA was studied in a box of finite volume with periodic boundary conditions and at finite temperature. The limit of infinite volume and low temperature was examined and three limiting regimes found. Two were equivalent to the unimproved 2PIEA and SI-2PIEA respectively. The third was a novel limit which had pathological behaviour. The zero mode of the Goldstone propagator was massless as expected, but modes with finite energy/momentum had a finite mass. Further, this mass *increases* as the constraint is more strongly imposed, contrary to intuition. The phase transition is strongly first order in the Hartree-Fock truncation, and there is a critical value of the stiffness parameter $\hat{\zeta}_c$ such that for $\hat{\zeta} < \hat{\zeta}_c$ (more strongly imposed constraints) solutions cease to exist for a range of temperatures below the critical temperature. As $\hat{\zeta} \to 0$ this range increases and at some finite value $\hat{\zeta}_\star$ the solution ceases to exist even at zero temperature. This loss of solution was confirmed both analytically and numerically. The 2PI, SI-2PI and novel SSI-2PI limits are all disconnected from each other: in an infinite volume there is no continuous parameter connecting any two of the limits. Thus the original goal of this program is not achievable in infinite volume. A paper based on this work has been published [5].

Chapter 6 investigated the linear response of a system in equilibrium subject to external perturbations in the SI-2PIEA formalism. This was based on novel work published by the author in [4]. It was shown that, beyond equilibrium, the two constraint schemes possible in the SI-2PIEA formalism are no longer equivalent. However, both schemes are inconsistent in that the symmetry improvement constraints are apparently incompatible with the dynamics generated by the 2PI equations of motion, at least in generic truncations. This result could be understood as a result of two contributions: the decoupling of the propagator fluctuation from the field fluctuation in the 2PIEA formalism and the failure of the higher order Ward identities (i.e., those involving vertex functions) which is unavoidable even in the SI-2PIEA. This latter source of error is present because an SI-nPIEA has only n independent variables, thus is only capable of imposing n Ward identities at most.

However, there is an infinite hierarchy of Ward identities involving higher order correlation functions, all of which are violated in truncations. When a system is out of equilibrium these higher order Ward identities feed back into the lower order equations of motion, generating an inconsistency. As a result, SI-nPIEA are invalid out of equilibrium, at least within the linear response approximation and for generic truncations.

7.2 Limitations of the results and directions for future work

The results in this thesis are limited in several respects. Most of the limitations are due to the primarily analytical nature of this thesis. All but the simplest (Hartree-Fock) truncation of nPIEA require numerical solutions which are significantly more difficult than anything attempted in this thesis. The reason for the great simplification of the Hartree-Fock truncation (which also requires numerical solution) is that the self-energies are momentum independent. Effectively, the thermal and quantum corrections contribute only to a shift of the mass of the particles involved. These masses can be found self-consistently. In other truncations the propagators are momentum dependent. Thus the effort is effectively multiplied by the number of lattice points in the spacetime discretisation used. The effort grows even more rapidly as the n in nPIEA is increased beyond $n = 2$. The effort can be reduced somewhat using lattice symmetries and algorithms based on the fast Fourier transform (see, e.g. [87, 125, 134]), however the increase in complexity over the Hartree-Fock case remains substantial. As such, while full numerical investigation of solutions of SI- and SSI-nPIEAs is certainly an intriguing prospect, it has been deferred for future work. Here follows a consideration of the limitations of each chapter.

The novel work of chapter 2 is the investigation of the analytical properties of the 2PIEA in contrast to standard resummation schemes. The chief limitation of this study is that it only applies directly to the zero dimensional toy model. Perhaps the most significant extension of this work achievable in the near term would a study along the same lines for the quantum mechanical anharmonic oscillator. As a one dimensional system there is a wide variety of relatively inexpensive techniques available to find numerical wavefunctions and energy levels to arbitrary accuracy. Also, individual terms of the perturbation series can be found, as well as accurate asymptotic expansions for the energy levels and wavefunctions in the complex coupling constant λ plane (see, e.g. [89, 90] for early studies of this system). In order to extend the study of chapter 2 to this case one must solve the 2PIEA in such a way as to obtain analytic information about the solution as a function of complex λ. Staying in zero dimensions there are several directions for further study. One can easily extend the analysis to the 4PIEA for the toy model by introducing a four-point source that effectively modifies λ. The most difficult step is performing the extra Legendre transform. The resulting effective action should embody an even more compact representation of the perturbation series than the 2PIEA, however it is not clear what new insights might come from this study. Similarly, all of the results of chapter 2 could be computed to higher order in the relevant truncations. It would also be interesting to study whether the hybrid 2PI-Padé scheme could be extended to a real theory.

Chapter 3 started the consideration of symmetries in this thesis. The chief limitation of chapter 3 is common to the rest of the thesis: only global bosonic symmetries are considered. No attention was paid in this thesis to gauge theories, supersymmetric theories or gravity (i.e. diffeomorphism invariant) theories, all of which are obviously of major importance in high energy and mathematical physics. The main problem hindering the extension to these theories is the proliferation of fields. Many of the existing treatments of nPIEA for these theories neglect or incorrectly treat important phenomena such as fermion-boson mixing. In general, for *any* two fields A and B in a theory one must have a self-consistent 2PI propagator Δ_{AB}, even if A and B have differing charges, spin or statistics. Similar remarks apply for arbitrary vertex functions V_{ABC} etc. Thus the number of quantities one must consider in the nPIEA formalism grows rapidly with the number of fields. Note that mixed fermion-boson propagators do indeed vanish on solutions of the equations of motion (as expected from Lorentz invariance, which is why they are often neglected), but *derivatives* of the effective action with respect to these mixed quantities do *not* in general vanish. Thus it is important to keep all mixed quantities during the intermediate steps of the computation [135, 136].

Chapter 4 discussed the symmetry improvement formalism. Apart from the overall limitation of this thesis to pure scalar theories with fields in a single fundamental representation of a global $\mathrm{O}(N)$ symmetry, the main limitations of this chapter are due to the numerical issues discussed above. The theory is considered in the SI-2PIEA and SI-3PIEA truncated to three loop order. Renormalised equations of motion are derived in the two loop truncation in both schemes in four dimensions, and the three loop truncation in three dimensions. Only the Hartree-Fock approximation is solved. Unfortunately, the most interesting case is the full three loop truncation of the SI-3PIEA in four dimensions. With these solutions a proper check could be made of the (a) existence of solutions, (b) sensitivity of the solutions to the infrared boundary conditions, (c) order and thermodynamics of the phase transition and (d) the predictions for the absorptive part of the Higgs propagator. Unfortunately, finding these solutions requires a numerical effort similar in scope to a large portion of this thesis. The main difficulty is that the renormalisation is not possible to carry out beforehand in four dimensions: quantum corrections to the vertex function alter its large momentum behaviour in such a way that the renormalisation must be carried out at the same time as the iterative solution of the equations of motion themselves. This is one of the main difficulties of nPIEA with $n \geq 3$ in general and this thesis has made no headway on this problem. Also, depending on the scheme, the regularisation method and renormalisation scheme may have to be redone. The only numerical 3PI solutions actually presented are given in the Hartree-Fock truncation, an extremely simplified and not fully self-consistent truncation that completely misses vertex corrections. As a result, the conclusions based on the Hartree-Fock results may not hold in the more physically relevant two and three loop truncations. A numerical effort to find these solutions is therefore strongly motivated. A conceptually straightforward, though probably laborious, extension would be to the SI-4PIEA or higher. This move is motivated theoretically because the 4PIEA at four loop order is necessary for a fully self-consistent treatment of non-abelian gauge theories. The Ward identity involving the four point vertex would have to be derived and enforced using a new set of Lagrange multipliers, then the equations of motion derived and the Lagrange multipliers eliminated through a suitable limit procedure. Then the equations of motion would have to be renormalised and solved numerically. Again, this represents a substantial effort in its own right and it is not clear in the present state of the theory what the pay off would be. In the spirit of chapter 2 the author would recommend a first study of higher symmetry improved nPIEAs for gauge theories to focus on one of the lower dimensional solvable models.

Chapter 5 introduced the soft symmetry improvement method. Again, the main limitation of this study was the focus on the analytically tractable (or nearly so) aspects of the formalism. Two studies are motivated by this chapter. The first is the numerical study of solutions of the novel limit in higher order truncations. It is possible (though in the author's opinion not very likely) that the unsatisfactory aspects of the solutions obtained are removed by higher order corrections. The second study is a numerical implementation of the method in finite volume with some sort of lattice or momentum cutoff. In this regime one *does* have a genuine interpolation between the unimproved and symmetry improved cases (since there are only a finite number of degrees of freedom the least squares term in the effective action must work as expected). Once the numerical method is implemented one must do a number of sensitivity studies to determine if, for physically relevant parameter values, there is a regime which (a) approaches the continuum limit, (b) has adequate symmetry properties and (c) is in a box of sufficiently large size for finite size effects to be unimportant.

Chapter 6 started the study of non-equilibrium aspects of symmetry improved effective actions. A limitation of this study is that conditions were identified that a truncation of the effective action must satisfy in order to have a satisfactory linear response approximation, but no truncation satisfying these conditions has been found so far. Indeed it is still an open question whether it is possible to satisfy all of the conditions. The corresponding study for the SSI-2PIEA is equally motivated theoretically, but has been deferred because it is significantly more complicated. The reason is that the symmetry constraints are only weakly enforced by the SSI-2PIEA. As a result, the dynamics of linear fluctuations are determined by a mixture of *both* the 2PIEA and the symmetry constraints (i.e., the dynamics are determined by the full SSI-2PIEA). Thus the full linear response equations must be solved. These equations are a linear system which is in principle straightforward to solve, but the actual equations turn out to be very bulky due to the form of the SSI term and its derivatives. Another issue is that, due to the infrared sensitivity of the formalism, the SSI investigation of Chapter 5 was carried out in a finite volume in Euclidean space. However, the linear response equations are formulated in an infinite volume and real time. Thus, an analytical continuation from imaginary to real time must be performed. In principle this is straightforward, although in practice it is complicated by the unusual behaviour of the zero mode in the SSI formalism. Thus, though the results of a linear response investigation for the SSI-2PIEA would certainly be interesting and perhaps be significant, the above considerations place it beyond the scope of this thesis.

An evaluation of the overall status of symmetry improvement methods is in order. SI methods have been applied with some success to the 2PIEA at the Hartree-Fock and two loop levels. However, the non-existence of solutions of the two loop truncation with certain cutoff regulators is deeply troubling. It implies that the symmetry improvement method is coupling short distance and long distance physics in ways that are still poorly understood. This defies the traditional understanding of the renormalisation group and perhaps relates to the long standing difficulty with the renormalisation of nPIEAs generally. It is clear from the results of Chapter 4 that SI methods can, at least formally, be extended to all nPIEAs straightforwardly. However, the properties of solutions of these SI-nPIEAs is still poorly understood. Only results for the Hartree-Fock truncation of the SI-3PIEA are known, and it is likely that these results will change qualitatively when higher order computations are done. The study of these methods is hampered by the difficulty with renormalisation in four dimensions, which has largely been sidestepped in this thesis. Similarly, the SSI method has only been applied in the Hartree-Fock truncation. It is certainly reasonable to hope that the difficulties with the SSI method are removed, or at least ameliorated, at higher orders. So far the hints are that carefully taking the infinite volume limit is critical in all cases, a result that is suggestive when seen alongside the IR problems of the SI-2PIEA at two loops. There is much that remains unknown about these methods. Progress is slow, but monotonic.

7.3 Closing remarks

A case could perhaps be made that this thesis is presenting a null result. After all, the dream of an analytically tractable and elegant, fully self-consistent, manifestly gauge invariant, non-perturbative and non-equilibrium formulation of non-abelian gauge theories with chiral fermion matter and Higgs fields remains just that: a dream. Perhaps it is a pipe dream. This thesis has not found the silver bullet for handling symmetries in non-perturbative quantum field theory. Nor does it seem likely that any of the techniques introduced in this thesis will open the door to a world of new discoveries any time soon. Rather this thesis is perhaps more in line with the normal, unglamorous, progress of science which is more akin to the mythic bird who, by aeons of persistence, gradually chips away at the mountain revealing one fleck of unweathered stone at a time.

This thesis has made a contribution to the literature on symmetries in quantum field theory less by finding a solution to the problems of the field and more by, hopefully, revealing some new faces of the same old problems. The n-particle irreducible effective action methods are certainly very formally elegant methods capable of handling non-equilibrium situations in quantum fields theories in a fully self-consistent, non-perturbative way which is derived directly from first principles. But difficulties persist.

The n-particle irreducible effective actions derive their strength by re-organising perturbation theory, but this very same act intimately couples the scales in a problem, so the problem of enforcing symmetries at large distances can no longer be separated from the problem of making the theory finite at short distances. Likewise, the theory punishes you for asking the wrong questions. If you ask for strictly massless Goldstone bosons you had better be prepared to work in infinite volume. The price to be paid is that, since scales are now coupled, solutions do not always exist. If you want to impose constraints on the states of the theory (the masses of particles), then the same constraints determine the dynamics (linear response), and the same constraints cannot get both aspects right.

It appears that non-perturbative quantum field theory is a world of trade-offs where no free lunch exists. The good news is that the journey is still just beginning. The techniques studied in this thesis were originally proposed less than four years ago as I write this. So it is that I, the author, hope that by navigating these treacherous waters (and perhaps running aground at points), those who come after may more easily find the clear path.

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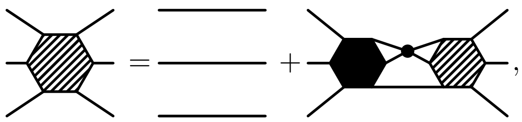
Appendix A

The auxiliary vertex and its renormalisation

As described in Section 4.7.3 the renormalisation of the three loop 3PIEA requires the definition of an auxiliary vertex V^{μ}_{abc} with the same asymptotic behaviour as the full self-consistent solution at large momentum. This auxiliary vertex can be found in terms of a six point kernel $\mathcal{K}_{abcdef}$ which obeys the integral equation (4.118), reproduced here

$$\mathcal{K}_{abcdef} = \delta_{ad}\delta_{be}\delta_{cf} + \frac{1}{3!}\sum_{\pi}\left(-\frac{3i\hbar}{2}\right)\delta_{\pi(a)h}W_{\pi(b)\pi(c)kg}\Delta^{\mu}_{ki}\Delta^{\mu}_{gj}\mathcal{K}_{hijdef}. \tag{A.1}$$

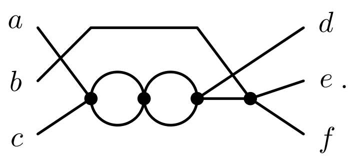
This can be written graphically as



(A.2)

where the shaded hexagonal blob is $\mathcal{K}$ and the solid hexagonal blob is the sum over all permutations of the lines coming in from the left and connecting to those on the right.

Solving (4.118) by iteration generates an infinite number of terms, one of which is



(A.3)

Each contribution is in one-to-one correspondence with the sequence of permutations $\pi_1\pi_2\cdots\pi_n\cdots$ of the propagator lines (read from left to right in relation to the diagram). The permutations fall into two classes: "stabilisers," for which $\pi(a) = a$, and "derangements," for which $\pi(a) = b$ or c. Any sequence of permutations is of the form of an alternating sequence of runs of (possibly zero) stabilisers, separated by derangements.

Consider a run of n stabilisers, $\cdots\pi_a(\pi_1\pi_2\cdots\pi_n)\pi_b\cdots$, where π_a and π_b are derangements and π_1 through π_n are all stabilisers. The case for $n = 2$ is shown above. Each stabiliser creates a logarithmically divergent loop on the bottom two lines $\sim -\lambda\mathcal{I}^{\mu}$. Derangements on the other hand, if they create loops at all, create loops with > 2 propagators, and hence are convergent. Thus all divergences in $\mathcal{K}_{abcdef}$ can be removed by rendering a single primitive divergence finite. Note that the whole series $\sum_{n=0}^{\infty}\cdots\pi_a\left(\prod_{i=1}^{n}\pi_i\right)\pi_b$, where again $\pi_{a,b}$ are derangements and $\{\pi_i\}$ are stabilisers, can be summed because the series is geometric. The result is that the six point kernel can be determined by an equation like (4.118), except that the sum over all permutations is replaced by a sum over derangements only, and the bare vertex W is replaced by a resummed four point kernel $\mathcal{K}^{(4)}_{abcd} \sim \lambda/\left(1+\lambda\mathcal{I}^{\mu}\right)$. Substituting the solution for $\mathcal{K}$ in (4.117) gives the solution for V^{μ}_{abc}.

This expression for V^{μ}_{abc} can be dramatically simplified in 3 or $1+2$ dimensions because $\mathcal{I}^{\mu}$ is finite and the geometric sum in $\mathcal{K}^{(4)}_{abcd}$ converges. Indeed $\mathcal{K}^{(4)}_{abcd}\left(p_1,p_2,p_3,p_1+p_2-p_3\right) \sim \lambda/\left[1+\lambda/\left(p_1+p_2\right)^{4-d}\right] \to \lambda$ as $p_{1,2,3,4}\to\infty$. Further, every loop integral in (4.117) likewise converges, and every loop yields a factor of $\sim 1/p^{4-d}$. Thus the dominant behaviour as $p\to\infty$ is just the tree level behaviour and the auxiliary vertex can be eliminated completely.

However, in 4 or $1+3$ dimensions V^{μ}_{abc} apparently cannot be simplified further. First $\mathcal{K}^{(4)}_{abcd}$ must be renormalised, then the bubble appearing in the non-trivial terms in (4.117) (or the equivalent integral equation) must be renormalised, then the resulting series must be summed (or the equivalent integral equation solved), noting that on the basis of power counting every term is apparently equally important. On this basis ones expect that no compact analytic expression for V^{μ}_{abc}, or even its asymptotic behaviour, exists and that the renormalisation must be accomplished as part of the self-consistent numerical solution of the full equations of motion.

This style of argument can be quickly generalised to many other theories, such as gauge theories, where the diagrammatic expansion has a similar combinatorial structure to scalar $\mathrm{O}(N)$ theory, showing up the well known problem of the renormalisation of nPIEA for $n\geq 3$ in four dimensions. The discussion here certainly does not solve this problem, which remains open, to the author's knowledge, though hopefully this discussion may be helpful.

Appendix B

Deriving counter-terms for three loop truncations

Here the renormalisation of the three loop truncation of the SI-3PIEA is carried out in $1+2$ dimensions as discussed in Section 4.7.3. The effective action is as in the two loop truncations in Appendix C except for a new counter-term $\delta\lambda \to \delta\lambda_C$ for the the Φ_2 term and the addition of the three loop diagrams

$$\begin{aligned}
\Phi_3 &= Z_V^4 Z_\Delta^6 \left[(N-1)\frac{i\hbar^3}{3!} V_N \left(\bar{V}\right)^3 (\Delta_H)^3 (\Delta_G)^3 + \frac{i\hbar^3}{4!} (V_N)^4 (\Delta_H)^6 \right. \\
&\quad \left. + (N-1)\frac{i\hbar^3}{8} \left(\bar{V}\right)^4 \Delta_H \Delta_H (\Delta_G)^4 \right], \qquad \text{(B.1)} \\
\Phi_4 &= \frac{i\hbar^3 (\lambda+\delta\lambda)}{24} Z_V^2 Z_\Delta^5 \left[2(N-1)\bar{V}V_N (\Delta_H)^3 \Delta_G\Delta_G + \left(N^2-1\right)\bar{V}\bar{V}\Delta_H (\Delta_G)^4 \right. \\
&\quad \left. + 3V_N V_N (\Delta_H)^5 + 2^2 (N-1)\bar{V}\bar{V} (\Delta_G)^3 \Delta_H\Delta_H \right], \qquad \text{(B.2)} \\
\Phi_5 &= \frac{i\hbar^3 (\lambda+\delta\lambda)^2}{144} Z_\Delta^4 \left\{ \left[(N-1)\Delta_G\Delta_G + \Delta_H\Delta_H\right]^2 + 2(N-1)(\Delta_G)^4 + 2(\Delta_H)^4 \right\}. \qquad \text{(B.3)}
\end{aligned}$$

The equations of motion following from $\Gamma^{(3)}$ are then

$$\begin{aligned}
\Delta_G^{-1} &= -\left(ZZ_\Delta \partial_\mu\partial^\mu + m^2 + \delta m_1^2 + Z_\Delta \frac{\lambda+\delta\lambda_1^A}{6} v^2 \right) \\
&\quad - \frac{\hbar}{6}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right] Z_\Delta^2 \mathcal{T}_G - \frac{\hbar}{6}\left(\lambda+\delta\lambda_2^A\right) Z_\Delta^2 \mathcal{T}_H \\
&\quad - i\hbar Z_V^2 Z_\Delta^3 \left[-2\frac{(\lambda+\delta\lambda_C) Z_V^{-1} v}{3} - \bar{V} \right] \Delta_H \Delta_G \bar{V} \\
&\quad + \hbar^2 Z_V^4 Z_\Delta^6 \left[V_N \left(\bar{V}\right)^3 (\Delta_H)^3 (\Delta_G)^2 + \left(\bar{V}\right)^4 \Delta_H\Delta_H (\Delta_G)^3 \right] \\
&\quad + \frac{\hbar^2 (\lambda+\delta\lambda)}{3} Z_V^2 Z_\Delta^5 \left[\bar{V}V_N (\Delta_H)^3 \Delta_G + (N+1)\bar{V}\bar{V}\Delta_H (\Delta_G)^3 + 3\bar{V}\bar{V}(\Delta_G)^2 \Delta_H\Delta_H \right] \\
&\quad + \frac{\hbar^2 (\lambda+\delta\lambda)^2}{18} Z_\Delta^4 \left[(N+1)(\Delta_G)^3 + \Delta_H\Delta_H\Delta_G \right], \qquad \text{(B.4)}
\end{aligned}$$

for the Goldstone propagator,

$$
\begin{aligned}
\bar{V} = & -\frac{(\lambda+\delta\lambda_C)\,v}{3} Z_V^{-1} + i\hbar Z_V^2 Z_\Delta^3 \left[V_N\left(\bar{V}\right)^2 (\Delta_H)^2 \Delta_G + \left(\bar{V}\right)^3 \Delta_H (\Delta_G)^2\right] \\
& + \frac{i\hbar(\lambda+\delta\lambda)}{6} Z_\Delta^2 \left[V_N (\Delta_H)^2 + (N+1)\,\bar{V}(\Delta_G)^2 + 4\bar{V}\Delta_G\Delta_H\right],
\end{aligned} \tag{B.5}
$$

for the Higgs-Goldstone-Goldstone vertex,

$$
\begin{aligned}
V_N = & -(\lambda+\delta\lambda_C)\,v Z_V^{-1} + i\hbar Z_V^2 Z_\Delta^3 \left[(N-1)\left(\bar{V}\right)^3 (\Delta_G)^3 + (V_N)^3 (\Delta_H)^3\right] \\
& + \frac{i\hbar(\lambda+\delta\lambda)}{2} Z_\Delta^2 \left[(N-1)\,\bar{V}\Delta_G\Delta_G + 3V_N (\Delta_H)^2\right],
\end{aligned} \tag{B.6}
$$

for the triple Higgs vertex, and finally

$$
0 = \Delta_G^{-1}(p=0)\,v, \tag{B.7}
$$

$$
0 = Z_V Z_\Delta \bar{V}(p,-p,0)\,v + \Delta_G^{-1}(p) - \Delta_H^{-1}(p), \tag{B.8}
$$

for the Ward identities.

Note that the only divergent integrals in these equations are the linearly divergent tadpole integrals $\mathcal{T}_{G/H}$ and the logarithmically divergent BBALL integrals (last line of B.4). By power counting one finds that the third, fourth, and fifth lines of B.4 produce finite self-energy contributions with leading asymptotics $\sim p^{-1}$, p^{-4}, and p^{-2} respectively. One can separate finite and divergent parts of Δ_G^{-1} as

$$
\Delta_G^{-1} = -\left(\partial_\mu\partial^\mu + m^2 + \frac{\lambda}{6}v^2\right) - \left[\Sigma_G^0(p) - \Sigma_G^0(m_G)\right] - \Sigma_G^\infty(p), \tag{B.9}
$$

where

$$
\begin{aligned}
-\Sigma_G^0(p) = & -\frac{\hbar}{6}(N+1)\,\lambda\left(\mathcal{T}_G - \mathcal{T}^\mu\right) - \frac{\hbar}{6}\lambda\left(\mathcal{T}_H - \mathcal{T}^\mu\right) \\
& - i\hbar\left[-2\frac{(\lambda+\delta\lambda_C)\,Z_V^{-1}v}{3} - \bar{V}\right]\Delta_H\Delta_G\bar{V} \\
& + \hbar^2\left[V_N\left(\bar{V}\right)^3 (\Delta_H)^3 (\Delta_G)^2 + \left(\bar{V}\right)^4 \Delta_H\Delta_H (\Delta_G)^3\right] \\
& + \frac{\hbar^2(\lambda+\delta\lambda)\,Z_\Delta^2}{3}\left[\bar{V}V_N (\Delta_H)^3 \Delta_G + (N+1)\,\bar{V}\bar{V}\Delta_H (\Delta_G)^3 + 3\bar{V}\bar{V}(\Delta_G)^2\Delta_H\Delta_H\right] \\
& + \frac{\hbar^2(\lambda+\delta\lambda)^2 Z_\Delta^4}{18}\left[(N+1)(\Delta_G)^3 + \Delta_H\Delta_H\Delta_G - (N+2)\,\mathcal{B}^\mu\right],
\end{aligned} \tag{B.10}
$$

and

$$
\begin{aligned}
-\Sigma_G^{\infty}(p) = & -\Sigma_G^0(m_G) - \left((ZZ_\Delta - 1)\partial_\mu\partial^\mu + \delta m_1^2 + \frac{\delta\lambda_1^A}{6}v^2 + (Z_\Delta - 1)\frac{\lambda + \delta\lambda_1^A}{6}v^2 \right) \\
& - \frac{\hbar}{6}(N+1)\lambda\mathcal{T}^\mu - \frac{\hbar}{6}\left[(N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right]\mathcal{T}_G \\
& - \frac{\hbar}{6}\left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right]\left(Z_\Delta^2 - 1\right)\mathcal{T}_G \\
& - \frac{\hbar}{6}\lambda\mathcal{T}^\mu - \frac{\hbar}{6}\delta\lambda_2^A\mathcal{T}_H - \frac{\hbar}{6}\left(\lambda + \delta\lambda_2^A\right)\left(Z_\Delta^2 - 1\right)\mathcal{T}_H \\
& - i\hbar\left(Z_V^2 Z_\Delta^3 - 1\right)\left[-2\frac{(\lambda + \delta\lambda_C)Z_V^{-1}v}{3} - \bar{V}\right]\Delta_H\Delta_G\bar{V} \\
& + \hbar^2\left(Z_V^4 Z_\Delta^6 - 1\right)\left[V_N\left(\bar{V}\right)^3(\Delta_H)^3(\Delta_G)^2 + \left(\bar{V}\right)^4\Delta_H\Delta_H(\Delta_G)^3\right] \\
& + \frac{\hbar^2(\lambda + \delta\lambda)Z_\Delta^2}{3}\left(Z_V^2 Z_\Delta^3 - 1\right) \\
& \times\left[\bar{V}V_N(\Delta_H)^3\Delta_G + (N+1)\bar{V}\bar{V}\Delta_H(\Delta_G)^3 + 3\bar{V}\bar{V}(\Delta_G)^2\Delta_H\Delta_H\right] \\
& + (N+2)\frac{\hbar^2(\lambda + \delta\lambda)^2 Z_\Delta^4}{18}\mathcal{B}^\mu,
\end{aligned}
\tag{B.11}
$$

are the finite and divergent parts respectively and we introduced the BBALL integral $\mathcal{B}^\mu = \int_{qp}\Delta^\mu(q)\Delta^\mu(p)\Delta^\mu(p+q)$. This split has already assumed that $(\lambda + \delta\lambda_C)Z_V^{-1}$ and $(\lambda + \delta\lambda)Z_\Delta^2$ are finite, which will be demonstrated to be consistent shortly. Renormalisation requires $\Sigma_G^\infty(p) = 0$. Note the explicit subtraction of $\Sigma_G^0(m_G)$ in order to fulfil the mass shell condition. Doing the same now for Δ_H^{-1} gives the pole condition

$$
0 = Z_V Z_\Delta\bar{V}(m_H, -m_H, 0)v + m_H^2 - m_G^2 - \Sigma_G^0(m_H), \tag{B.12}
$$

which requires

$$
Z_V Z_\Delta = \frac{m_G^2 + \Sigma_G^0(m_H) - m_H^2}{\bar{V}(m_H, -m_H, 0)v} \equiv \kappa, \tag{B.13}
$$

which is finite. Taking for the other renormalisation conditions the separate vanishing of kinematically independent divergences, implies

$$
\begin{aligned}
ZZ_\Delta &= 1, && \text{(B.14)}\\
\delta m_1^2 &= -\Sigma_G^0(m_G) - \frac{\hbar}{6}(N+2)\lambda\mathcal{T}^\mu + (N+2)\frac{\hbar^2\lambda^2}{18}\mathcal{B}^\mu, && \text{(B.15)}\\
\delta\lambda_1^A &= -\frac{(Z_\Delta - 1)\lambda}{Z_\Delta} && \text{(B.16)}\\
Z_V^2 Z_\Delta^3 &= 1, && \text{(B.17)}\\
0 &= (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B \\
&\quad + \left[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B\right]\left(Z_\Delta^2 - 1\right), && \text{(B.18)}\\
0 &= \delta\lambda_2^A + \left(\lambda + \delta\lambda_2^A\right)\left(Z_\Delta^2 - 1\right). && \text{(B.19)}
\end{aligned}
$$

The conditions

$$(\lambda + \delta\lambda)\, Z_\Delta^2 = \lambda, \tag{B.20}$$

$$(\lambda + \delta\lambda_C)\, Z_V^{-1} = \lambda, \tag{B.21}$$

can be used to recover the tree level asymptotics for $\bar{V}$ and V_N. These conditions give a closed system of nine equations for the nine quantities Z, Z_Δ, Z_V, δm_1^2, $\delta\lambda_1^A$, $\delta\lambda_2^{A/B}$, $\delta\lambda$, and $\delta\lambda_C$.

The solution of all these equations can be found in Appendix C, with the result that

$$\delta\lambda_2^A = \delta\lambda_2^B = -\frac{\lambda\left(Z_\Delta^2 - 1\right)}{Z_\Delta^2} = \left(\kappa^2 - 1\right)\lambda, \tag{B.22}$$

$$Z_V = \kappa^3, \tag{B.23}$$

$$Z_\Delta = \kappa^{-2}, \tag{B.24}$$

$$Z = \kappa^2, \tag{B.25}$$

$$\delta\lambda = \left(\kappa^4 - 1\right)\lambda, \tag{B.26}$$

$$\delta\lambda_C = \left(\kappa^3 - 1\right)\lambda. \tag{B.27}$$

Note that if $\kappa = 1$ all of the counter-terms except δm_1^2 vanish. This is a manifestation of the super-renormalisability of ϕ^4 theory in $1+2$ dimensions. The non-zero, indeed finite, values of all of the other counter-terms are not required to UV-renormalise the theory, but only to maintain the pole condition for the Higgs propagator despite the vertex Ward identity.

Appendix C

Mathematica notebooks for renormalisation

Many of the renormalisation calculations must be performed using a computer algebra package owing to the unwieldy form of the intermediate results. For reference purposes the Mathematica notebooks used to produce the results in this thesis are included here. The text of these notebooks is bulky, however, if one wants the explicit expression for the counter-terms in some approximation, or the explicit steps used to derive them, they can be found in the following pages. The full notebooks will be made available in machine readable form online at https://github.com/mikejbrown/thesis.

Renormalization of 2PIEA gap equations in the Hartree-Fock approximation

Author: Michael Brown
Supplement to thesis Chapter 3

Mathematica notebook to compute couter-terms for two loop truncations of the two particle irreducible effective action

```
ClearAll[veom, geom, neom, divergentpartrules, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the thesis are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
mn2 is the Higgs mass squared $m_H{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_0{}^2$, $\delta m_1{}^2$ are its counter-terms,
λ is the (renormalized) four point coupling,
$\delta\lambda_0$, $\delta\lambda_{1\,a}$, $\delta\lambda_{1\,b}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

Equations of motion

Vev equation of motion

$$\mathbf{veom = Z\Delta^{-1}\left(m^2+\delta m_0{}^2\right) v + \frac{\lambda+\delta\lambda_0}{6} v^3 + \frac{\hbar}{6} Z\Delta\left(n-1\right)(\lambda+\delta\lambda_{1\,a})\, v\,(t\infty g + tfing) + \frac{\hbar}{6} Z\Delta\left(3\lambda+\delta\lambda_{1\,a}+2\,\delta\lambda_{1\,b}\right) v\,(t\infty n + tfinn)}$$

$$\mathbf{finveom = m^2\, v + \frac{\lambda}{6} v^3 + \frac{\hbar}{6}\left(n-1\right)\lambda\, v\, tfing + \frac{\hbar}{2}\lambda\, v\, tfinn}$$

$$\frac{v\left(m^2+\delta m_0^2\right)}{Z\Delta} + \frac{1}{6} v^3 (\lambda+\delta\lambda_0) + \frac{1}{6}(-1+n)\,(tfing + t\infty g)\, v\, Z\Delta\,\hbar\,(\lambda+\delta\lambda_a) + \frac{1}{6}(tfinn + t\infty n)\, v\, Z\Delta\,\hbar\,(3\lambda+\delta\lambda_a+2\,\delta\lambda_b)$$

$$m^2\, v + \frac{v^3\lambda}{6} + \frac{1}{6}(-1+n)\, tfing\, v\,\lambda\,\hbar + \frac{1}{2} tfinn\, v\,\lambda\,\hbar$$

Goldstone equation of motion

$$\mathbf{geom = p^2 - mg2 == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda+\delta\lambda_{1\,a}}{6} v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda + \left(n-1\right)\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(t\infty g + tfing) - \frac{\hbar}{6}(\lambda+\delta\lambda_{2\,a})\, Z\Delta^2\,(t\infty n + tfinn)}$$

$$\mathbf{finmg2 = mg2\ /.\ Solve\left[p^2 - mg2 == p^2 - m^2 - \frac{\lambda}{6} v^2 - \frac{\hbar}{6}\left(n+1\right)\lambda\, tfing - \frac{\hbar}{6}\lambda\, tfinn,\ mg2\right][[1]]}$$

$$-mg2 + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6} v^2\, Z\Delta\,(\lambda+\delta\lambda_a) - \frac{1}{6}(tfinn + t\infty n)\, Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a}) - \frac{1}{6}(tfing + t\infty g)\, Z\Delta^2\,\hbar\,((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})$$

$$\frac{1}{6}\left(6\, m^2 + v^2\lambda + tfing\,\lambda\,\hbar + n\, tfing\,\lambda\,\hbar + tfinn\,\lambda\,\hbar\right)$$

Higgs equation of motion

$$\mathbf{neom = p^2 - mn2 == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta\, v^2\,\frac{\left(3\lambda+\delta\lambda_{1\,a}+2\,\delta\lambda_{1\,b}\right)}{6} - \frac{\hbar}{6}(\lambda+\delta\lambda_{2\,a})\left(n-1\right) Z\Delta^2\,(t\infty g + tfing) - \frac{\hbar}{6}\left(3\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(t\infty n + tfinn)}$$

$$\mathbf{finmn2 = mn2\ /.\ Solve\left[p^2 - mn2 == p^2 - m^2 - v^2\,\frac{\lambda}{2} - \frac{\hbar}{6}\lambda\left(n-1\right) tfing - \frac{\hbar}{2}\lambda\, tfinn,\ mn2\right][[1]]}$$

$$-mn2 + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6}(-1+n)\,(tfing + t\infty g)\, Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a}) - \frac{1}{6} v^2\, Z\Delta\,(3\lambda+\delta\lambda_a+2\,\delta\lambda_b) - \frac{1}{6}(tfinn + t\infty n)\, Z\Delta^2\,\hbar\,(3\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})$$

$$\frac{1}{6}\left(6\, m^2 + 3\, v^2\lambda - tfing\,\lambda\,\hbar + n\, tfing\,\lambda\,\hbar + 3\, tfinn\,\lambda\,\hbar\right)$$

Infinite parts of tadpoles

c0, c1, Λ and μ are regularisation/renormalisation scheme dependent quantities

divergentpartrules = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}

$$\left\{t\infty g \to c0\,\Lambda^2 + c1\,mg2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ t\infty n \to c0\,\Lambda^2 + c1\,mn2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

mn2fromneom = Solve[neom /. divergentpartrules, mn2][[1]]

$$\left\{mn2 \to \left(-m^2 - p^2 + p^2\,Z\,Z\Delta - \delta m_1^2 - \frac{1}{6}(-1+n)\,Z\Delta^2\,\hbar\left(tfing + c0\,\Lambda^2 + c1\,mg2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)(\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}v^2\,Z\Delta\,(3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b) - \frac{1}{6}tfinn\,Z\Delta^2\,\hbar\,(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \frac{1}{6}c0\,Z\Delta^2\,\Lambda^2\,\hbar\,(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right) \Big/ \left(-1 + \frac{1}{6}c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right\}$$

```
mg2soln = mg2 /. (geom /. divergentpartrules /. mn2fromneom // Solve[#, mg2][[1]] &)
```

$$\Bigg(-m^2 - p^2 + p^2\, Z\, Z\triangle - \delta m_1^2 - \frac{1}{6} v^2\, Z\triangle\, (\lambda + \delta\lambda_a) - \frac{1}{6}\, \text{tfinn}\, Z\triangle^2\, \hbar\, (\lambda + \delta\lambda_{2a}) - \frac{1}{6}\, \text{c0}\, Z\triangle^2\, \Lambda^2\, \hbar\, (\lambda + \delta\lambda_{2a}) - \frac{1}{6}\, \text{tfing}\, Z\triangle^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}) - \frac{1}{6}\, \text{c0}\, Z\triangle^2\, \Lambda^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}) + \frac{\text{c1}\, m^2\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c1}\, p^2\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} - \frac{\text{c1}\, p^2\, Z\, Z\triangle^3\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] \delta m_1^2\, (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c1}\, (-1+n)\, \text{tfing}\, Z\triangle^4\, \hbar^2\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})^2}{36\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c0}\,\text{c1}\, (-1+n)\, Z\triangle^4\, \Lambda^2\, \hbar^2\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})^2}{36\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c1}\, v^2\, Z\triangle^3\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})\, (3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b)}{36\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c1}\,\text{tfinn}\, Z\triangle^4\, \hbar^2\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})\, (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})}{36\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)} + \frac{\text{c0}\,\text{c1}\, Z\triangle^4\, \Lambda^2\, \hbar^2\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})\, (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})}{36\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)}\Bigg) \Bigg/ \Bigg(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}) - \frac{\text{c1}^2\, (-1+n)\, Z\triangle^4\, \hbar^2\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 (\lambda + \delta\lambda_{2a})^2}{36\left(-1 + \frac{1}{6}\,\text{c1}\, Z\triangle^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)}\Bigg)$$

```
mn2soln = mn2 /. mn2fromneom /. mg2 → mg2soln // Simplify
```

$$-\frac{1}{6\left(-1+\frac{1}{6}\,\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}$$

$$\Big(6\,m^2+6\,p^2-6\,p^2\,Z\,Z\Delta+6\,\delta m_1^2+v^2\,Z\Delta\,(3\,\lambda+\delta\lambda_a+2\,\delta\lambda_b)+$$
$$\text{tfinn}\,Z\Delta^2\,\hbar\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})+\text{c0}\,Z\Delta^2\,\Lambda^2\,\hbar\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})+$$
$$\Big((-1+n)\,Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})\,\Big(18\,\text{tfing}+18\,\text{c0}\,\Lambda^2+18\,\text{c1}\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+18\,\text{c1}\,p^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$
$$18\,\text{c1}\,p^2\,Z\,Z\Delta\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+3\,\text{c1}\,v^2\,Z\Delta\,\lambda\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-9\,\text{c1}\,\text{tfing}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$
$$3\,\text{c1}\,\text{tfinn}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-6\,\text{c0}\,\text{c1}\,Z\Delta^2\,\lambda\,\Lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-6\,\text{c1}^2\,m^2\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$
$$6\,\text{c1}^2\,p^2\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+6\,\text{c1}^2\,p^2\,Z\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-3\,\text{c1}\,\text{tfing}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]$$
$$\delta\lambda_{2\,a}+3\,\text{c1}\,\text{tfinn}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,a}+\text{c1}^2\,v^2\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,a}+\text{c1}^2\,v^2\,Z\Delta^3\,\lambda$$
$$\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_b+\text{c1}^2\,v^2\,Z\Delta^3\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,a}\,\delta\lambda_b-6\,\text{c1}\,\text{tfing}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}-$$
$$6\,\text{c0}\,\text{c1}\,Z\Delta^2\,\Lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}-6\,\text{c1}^2\,m^2\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,b}-6\,\text{c1}^2\,p^2\,Z\Delta^2\,\hbar$$
$$\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,b}+6\,\text{c1}^2\,p^2\,Z\,Z\Delta^3\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,b}-\text{c1}^2\,v^2\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,b}-$$
$$6\,\text{c1}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta m_1^2\,\Big(-3+\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\Big)-$$
$$\text{c1}\,v^2\,Z\Delta\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_a\,\Big(-3+\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\Big)\Big)\Big)\Big/$$
$$\Big(\Big(-3+\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\Big)\,\Big(-6+2\,\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$
$$\text{c1}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,a}+2\,\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\Big)\Big)\Big)$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

```
cteq = ((CoefficientList[mg2soln - finmg2, {p, v, tfing, tfinn}] // Flatten) //
    DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

$$\Big\{-\Bigg(\bigg(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda + n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\bigg)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\Bigg) == 0,$$

$$-\frac{\lambda\,\hbar}{6} + \left(3\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2a})\right)\Big/\Bigg(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\Bigg) == 0,$$

$$-\frac{1}{6\,c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\Bigg(6 + c1\,(1+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{18}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} + (36\,(-1+n))\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right)\Bigg) == 0,\ \text{True},$$

$$-\frac{\lambda}{6} + \frac{Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} - \left(Z\Delta\,\left((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b\right)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right) == 0,$$

$$(-6 + 6\,Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right) == 0\Big\}$$

```
cteq2 = ((CoefficientList[mn2soln - finmn2, {p, v, tfing, tfinn}] // Flatten) //
    DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

$$\left\{-\left(\left(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\,((2+n)\,\lambda + n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)\right) == 0,\right.$$

$$-\frac{1}{2\,c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(2 + c1\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{6\,(-1+n)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2b}\right)} + 12\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)\right)\right) == 0,$$

$$\left((-1+n)\,\hbar\left(-Z\Delta^2\,\delta\lambda_{2a}\left(-18 + c1\,n\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) + c1^2\,n\,Z\Delta^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2b}\right) + \lambda\left(18\,(-1 + Z\Delta^2) + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(3\,(4+n) - c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) - c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2b}\left(-12 + c1\,(4+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + 2\,c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2b}\right)\right)\right)\right)\Big/\left(6\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)\right) == 0,$$

$$\text{True},\ -\frac{\lambda}{2} - \frac{(-1+n)\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2b}\right)} - (Z\Delta\,((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b))\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)\right) == 0,$$

$$\left.(-6 + 6\,Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right) == 0\right\}$$

Solve for counterterms

Find counter-terms from the gap equations

```
cteqs = {cteq, cteq2} // Flatten // FullSimplify // DeleteDuplicates
```

$$\Big\{\left(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda + n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right) == 0,$$

$$\hbar\left(\lambda - \left(18\,Z\Delta^2\,(\lambda + \delta\lambda_{2a})\right)\Big/\left(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right)\right) == 0,$$

$$\frac{1}{c1\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(6 + c1\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{18}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} + (36\,(-1+n))\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right)\right) == 0,$$

$$\mathrm{True},\ \lambda + \left(6\,Z\Delta\,\left((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b\right)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right) == \frac{6\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)},$$

$$(-1 + Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right) == 0,$$

$$\frac{1}{c1\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(2 + c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{6\,(-1+n)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} + 12\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right)\right) == 0,$$

$$(-1+n)\,\hbar\left(\lambda - \left(18\,Z\Delta^2\,(\lambda + \delta\lambda_{2a})\right)\Big/\left(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right)\right) == 0,$$

$$\lambda + \frac{2\,(-1+n)\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} + \left(2\,Z\Delta\,\left((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b\right)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right) == 0\Big\}$$

```
cts = {δm₁², δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ} /. Solve[cteqs,
    {δm₁, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] // FullSimplify // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$

$$\lambda\left(-1+\frac{6\,(2+n)}{n\,Z\Delta\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}-\frac{6}{3\,n\,Z\Delta+c1\,n\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),$$

$$\lambda\left(-1+\frac{18}{Z\Delta^2\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$

$$\left.\left.\lambda\left(-1+\frac{3}{3\,Z\Delta+c1\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{3}{Z\Delta^2\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

ZΔ is redundant in this truncation, can remove it :

```
cts /. ZΔ → 1 // FullSimplify
```

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$

$$\lambda\left(-1-\frac{6}{3\,n+c1\,n\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}+\frac{6\,(2+n)}{n\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$

$$\lambda\left(-1+\frac{18}{\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$

$$\left.\left.\lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ 1,\ 1\right\}\right\}$$

Verify that the finite gap equations come out right

```
finmg2 ==
  (mg2soln /. Solve[cteqs, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] /. ZΔ → 1 // FullSimplify //
     DeleteDuplicates)[[2]] // Simplify
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

FullSimplify::infd : Expression $\Bigg(-m^2 - \frac{-c0\,\Lambda^2 - c1\,m^2\,\text{Log[Power[≪2≫] Power[≪2≫]]}}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]} - \frac{v^2\,\lambda\,(\lambda + ≪1≫)}{2 ≪1≫ (\lambda + ≪1≫)} - \frac{3\,\text{tfing} ≪1≫ \hbar\left(-(-1+n)\,\lambda + (1+n)\,\lambda + \frac{2\,(≪1≫)}{3 ≪3≫ \text{Log}[≪1≫]}\right)}{2\,(3 + c1 ≪2≫ \text{Log}[≪1≫])^2\,(\lambda + \delta\lambda_b)^2} - \Big(3\,c0\,\lambda^2\,\Lambda^2\,\hbar\,\big(-(-1+n)\,\lambda + (1+n)\,\lambda + (2\,(9\,\text{Power}[≪2≫] + ≪7≫ + \text{Power}[≪2≫]\,\text{Power}[≪2≫] ≪1≫ \text{Power}[≪2≫]\,\text{Power}[≪2≫]))\big/(3\,c1\,\lambda^2\,\hbar\,\text{Log[Times[≪2≫]]})\big)\Big)\Big/\big(2\,(3 + c1\,\lambda\,\hbar\,\text{Log[Times[≪2≫]]})^2\,(\lambda + \delta\lambda_b)^2\big)\Bigg)\Bigg/\Bigg(-1 + \Big(3\,c1\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\big(-(-1+n)\,\lambda + (1+n)\,\lambda + (2\,(\text{Times}[≪2≫] + \text{Times}[≪5≫] + ≪5≫ + \text{Times}[≪6≫] + \text{Times}[≪5≫]))\big/(3\,c1\,\lambda^2\,\hbar\,\text{Log}[≪1≫])\big)\Big)\Big/\big(2\,(3 + c1\,\lambda\,\hbar\,\text{Log}[≪1≫])^2\,(\lambda + \delta\lambda_b)^2\big)\Bigg)$

simplified to ComplexInfinity. ≫

```
True
```

```
finmn2 == mn2 /.
  ((neom /. divergentpartrules /. mg2 → mg2soln /. Solve[cteqs, {δm1, δλ1a,
          δλ2a, δλ1b, δλ2b, Z, ZΔ}] /. ZΔ → 1 // FullSimplify //
      DeleteDuplicates) // Solve[#, mn2] &) // FullSimplify
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

```
{True}
```

Find counter-terms for vev equation

```
rnveom = veom /. {mg2 → finmg2, mn2 → finmn2} // Simplify // DeleteDuplicates
```

$$\frac{1}{6}\,v\left(\frac{6\,(m^2 + \delta m_0^2)}{Z\Delta} + v^2\,(\lambda + \delta\lambda_0) + (-1+n)\,(\text{tfing} + t\infty g)\,Z\Delta\,\hbar\,(\lambda + \delta\lambda_a) + (\text{tfinn} + t\infty n)\,Z\Delta\,\hbar\,(3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b)\right)$$

```
ctegs3 =
 ((((CoefficientList[(1/v rnveom - 1/v finveom) /. divergentpartrules /. {mg2 → finmg2,
            mn2 → finmn2} // Simplify // Expand // FullSimplify, {v,
          tfing, tfinn}] // Simplify // Flatten) // DeleteDuplicates //
       Simplify // FullSimplify // DeleteDuplicates) == 0 // Thread) /.
     Solve[cteqs, {δm₁, δλ1a, δλ2a, δλ1b, δλ2b, Z}] /. ZΔ → 1 // Simplify //
    FullSimplify // DeleteDuplicates) [[1]] // Flatten // DeleteDuplicates
```

$$\left\{\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}+\delta m_0^2==0,\ \text{True},\right.$$
$$\left.3\,\lambda\left(1+\frac{2-2\,n}{3\,n+c1\,n\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}-\frac{2\,(2+n)}{n\left(6+c1\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right)+\delta\lambda_0==0\right\}$$

Verify counter-term expressions in text

```
{δm₁² == δm₀², δλ1a == δλ2a, δλ1b == δλ2b} /.
    Solve[cteqs, {δm₁, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] /. Solve[ctegs3, {δm₀, δλ₀}] /.
  ZΔ → 1 // FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

{True}

```
{δλ1a / δλ1b} /. Solve[cteqs, {δm₁, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] /.
   Solve[ctegs3, {δm₀, δλ₀}] /. ZΔ → 1 // FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{1+\frac{3\,(2+n)}{6+c1\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right\}$$

```
δλ1b /. Solve[cteqs, {δm₁, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] /. ZΔ → 1 // FullSimplify //
 DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right)\right\}$$

```
{δλ₀ == 1 δλ1a + 2 δλ1b} /. Solve[cteqs, {δm₁, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] /.
   Solve[ctegs3, {δm₀, δλ₀}] /. ZΔ → 1 // FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

{True}

```
{δm0^2 == -((n + 2) λ ℏ (c0 Λ^2 + c1 m^2 Log[Λ^2 / μ^2]))/6 (δλ1 a + λ)/(δλ1 b + λ)} /.
      Solve[cteqs, {δm1, δλ1 a, δλ2 a, δλ1 b, δλ2 b, Z, ZΔ}] /. Solve[ctegs3, {δm0, δλ0}] /.
    ZΔ → 1 // FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

```
{True}
```

Total number of independent counter-term equations

```
Length[{cteqs, ctegs3} // Flatten // FullSimplify // DeleteDuplicates] -
 1 (* -1 because one of the "equations" is identically "True" *)
```

```
10
```

Renormalization of SI-2PIEA gap equations in the Hartree-Fock approximation

Author: Michael Brown
Supplement to thesis Chapter 3

Mathematica notebook to compute couter-terms for two loop truncations of the two particle irreducible effective action

In[90]:=
```
ClearAll[veom, geom, neom, regularisedtadpoles, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the thesis are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
mn2 is the Higgs mass squared $m_H{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_0{}^2$, $\delta m_1{}^2$ are its counter-terms,
λ is the (renormalized) four point coupling,
$\delta\lambda_0$, $\delta\lambda_{1\,a}$, $\delta\lambda_{1\,b}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

Equations of motion

Vev equation of motion

In[91]:=

(*veom=

$$Z\Delta^{-1}\left(m^2+\delta m_0{}^2\right)v+\frac{\lambda+\delta\lambda_0}{6}v^3+\frac{\hbar}{6}Z\Delta\left(n-1\right)(\lambda+\delta\lambda_{1a})\,v\,(t\infty g+tfing)+\frac{\hbar}{6}Z\Delta\ \left(3\lambda+\delta\lambda_{1a}+2\delta\lambda_{1b}\right)v\,(t\infty n+tfinn)$$

$$finveom=m^2v+\frac{\lambda}{6}v^3+\frac{\hbar}{6}\left(n-1\right)\lambda\ v\ tfing+\frac{\hbar}{2}\lambda\ v\ tfinn*)$$

```
veom = v mg2
```

Out[91]= mg2 v

Goldstone equation of motion

In[92]:=

```
geom = p^2 - mg2 == Z ZΔ p^2 - m^2 - δm1^2 - ZΔ (λ + δλ1a)/6 v^2 -
   ħ/6 ((n + 1) λ + (n - 1) δλ2a + 2 δλ2b) ZΔ^2 (t∞g + tfing) - ħ/6 (λ + δλ2a) ZΔ^2 (t∞n + tfinn)
finmg2 = mg2 /. Solve[p^2 - mg2 == p^2 - m^2 - λ/6 v^2 - ħ/6 (n + 1) λ tfing - ħ/6 λ tfinn, mg2][[1]]
```

Out[92]= $-\mathrm{mg2} + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6} v^2 Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6} (\mathrm{tfinn} + \mathrm{t}\infty\mathrm{n})\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2a}) - \frac{1}{6} (\mathrm{tfing} + \mathrm{t}\infty\mathrm{g})\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2a} + 2\, \delta\lambda_{2b})$

Out[93]= $\frac{1}{6} \left(6\, m^2 + v^2 \lambda + \mathrm{tfing}\, \lambda\, \hbar + n\, \mathrm{tfing}\, \lambda\, \hbar + \mathrm{tfinn}\, \lambda\, \hbar\right)$

Higgs equation of motion

In[94]:=

```
neom = p^2 - mn2 == Z ZΔ p^2 - m^2 - δm1^2 - ZΔ v^2 (3 λ + δλ1a + 2 δλ1b)/6 -
   ħ/6 (λ + δλ2a) (n - 1) ZΔ^2 (t∞g + tfing) - ħ/6 (3 λ + δλ2a + 2 δλ2b) ZΔ^2 (t∞n + tfinn)
finmn2 = mn2 /. Solve[p^2 - mn2 == p^2 - m^2 - v^2 λ/2 - ħ/6 λ (n - 1) tfing - ħ/2 λ tfinn, mn2][[1]]
```

Out[94]= $-\mathrm{mn2} + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6} (-1 + n)\, (\mathrm{tfing} + \mathrm{t}\infty\mathrm{g})\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2a}) - \frac{1}{6} v^2 Z\Delta\, (3\lambda + \delta\lambda_a + 2\, \delta\lambda_b) - \frac{1}{6} (\mathrm{tfinn} + \mathrm{t}\infty\mathrm{n})\, Z\Delta^2\, \hbar\, (3\lambda + \delta\lambda_{2a} + 2\, \delta\lambda_{2b})$

Out[95]= $\frac{1}{6} \left(6\, m^2 + 3\, v^2 \lambda - \mathrm{tfing}\, \lambda\, \hbar + n\, \mathrm{tfing}\, \lambda\, \hbar + 3\, \mathrm{tfinn}\, \lambda\, \hbar\right)$

Infinite parts of tadpoles

c0, c1, Λ and μ are regularisation/renormalisation scheme dependent quantities

In[96]:=

```
regularisedtadpoles = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}
```

Out[96]= $\left\{\mathrm{t}\infty\mathrm{g} \to \mathrm{c0}\, \Lambda^2 + \mathrm{c1}\, \mathrm{mg2}\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \mathrm{t}\infty\mathrm{n} \to \mathrm{c0}\, \Lambda^2 + \mathrm{c1}\, \mathrm{mn2}\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

In[97]:=
```
mn2fromneom = Solve[neom /. regularisedtadpoles, mn2][[1]]
```

Out[97]=

$$\left\{\text{mn2} \to \left(-m^2 - p^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6}(-1+n)\, Z\Delta^2\, \hbar \left(\text{tfing} + c0\, \Lambda^2 + c1\, \text{mg2}\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)(\lambda + \delta\lambda_{2a}) - \frac{1}{6} v^2\, Z\Delta\, (3\lambda + \delta\lambda_a + 2\,\delta\lambda_b) - \frac{1}{6}\,\text{tfinn}\, Z\Delta^2\, \hbar\, (3\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b}) - \frac{1}{6}\, c0\, Z\Delta^2\, \Lambda^2\, \hbar\, (3\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right) \Big/ \left(-1 + \frac{1}{6}\, c1\, Z\Delta^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})\right)\right\}$$

In[98]:= **mg2soln = mg2 /. (geom /. regularisedtadpoles /. mn2fromneom // Solve[#, mg2][[1]] &)**

Out[98]=

$$\left(-m^2 - p^2 + p^2 Z Z\Delta - \delta m_1^2 - \frac{1}{6} v^2 Z\Delta (\lambda + \delta\lambda_a) - \frac{1}{6} \text{tfinn}\, Z\Delta^2 \hbar (\lambda + \delta\lambda_{2a}) - \frac{1}{6} \text{c0}\, Z\Delta^2 \Lambda^2 \hbar (\lambda + \delta\lambda_{2a}) - \frac{1}{6} \text{tfing}\, Z\Delta^2 \hbar ((1+n)\lambda + (-1+n)\delta\lambda_{2a} + 2\delta\lambda_{2b}) - \frac{1}{6} \text{c0}\, Z\Delta^2 \Lambda^2 \hbar ((1+n)\lambda + (-1+n)\delta\lambda_{2a} + 2\delta\lambda_{2b}) + \frac{\text{c1}\, m^2 Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c1}\, p^2 Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} - \frac{\text{c1}\, p^2 Z Z\Delta^3 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] \delta m_1^2 (\lambda + \delta\lambda_{2a})}{6\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c1}\, (-1+n)\, \text{tfing}\, Z\Delta^4 \hbar^2 \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})^2}{36\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c0}\, \text{c1}\, (-1+n)\, Z\Delta^4 \Lambda^2 \hbar^2 \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})^2}{36\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c1}\, v^2 Z\Delta^3 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})(3\lambda + \delta\lambda_a + 2\delta\lambda_b)}{36\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c1}\, \text{tfinn}\, Z\Delta^4 \hbar^2 \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})(3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})}{36\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)} + \frac{\text{c0}\, \text{c1}\, Z\Delta^4 \Lambda^2 \hbar^2 \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2a})(3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})}{36\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)}\right) \Bigg/ \left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]((1+n)\lambda + (-1+n)\delta\lambda_{2a} + 2\delta\lambda_{2b}) - \frac{\text{c1}^2 (-1+n)\, Z\Delta^4 \hbar^2 \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 (\lambda + \delta\lambda_{2a})^2}{36\left(-1 + \frac{1}{6}\text{c1}\, Z\Delta^2 \hbar \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](3\lambda + \delta\lambda_{2a} + 2\delta\lambda_{2b})\right)}\right)$$

In[99]:=
```
mn2soln = mn2 /. mn2fromneom /. mg2 → mg2soln // Simplify
```

Out[99]=

$$
\begin{aligned}
-\Bigg(\Bigg(\Bigg(& 6\,m^2 + 6\,p^2 - 6\,p^2\,Z\,Z\Delta + 6\,\delta m_1^2 + v^2\,Z\Delta\,(3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b) + {} \\
& \text{tfinn}\,Z\Delta^2\,\hbar\,(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) + \text{c0}\,Z\Delta^2\,\Lambda^2\,\hbar\,(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) + {} \\
& \Bigg((-1 + n)\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2\,a})\,\Bigg(18\,\text{tfing} + 18\,\text{c0}\,\Lambda^2 + 18\,\text{c1}\,m^2\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + 18\,\text{c1}\,p^2\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] - {} \\
& 18\,\text{c1}\,p^2\,Z\,Z\Delta\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + 3\,\text{c1}\,v^2\,Z\Delta\,\lambda\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] - 9\,\text{c1}\,\text{tfing}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + {} \\
& 3\,\text{c1}\,\text{tfinn}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] - 6\,\text{c0}\,\text{c1}\,Z\Delta^2\,\lambda\,\Lambda^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] - 6\,\text{c1}^2\,m^2\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2 - {} \\
& 6\,\text{c1}^2\,p^2\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2 + 6\,\text{c1}^2\,p^2\,Z\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2 - 3\,\text{c1}\,\text{tfing}\,Z\Delta^2\,\hbar \\
& \text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,a} + 3\,\text{c1}\,\text{tfinn}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,a} + \text{c1}^2\,v^2\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,a} + {} \\
& \text{c1}^2\,v^2\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_b + \text{c1}^2\,v^2\,Z\Delta^3\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,a}\,\delta\lambda_b - {} \\
& 6\,\text{c1}\,\text{tfing}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b} - 6\,\text{c0}\,\text{c1}\,Z\Delta^2\,\Lambda^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b} - {} \\
& 6\,\text{c1}^2\,m^2\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,b} - 6\,\text{c1}^2\,p^2\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,b} + {} \\
& 6\,\text{c1}^2\,p^2\,Z\,Z\Delta^3\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,b} - \text{c1}^2\,v^2\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,b} - {} \\
& 6\,\text{c1}\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta m_1^2\,\Big(-3 + \text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + \text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big) - {} \\
& \text{c1}\,v^2\,Z\Delta\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_a\,\Big(-3 + \text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + \text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)\Bigg)\Bigg)\Bigg)\Bigg/ \\
& \Bigg(\Big(-3 + \text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + \text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)\,\Big(-6 + 2\,\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + {} \\
& \text{c1}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big] + \text{c1}\,n\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,a} + 2\,\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)\Bigg)\Bigg)\Bigg/ \\
& \Bigg(6\,\Big(-1 + \frac{1}{6}\,\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Bigg)\Bigg)
\end{aligned}
$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

In[100]:= **cteq = ((CoefficientList[mg2soln - finmg2, {p, v, tfing, tfinn}] // Flatten) // DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread**

Out[100]=

$$\Bigg\{-\Bigg(\Bigg(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\,((2+n)\,\lambda + n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Bigg)\Big/ \left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\Bigg) == 0,$$

$$-\frac{\lambda\,\hbar}{6} + \left(3\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2\,a})\right)\Big/\Bigg(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right) \left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\Bigg) == 0,$$

$$-\frac{1}{6\,c1\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\Bigg(6 + c1\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{18}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)} + (36\,(-1+n))\Big/ \left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right)\Bigg) == 0,\ \mathrm{True},$$

$$-\frac{\lambda}{6} + \frac{Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)} - (Z\Delta\,((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b))\Big/ \left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right) == 0,$$

$$(-6 + 6\,Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right) == 0\Bigg\}$$

In[101]:=
```
cteq2 = ((CoefficientList[mn2soln - finmn2, {p, v, tfing, tfinn}] // Flatten) //
      DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

Out[101]=

$$\left\{-\left(\left(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda + n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right) == 0,\right.$$

$$-\frac{1}{2\,c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(2 + c1\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{6\,(-1+n)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} + 12\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right)\right) == 0,$$

$$\left((-1+n)\,\hbar\left(-Z\Delta^2\,\delta\lambda_{2a}\left(-18 + c1\,n\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) + c1^2\,n\,Z\Delta^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2b}\right) + \lambda\left(18\left(-1 + Z\Delta^2\right) + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(3\,(4+n) - c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) - c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\left(-12 + c1\,(4+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + 2\,c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)\right)\right)\Big/\left(6\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right) == 0,$$

$$\text{True},\ -\frac{\lambda}{2} - \frac{(-1+n)\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2b}\right)} - \left(Z\Delta\left((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b\right)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\right) == 0,$$

$$\left.(-6 + 6\,Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right) == 0\right\}$$

Solve for counterterms

Find counter-terms from the gap equations

In[102]:=

```
cteqs = {cteq, cteq2} // Flatten // FullSimplify // DeleteDuplicates
```

Out[102]=

$$\Bigg\{\left(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda + n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right) == 0,$$

$$\hbar\left(\lambda - \left(18\,Z\Delta^2\,(\lambda + \delta\lambda_{2\,a})\right)\Big/\left(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right)\right) == 0,$$

$$\frac{1}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(6 + c1\,(1+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{18}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)} + \left(36\,(-1+n)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right)\right) == 0,$$

$$\text{True},\ \lambda + \left(6\,Z\Delta\left((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b\right)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right) == \frac{6\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)},$$

$$(-1 + Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right) == 0,$$

$$\frac{1}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(2 + c1\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{6\,(-1+n)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)} + 12\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right)\right) == 0,$$

$$(-1+n)\,\hbar\left(\lambda - \left(18\,Z\Delta^2\,(\lambda + \delta\lambda_{2\,a})\right)\Big/\left(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right)\right) == 0,$$

$$\lambda + \frac{2\,(-1+n)\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)} + \left(2\,Z\Delta\left((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b\right)\right)\Big/\left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right) == 0\Bigg\}$$

In[103]:=
```
cts = {δm1^2, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ} /. Solve[cteqs,
    {δm1, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] // FullSimplify // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[103]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$
$$\lambda\left(-1+\frac{6\,(2+n)}{n\,Z\Delta\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}-\frac{6}{3\,n\,Z\Delta+c1\,n\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),$$
$$\lambda\left(-1+\frac{18}{Z\Delta^2\,\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\left.\left.\lambda\left(-1+\frac{3}{3\,Z\Delta+c1\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{3}{Z\Delta^2\,\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

ZΔ is redundant in this truncation, can remove it :

In[104]:=
```
cts /. ZΔ → 1 // FullSimplify
```

Out[104]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$
$$\lambda\left(-1-\frac{6}{3\,n+c1\,n\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}+\frac{6\,(2+n)}{n\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\lambda\left(-1+\frac{18}{\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\left.\left.\lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ 1,\ 1\right\}\right\}$$

Verify that the finite gap equations come out right

In[105]:=
```
finmg2 ==
  (mg2soln /. Solve[cteqs, {δm1, δλ1 a, δλ2 a, δλ2 b, Z, ZΔ}] /. ZΔ → 1 // FullSimplify //
     DeleteDuplicates)[[2]] // Simplify
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

FullSimplify::infd : Expression $\Bigg(-m^2-\frac{-c0\,\Lambda^2-c1\,m^2\,\text{Log[Power[≪2≫] Power[≪2≫]]}}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}-\frac{≪1≫}{≪1≫}-\frac{3\,\text{tfing}\,≪1≫\,\hbar\,(≪1≫)}{2\,≪1≫^2\,(≪1≫)^2}-\frac{3\,c0\,\lambda^2\,\Lambda^2\,\hbar\left(-(-1+n)\,\lambda+(1+n)\,\lambda+\frac{2\,(9\,\text{Power[≪2≫]}+≪7≫+≪1≫)}{3\,c1\,\lambda^2\,\hbar\,\text{Log[Times[≪2≫]]}}\right)}{2\,(3+c1\,\lambda\,\hbar\,\text{Log[Times[≪2≫]]})^2\,(\lambda+\delta\lambda_b)^2}\Bigg)\Bigg/\Bigg(-1+\Bigg(3\,c1\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\Bigg(-(-1+n)\,\lambda+(1+n)\,\lambda+\frac{2\,(\text{Times[≪2≫]}+≪7≫+\text{Times[≪5≫]})}{3\,c1\,\lambda^2\,\hbar\,\text{Log[≪1≫]}}\Bigg)\Bigg)\Big/\left(2\,(3+c1\,\lambda\,\hbar\,\text{Log[≪1≫]})^2\,(\lambda+\delta\lambda_b)^2\right)\Bigg)$ simplified to ComplexInfinity. ≫

Out[105]= True

In[106]:=
```
finmn2 == mn2 /.
  ((neom /. regularisedtadpoles /. mg2 → mg2soln /. Solve[cteqs, {δm1, δλ1 a,
           δλ2 a, δλ1 b, δλ2 b, Z, ZΔ}] /. ZΔ → 1 // FullSimplify //
      DeleteDuplicates) // Solve[#, mn2] &) // FullSimplify
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[106]= {True}

Verify counter-term expressions in text

In[107]:= $\left\{\delta m_1{}^2 == \frac{-\hbar\,\lambda\,(n+2)}{6}\left(c0\,\Lambda^2+c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\frac{\delta\lambda_{1\,a}+\lambda}{\delta\lambda_{1\,b}+\lambda}\right\}$ /.
```
    Solve[cteqs, {δm1, δλ1 a, δλ2 a, δλ1 b, δλ2 b, Z, ZΔ}] /. ZΔ → 1 //
  FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[107]= {True}

In[108]:=
```
{δλ1 a == δλ2 a, δλ1 b == δλ2 b} /. Solve[cteqs, {δm1, δλ1 a, δλ2 a, δλ1 b, δλ2 b, Z, ZΔ}] /.
    ZΔ → 1 // FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[108]= {True}

In[109]:=
```
{δλ1a / δλ1b} /. Solve[cteqs, {δm1, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] /. ZΔ → 1 //
   FullSimplify // Flatten // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[109]= $\left\{1 + \frac{3\,(2+n)}{6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Delta^2}{\mu^2}\right]}\right\}$

In[110]:=
```
δλ1b /. Solve[cteqs, {δm1, δλ1a, δλ2a, δλ1b, δλ2b, Z, ZΔ}] /. ZΔ → 1 // FullSimplify //
  DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[110]= $\left\{\lambda\left(-1 + \frac{3}{3 + c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Delta^2}{\mu^2}\right]}\right)\right\}$

Total number of independent counter-term equations

In[111]:=
```
Length[{cteqs} // Flatten // FullSimplify // DeleteDuplicates] -
  1 (* -1 because one of the "equations" is identically "True" *)
```

Out[111]= 8

Renormalization of Symmetry Improved 2PIEA gap equations at 2 loops

Supplement to chapter 4 of thesis by Michael J. Brown.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Chapter 4.

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, msbarrules, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1a}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.
Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\rm NG}(p)$
Ifingp is the finite sunset integral $I^{\rm fin}_{\rm NG}(p)$,
Ifing0 is $I^{\rm fin}_{\rm NG}(m_G)$,
Ifingn is $I^{\rm fin}_{\rm NG}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\mathbf{geom} = p^2 - mg2 + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\mathrm{Ifingp} - \mathrm{Ifing0}) == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right) Z\Delta^2\,(tg) - \frac{\hbar}{6}(\lambda + \delta\lambda_{2a})\,Z\Delta^2\,(tn) + i\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\mathrm{Ing}$$

$$-mg2 + p^2 + \frac{1}{9}\,i\,(-\mathrm{Ifing0} + \mathrm{Ifingp})\,v^2\,\lambda^2\,\hbar == -m^2 + p^2\,Z\,Z\Delta + \frac{1}{9}\,i\,\mathrm{Ing}\,v^2\,Z\Delta^3\,(\delta\lambda + \lambda)^2\,\hbar - \delta m_1^2 - \frac{1}{6}\,v^2\,Z\Delta\,(\lambda + \delta\lambda_a) - \frac{1}{6}\,tn\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2a}) - \frac{1}{6}\,tg\,Z\Delta^2\,\hbar\,((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})$$

$$\mathbf{neom} = p^2 - mn2 + \frac{i\hbar}{2}\left(\frac{(\lambda)\,v}{3}\right)^2(n-1)\,(\mathrm{Ifinggp} - \mathrm{Ifinggn}) + \frac{i\hbar}{2}(\lambda)^2\,v^2\,(\mathrm{Ifinhhp} - \mathrm{Ifinhhn}) == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{3\,\lambda + \delta\lambda_{1a} + 2\,\delta\lambda_{1b}}{6}\,v^2 - \frac{\hbar}{6}\left(3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right) Z\Delta^2\,tn - \frac{\hbar}{6}(\lambda + \delta\lambda_{2a})\,Z\Delta^2\,(n-1)\,tg + \frac{i\hbar}{2}\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,(n-1)\,\mathrm{Igg} + \frac{i\hbar}{2}(\lambda+\delta\lambda)^2\,v^2\,Z\Delta^3\,\mathrm{Ihh}$$

$$-mn2 + p^2 + \frac{1}{2}\,i\,(-\mathrm{Ifinhhn} + \mathrm{Ifinhhp})\,v^2\,\lambda^2\,\hbar + \frac{1}{18}\,i\,(-\mathrm{Ifinggn} + \mathrm{Ifinggp})\,(-1+n)\,v^2\,\lambda^2\,\hbar == -m^2 + p^2\,Z\,Z\Delta + \frac{1}{2}\,i\,\mathrm{Ihh}\,v^2\,Z\Delta^3\,(\delta\lambda+\lambda)^2\,\hbar + \frac{1}{18}\,i\,\mathrm{Igg}\,(-1+n)\,v^2\,Z\Delta^3\,(\delta\lambda+\lambda)^2\,\hbar - \delta m_1^2 - \frac{1}{6}\,(-1+n)\,tg\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2a}) - \frac{1}{6}\,v^2\,Z\Delta\,(3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b) - \frac{1}{6}\,tn\,Z\Delta^2\,\hbar\,(3\,\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b})$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\mathbf{intrules} = \left\{\mathrm{Ing} \to I\mu + \mathrm{Ifingp},\ \mathrm{Igg} \to I\mu + \mathrm{Ifinggp},\ \mathrm{Ihh} \to I\mu + \mathrm{Ifinhhp},\ tg \to t\mu - i\,(mg2 - \mu^2)\,I\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 c\mu + \mathrm{tfing},\ tn \to t\mu - i\,(mn2 - \mu^2)\,I\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 c\mu + \mathrm{tfinn}\right\}$$

$$\left\{\mathrm{Ing} \to \mathrm{Ifingp} + I\mu,\ \mathrm{Igg} \to \mathrm{Ifinggp} + I\mu,\ \mathrm{Ihh} \to \mathrm{Ifinhhp} + I\mu,\ tg \to \mathrm{tfing} + t\mu - i\,I\mu\,(mg2 - \mu^2) + \frac{1}{9}\,c\mu\,v^2\,(\delta\lambda+\lambda)^2\,\hbar,\ tn \to \mathrm{tfinn} + t\mu - i\,I\mu\,(mn2 - \mu^2) + \frac{1}{9}\,c\mu\,v^2\,(\delta\lambda+\lambda)^2\,\hbar\right\}$$

$$\mathbf{msbarrules} = \left\{I\mu \to c2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ t\mu \to c0\,\Lambda^2 + c1\,\mu^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ c\mu \to a0\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + a1\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

$$\left\{I\mu \to c2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ t\mu \to c0\,\Lambda^2 + c1\,\mu^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ c\mu \to a1\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + a0\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
{mg2soln, mn2soln} =
  (Solve[{geom, neom} /. intrules, {mg2, mn2}] // ExpandAll // Simplify)[[1]]
```

PowerMod::ninv : 0 is not invertible modulo 8387. >>

$$
\begin{aligned}
&\{\mathrm{mg2} \to \\
&-\Big(\Big(324\, m^2 + 324\, p^2 - 324\, p^2 Z\, Z\Delta + 54\, v^2 Z\Delta\, \lambda - 36\, \mathbb{i}\, \mathrm{Ifingp}\, v^2 Z\Delta^3\, \delta\lambda^2\, \hbar - 36\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \delta\lambda^2\, \hbar + \\
&108\, \mathbb{i}\, \mathrm{I}\mu\, m^2 Z\Delta^2\, \lambda\, \hbar + 108\, \mathbb{i}\, \mathrm{I}\mu\, p^2 Z\Delta^2\, \lambda\, \hbar + 54\, \mathrm{tfing}\, Z\Delta^2\, \lambda\, \hbar + 54\, n\, \mathrm{tfing}\, Z\Delta^2\, \lambda\, \hbar + \\
&54\, \mathrm{tfinn}\, Z\Delta^2\, \lambda\, \hbar + 108\, \mathrm{t}\mu\, Z\Delta^2\, \lambda\, \hbar + 54\, n\, \mathrm{t}\mu\, Z\Delta^2\, \lambda\, \hbar - 108\, \mathbb{i}\, \mathrm{I}\mu\, p^2 Z\, Z\Delta^3\, \lambda\, \hbar - \\
&72\, \mathbb{i}\, \mathrm{Ifingp}\, v^2 Z\Delta^3\, \delta\lambda\, \lambda\, \hbar - 72\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \delta\lambda\, \lambda\, \hbar - 36\, \mathbb{i}\, \mathrm{Ifing0}\, v^2\, \lambda^2\, \hbar + \\
&36\, \mathbb{i}\, \mathrm{Ifingp}\, v^2\, \lambda^2\, \hbar - 36\, \mathbb{i}\, \mathrm{Ifingp}\, v^2 Z\Delta^3\, \lambda^2\, \hbar - 36\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \lambda^2\, \hbar + \\
&108\, \mathbb{i}\, \mathrm{I}\mu\, Z\Delta^2\, \lambda\, \mu^2\, \hbar + 54\, \mathbb{i}\, \mathrm{I}\mu\, n\, Z\Delta^2\, \lambda\, \mu^2\, \hbar + 12\, \mathrm{c}\mu\, v^2 Z\Delta^2\, \delta\lambda^2\, \lambda\, \hbar^2 + \\
&6\, \mathrm{c}\mu\, n\, v^2 Z\Delta^2\, \delta\lambda^2\, \lambda\, \hbar^2 + 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 18\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 - \\
&27\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 - 6\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 - 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 - \\
&3\, \mathrm{I}\mu^2\, n\, v^2 Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 36\, \mathbb{i}\, \mathrm{I}\mu\, \mathrm{tfing}\, Z\Delta^4\, \lambda^2\, \hbar^2 + 18\, \mathbb{i}\, \mathrm{I}\mu\, n\, \mathrm{tfing}\, Z\Delta^4\, \lambda^2\, \hbar^2 + \\
&36\, \mathbb{i}\, \mathrm{I}\mu\, \mathrm{t}\mu\, Z\Delta^4\, \lambda^2\, \hbar^2 + 18\, \mathbb{i}\, \mathrm{I}\mu\, n\, \mathrm{t}\mu\, Z\Delta^4\, \lambda^2\, \hbar^2 + 24\, \mathrm{c}\mu\, v^2 Z\Delta^2\, \delta\lambda\, \lambda^2\, \hbar^2 + \\
&12\, \mathrm{c}\mu\, n\, v^2 Z\Delta^2\, \delta\lambda\, \lambda^2\, \hbar^2 + 6\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 36\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 - \\
&54\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 - 12\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 - 6\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 - \\
&6\, \mathrm{I}\mu^2\, n\, v^2 Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 12\, \mathrm{c}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 + 18\, \mathrm{Ifing0}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 + \\
&3\, \mathrm{Ifinggn}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 - 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 - 18\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 - \\
&27\, \mathrm{Ifinhhn}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 + 27\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 + 6\, \mathrm{c}\mu\, n\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 - \\
&3\, \mathrm{Ifinggn}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 + 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^2\, \lambda^3\, \hbar^2 + 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \lambda^3\, \hbar^2 + \\
&18\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \lambda^3\, \hbar^2 - 27\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \lambda^3\, \hbar^2 - 6\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \lambda^3\, \hbar^2 - \\
&3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^5\, \lambda^3\, \hbar^2 - 3\, \mathrm{I}\mu^2\, n\, v^2 Z\Delta^5\, \lambda^3\, \hbar^2 - 36\, \mathrm{I}\mu^2\, Z\Delta^4\, \lambda^2\, \mu^2\, \hbar^2 - \\
&18\, \mathrm{I}\mu^2\, n\, Z\Delta^4\, \lambda^2\, \mu^2\, \hbar^2 + 4\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, v^2 Z\Delta^4\, \delta\lambda^2\, \lambda^2\, \hbar^3 + 2\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, n\, v^2 Z\Delta^4\, \delta\lambda^2\, \lambda^2\, \hbar^3 + \\
&8\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, v^2 Z\Delta^4\, \delta\lambda\, \lambda^3\, \hbar^3 + 4\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, n\, v^2 Z\Delta^4\, \delta\lambda\, \lambda^3\, \hbar^3 + 4\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, v^2 Z\Delta^4\, \lambda^4\, \hbar^3 + \\
&2\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, n\, v^2 Z\Delta^4\, \lambda^4\, \hbar^3 - 54\, \mathrm{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} + 54\, n\, \mathrm{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} + \\
&54\, \mathrm{tfinn}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} + 54\, n\, \mathrm{t}\mu\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 18\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \lambda\, \hbar\, \delta\lambda_{2\,a} + \\
&54\, \mathbb{i}\, \mathrm{I}\mu\, n\, Z\Delta^2\, \mu^2\, \hbar\, \delta\lambda_{2\,a} + 6\, \mathrm{c}\mu\, n\, v^2 Z\Delta^2\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + \\
&6\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 27\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - \\
&18\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 3\, \mathrm{I}\mu^2\, n\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + \\
&18\, \mathbb{i}\, \mathrm{I}\mu\, n\, \mathrm{tfing}\, Z\Delta^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 18\, \mathbb{i}\, \mathrm{I}\mu\, n\, \mathrm{t}\mu\, Z\Delta^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 12\, \mathrm{c}\mu\, n\, v^2 Z\Delta^2\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + \\
&6\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 12\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} - \\
&54\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} - 36\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} - \\
&6\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} - 6\, \mathrm{I}\mu^2\, n\, v^2 Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + \\
&6\, \mathrm{Ifing0}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \mathrm{Ifinggn}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - \\
&3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 6\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - \\
&27\, \mathrm{Ifinhhn}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 27\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + \\
&6\, \mathrm{c}\mu\, n\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 3\, \mathrm{Ifinggn}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + \\
&3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + \\
&6\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 27\, \mathrm{Ifinhhp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 18\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - \\
&3\, \mathrm{Ifinggp}\, \mathrm{I}\mu\, n\, v^2 Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 3\, \mathrm{I}\mu^2\, n\, v^2 Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 18\, \mathrm{I}\mu^2\, n\, Z\Delta^4\, \lambda\, \mu^2\, \hbar^2\, \delta\lambda_{2\,a} + \\
&2\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, n\, v^2 Z\Delta^4\, \delta\lambda^2\, \lambda\, \hbar^3\, \delta\lambda_{2\,a} + 4\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, n\, v^2 Z\Delta^4\, \delta\lambda\, \lambda^2\, \hbar^3\, \delta\lambda_{2\,a} + \\
&2\, \mathbb{i}\, \mathrm{c}\mu\, \mathrm{I}\mu\, n\, v^2 Z\Delta^4\, \lambda^3\, \hbar^3\, \delta\lambda_{2\,a} - 18\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \lambda\, \hbar\, \delta\lambda_{b} - 18\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \hbar\, \delta\lambda_{2\,a}\, \delta\lambda_{b} + \\
&108\, \mathbb{i}\, \mathrm{I}\mu\, m^2 Z\Delta^2\, \hbar\, \delta\lambda_{2\,b} + 108\, \mathbb{i}\, \mathrm{I}\mu\, p^2 Z\Delta^2\, \hbar\, \delta\lambda_{2\,b} + 108\, \mathrm{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,b} + 108\, \mathrm{t}\mu\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,b} - \\
&108\, \mathbb{i}\, \mathrm{I}\mu\, p^2 Z\, Z\Delta^3\, \hbar\, \delta\lambda_{2\,b} + 18\, \mathbb{i}\, \mathrm{I}\mu\, v^2 Z\Delta^3\, \lambda\, \hbar\, \delta\lambda_{2\,b} + 108\, \mathbb{i}\, \mathrm{I}\mu\, Z\Delta^2\, \mu^2\, \hbar\, \delta\lambda_{2\,b} + \\
&12\, \mathrm{c}\mu\, v^2 Z\Delta^2\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 12\, \mathrm{Ifingp}\, \mathrm{I}\mu\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 12\, \mathrm{I}\mu^2\, v^2 Z\Delta^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + \\
&72\, \mathbb{i}\, \mathrm{I}\mu\, \mathrm{tfing}\, Z\Delta^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 18\, \mathbb{i}\, \mathrm{I}\mu\, n\, \mathrm{tfing}\, Z\Delta^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 72\, \mathbb{i}\, \mathrm{I}\mu\, \mathrm{t}\mu\, Z\Delta^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} +
\end{aligned}
$$

$18\, i\, I\mu\, n\, t\mu\, Z\Delta^4\, \lambda\, \hbar^2\, \delta\lambda_{2b} + 24\, c\mu\, v^2\, Z\Delta^2\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2b} + 24\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2b} +$
$24\, I\mu^2\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2b} + 12\, c\mu\, v^2\, Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2b} + 12\, \mathrm{Ifing0}\, I\mu\, v^2\, Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2b} -$
$12\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2b} + 12\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2b} + 12\, I\mu^2\, v^2\, Z\Delta^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2b} -$
$72\, I\mu^2\, Z\Delta^4\, \lambda\, \mu^2\, \hbar^2\, \delta\lambda_{2b} - 18\, I\mu^2\, n\, Z\Delta^4\, \lambda\, \mu^2\, \hbar^2\, \delta\lambda_{2b} + 8\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \delta\lambda^2\, \lambda\, \hbar^3\, \delta\lambda_{2b} +$
$2\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \delta\lambda^2\, \lambda\, \hbar^3\, \delta\lambda_{2b} + 16\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \delta\lambda\, \lambda^2\, \hbar^3\, \delta\lambda_{2b} +$
$4\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \delta\lambda\, \lambda^2\, \hbar^3\, \delta\lambda_{2b} + 8\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \lambda^3\, \hbar^3\, \delta\lambda_{2b} + 2\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \lambda^3\, \hbar^3\, \delta\lambda_{2b} +$
$18\, i\, I\mu\, n\, \mathrm{tfing}\, Z\Delta^4\, \hbar^2\, \delta\lambda_{2a}\, \delta\lambda_{2b} + 18\, i\, I\mu\, n\, t\mu\, Z\Delta^4\, \hbar^2\, \delta\lambda_{2a}\, \delta\lambda_{2b} -$
$18\, I\mu^2\, n\, Z\Delta^4\, \mu^2\, \hbar^2\, \delta\lambda_{2a}\, \delta\lambda_{2b} + 2\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \delta\lambda^2\, \hbar^3\, \delta\lambda_{2a}\, \delta\lambda_{2b} +$
$4\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \delta\lambda\, \lambda\, \hbar^3\, \delta\lambda_{2a}\, \delta\lambda_{2b} + 2\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \lambda^2\, \hbar^3\, \delta\lambda_{2a}\, \delta\lambda_{2b} +$
$36\, i\, I\mu\, \mathrm{tfing}\, Z\Delta^4\, \hbar^2\, \delta\lambda_{2b}^2 + 36\, i\, I\mu\, t\mu\, Z\Delta^4\, \hbar^2\, \delta\lambda_{2b}^2 - 36\, I\mu^2\, Z\Delta^4\, \mu^2\, \hbar^2\, \delta\lambda_{2b}^2 +$
$4\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \delta\lambda^2\, \hbar^3\, \delta\lambda_{2b}^2 + 8\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \delta\lambda\, \lambda\, \hbar^3\, \delta\lambda_{2b}^2 +$
$4\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \lambda^2\, \hbar^3\, \delta\lambda_{2b}^2 + 108\, i\, \delta m_1^2\, \left(-3\, i + I\mu\, Z\Delta^2\, \lambda\, \hbar + I\mu\, Z\Delta^2\, \hbar\, \delta\lambda_{2b}\right) +$
$18\, i\, v^2\, Z\Delta\, \delta\lambda_a\, \left(-3\, i + I\mu\, Z\Delta^2\, \lambda\, \hbar + I\mu\, Z\Delta^2\, \hbar\, \delta\lambda_{2b}\right)\Big) \Big/ \Big(18\, \left(-3\, i + I\mu\, Z\Delta^2\, \lambda\, \hbar + I\mu\, Z\Delta^2\, \hbar\, \delta\lambda_{2b}\right)$
$\left(-6\, i + 2\, I\mu\, Z\Delta^2\, \lambda\, \hbar + I\mu\, n\, Z\Delta^2\, \lambda\, \hbar + I\mu\, n\, Z\Delta^2\, \hbar\, \delta\lambda_{2a} + 2\, I\mu\, Z\Delta^2\, \hbar\, \delta\lambda_{2b}\right)\Big)\Big),$
$\mathrm{mn2} \to -\Big(\Big(324\, m^2 + 324\, p^2 - 324\, p^2\, Z\, Z\Delta + 162\, v^2\, Z\Delta\, \lambda + 18\, i\, \mathrm{Ifinggp}\, v^2\, Z\Delta^3\, \delta\lambda^2\, \hbar -$
$162\, i\, \mathrm{Ifinhhp}\, v^2\, Z\Delta^3\, \delta\lambda^2\, \hbar - 144\, i\, I\mu\, v^2\, Z\Delta^3\, \delta\lambda^2\, \hbar - 18\, i\, \mathrm{Ifinggp}\, n\, v^2\, Z\Delta^3\, \delta\lambda^2\, \hbar -$
$18\, i\, I\mu\, n\, v^2\, Z\Delta^3\, \delta\lambda^2\, \hbar + 108\, i\, I\mu\, m^2\, Z\Delta^2\, \lambda\, \hbar + 108\, i\, I\mu\, p^2\, Z\Delta^2\, \lambda\, \hbar - 54\, \mathrm{tfing}\, Z\Delta^2\, \lambda\, \hbar +$
$54\, n\, \mathrm{tfing}\, Z\Delta^2\, \lambda\, \hbar + 162\, \mathrm{tfinn}\, Z\Delta^2\, \lambda\, \hbar + 108\, t\mu\, Z\Delta^2\, \lambda\, \hbar + 54\, n\, t\mu\, Z\Delta^2\, \lambda\, \hbar -$
$108\, i\, I\mu\, p^2\, Z\, Z\Delta^3\, \lambda\, \hbar + 36\, i\, \mathrm{Ifinggp}\, v^2\, Z\Delta^3\, \delta\lambda\, \lambda\, \hbar - 324\, i\, \mathrm{Ifinhhp}\, v^2\, Z\Delta^3\, \delta\lambda\, \lambda\, \hbar -$
$288\, i\, I\mu\, v^2\, Z\Delta^3\, \delta\lambda\, \lambda\, \hbar - 36\, i\, \mathrm{Ifinggp}\, n\, v^2\, Z\Delta^3\, \delta\lambda\, \lambda\, \hbar - 36\, i\, I\mu\, n\, v^2\, Z\Delta^3\, \delta\lambda\, \lambda\, \hbar +$
$18\, i\, \mathrm{Ifinggn}\, v^2\, \lambda^2\, \hbar - 18\, i\, \mathrm{Ifinggp}\, v^2\, \lambda^2\, \hbar - 162\, i\, \mathrm{Ifinhhn}\, v^2\, \lambda^2\, \hbar +$
$162\, i\, \mathrm{Ifinhhp}\, v^2\, \lambda^2\, \hbar - 18\, i\, \mathrm{Ifinggn}\, n\, v^2\, \lambda^2\, \hbar + 18\, i\, \mathrm{Ifinggp}\, n\, v^2\, \lambda^2\, \hbar +$
$18\, i\, \mathrm{Ifinggp}\, v^2\, Z\Delta^3\, \lambda^2\, \hbar - 162\, i\, \mathrm{Ifinhhp}\, v^2\, Z\Delta^3\, \lambda^2\, \hbar - 108\, i\, I\mu\, v^2\, Z\Delta^3\, \lambda^2\, \hbar -$
$18\, i\, \mathrm{Ifinggp}\, n\, v^2\, Z\Delta^3\, \lambda^2\, \hbar + 108\, i\, I\mu\, Z\Delta^2\, \lambda\, \mu^2\, \hbar + 54\, i\, I\mu\, n\, Z\Delta^2\, \lambda\, \mu^2\, \hbar +$
$12\, c\mu\, v^2\, Z\Delta^2\, \delta\lambda^2\, \lambda\, \hbar^2 + 6\, c\mu\, n\, v^2\, Z\Delta^2\, \delta\lambda^2\, \lambda\, \hbar^2 - 3\, \mathrm{Ifinggp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 +$
$6\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 27\, \mathrm{Ifinhhp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 30\, I\mu^2\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 -$
$6\, \mathrm{Ifingp}\, I\mu\, n\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 27\, \mathrm{Ifinhhp}\, I\mu\, n\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 21\, I\mu^2\, n\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 +$
$3\, \mathrm{Ifinggp}\, I\mu\, n^2\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 3\, I\mu^2\, n^2\, v^2\, Z\Delta^5\, \delta\lambda^2\, \lambda\, \hbar^2 + 36\, i\, I\mu\, \mathrm{tfinn}\, Z\Delta^4\, \lambda^2\, \hbar^2 +$
$18\, i\, I\mu\, n\, \mathrm{tfinn}\, Z\Delta^4\, \lambda^2\, \hbar^2 + 36\, i\, I\mu\, t\mu\, Z\Delta^4\, \lambda^2\, \hbar^2 + 18\, i\, I\mu\, n\, t\mu\, Z\Delta^4\, \lambda^2\, \hbar^2 +$
$24\, c\mu\, v^2\, Z\Delta^2\, \delta\lambda\, \lambda^2\, \hbar^2 + 12\, c\mu\, n\, v^2\, Z\Delta^2\, \delta\lambda\, \lambda^2\, \hbar^2 - 6\, \mathrm{Ifinggp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 +$
$12\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 54\, \mathrm{Ifinhhp}\, I\mu\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 60\, I\mu^2\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 -$
$12\, \mathrm{Ifingp}\, I\mu\, n\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 54\, \mathrm{Ifinhhp}\, I\mu\, n\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 42\, I\mu^2\, n\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 +$
$6\, \mathrm{Ifinggp}\, I\mu\, n^2\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 6\, I\mu^2\, n^2\, v^2\, Z\Delta^5\, \delta\lambda\, \lambda^2\, \hbar^2 + 12\, c\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 +$
$6\, \mathrm{Ifing0}\, I\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 - 3\, \mathrm{Ifinggn}\, I\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 + 3\, \mathrm{Ifinggp}\, I\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 -$
$6\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 + 27\, \mathrm{Ifinhhn}\, I\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 - 27\, \mathrm{Ifinhhp}\, I\mu\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 +$
$6\, c\mu\, n\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 - 6\, \mathrm{Ifing0}\, I\mu\, n\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 + 6\, \mathrm{Ifingp}\, I\mu\, n\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 +$
$27\, \mathrm{Ifinhhn}\, I\mu\, n\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 - 27\, \mathrm{Ifinhhp}\, I\mu\, n\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 + 3\, \mathrm{Ifinggn}\, I\mu\, n^2\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 -$
$3\, \mathrm{Ifinggp}\, I\mu\, n^2\, v^2\, Z\Delta^2\, \lambda^3\, \hbar^2 - 3\, \mathrm{Ifinggp}\, I\mu\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 + 6\, \mathrm{Ifingp}\, I\mu\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 +$
$27\, \mathrm{Ifinhhp}\, I\mu\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 + 30\, I\mu^2\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 - 6\, \mathrm{Ifingp}\, I\mu\, n\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 +$
$27\, \mathrm{Ifinhhp}\, I\mu\, n\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 + 21\, I\mu^2\, n\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 + 3\, \mathrm{Ifinggp}\, I\mu\, n^2\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 +$
$3\, I\mu^2\, n^2\, v^2\, Z\Delta^5\, \lambda^3\, \hbar^2 - 36\, I\mu^2\, Z\Delta^4\, \lambda^2\, \mu^2\, \hbar^2 - 18\, I\mu^2\, n\, Z\Delta^4\, \lambda^2\, \mu^2\, \hbar^2 + 4\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \delta\lambda^2\, \lambda^2\, \hbar^3 +$
$2\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \delta\lambda^2\, \lambda^2\, \hbar^3 + 8\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \delta\lambda\, \lambda^3\, \hbar^3 + 4\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \delta\lambda\, \lambda^3\, \hbar^3 +$
$4\, i\, c\mu\, I\mu\, v^2\, Z\Delta^4\, \lambda^4\, \hbar^3 + 2\, i\, c\mu\, I\mu\, n\, v^2\, Z\Delta^4\, \lambda^4\, \hbar^3 - 54\, \mathrm{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2a} +$

$54\, n\, \text{tfing}\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,a} + 54\, \text{tfinn}\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,a} + 54\, n\, t\mu\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,a} - 18\, \mathbb{i}\, I\mu\, v^2\, Z\triangle^3\, \lambda\, \hbar\, \delta\lambda_{2\,a} +$

$18\, \mathbb{i}\, I\mu\, n\, v^2\, Z\triangle^3\, \lambda\, \hbar\, \delta\lambda_{2\,a} + 54\, \mathbb{i}\, I\mu\, n\, Z\triangle^2\, \mu^2\, \hbar\, \delta\lambda_{2\,a} + 6\, c\mu\, n\, v^2\, Z\triangle^2\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$3\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 6\, \text{Ifingp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$27\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 18\, I\mu^2\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$6\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 6\, \text{Ifingp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$27\, \text{Ifinhhp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 15\, I\mu^2\, n\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$3\, \text{Ifinggp}\, I\mu\, n^2\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, I\mu^2\, n^2\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$18\, \mathbb{i}\, I\mu\, n\, \text{tfinn}\, Z\triangle^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 18\, \mathbb{i}\, I\mu\, n\, t\mu\, Z\triangle^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 12\, c\mu\, n\, v^2\, Z\triangle^2\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} +$

$6\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 12\, \text{Ifingp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} -$

$54\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} - 36\, I\mu^2\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} -$

$12\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} - 12\, \text{Ifingp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} +$

$54\, \text{Ifinhhp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 30\, I\mu^2\, n\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} +$

$6\, \text{Ifinggp}\, I\mu\, n^2\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} + 6\, I\mu^2\, n^2\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,a} +$

$6\, \text{Ifing0}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \text{Ifinggn}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$3\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 6\, \text{Ifingp}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$27\, \text{Ifinhhn}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 27\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$6\, c\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 6\, \text{Ifing0}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$6\, \text{Ifinggn}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 6\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$6\, \text{Ifingp}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 27\, \text{Ifinhhn}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$27\, \text{Ifinhhp}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \text{Ifinggn}\, I\mu\, n^2\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$3\, \text{Ifinggp}\, I\mu\, n^2\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$6\, \text{Ifingp}\, I\mu\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 27\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$18\, I\mu^2\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 6\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} -$

$6\, \text{Ifingp}\, I\mu\, n\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 27\, \text{Ifinhhp}\, I\mu\, n\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$15\, I\mu^2\, n\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} + 3\, \text{Ifinggp}\, I\mu\, n^2\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} +$

$3\, I\mu^2\, n^2\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,a} - 18\, I\mu^2\, n\, Z\triangle^4\, \lambda\, \mu^2\, \hbar^2\, \delta\lambda_{2\,a} + 2\, \mathbb{i}\, c\mu\, I\mu\, n\, v^2\, Z\triangle^4\, \delta\lambda^2\, \lambda\, \hbar^3\, \delta\lambda_{2\,a} +$

$4\, \mathbb{i}\, c\mu\, I\mu\, n\, v^2\, Z\triangle^4\, \delta\lambda\, \lambda^2\, \hbar^3\, \delta\lambda_{2\,a} + 2\, \mathbb{i}\, c\mu\, I\mu\, n\, v^2\, Z\triangle^4\, \lambda^3\, \hbar^3\, \delta\lambda_{2\,a} + 108\, v^2\, Z\triangle\, \delta\lambda_b +$

$18\, \mathbb{i}\, I\mu\, v^2\, Z\triangle^3\, \lambda\, \hbar\, \delta\lambda_b + 18\, \mathbb{i}\, I\mu\, n\, v^2\, Z\triangle^3\, \lambda\, \hbar\, \delta\lambda_b - 18\, \mathbb{i}\, I\mu\, v^2\, Z\triangle^3\, \hbar\, \delta\lambda_{2\,a}\, \delta\lambda_b +$

$18\, \mathbb{i}\, I\mu\, n\, v^2\, Z\triangle^3\, \hbar\, \delta\lambda_{2\,a}\, \delta\lambda_b + 108\, \mathbb{i}\, I\mu\, m^2\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,b} + 108\, \mathbb{i}\, I\mu\, p^2\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,b} +$

$108\, \text{tfinn}\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,b} + 108\, t\mu\, Z\triangle^2\, \hbar\, \delta\lambda_{2\,b} - 108\, \mathbb{i}\, I\mu\, p^2\, Z\, Z\triangle^3\, \hbar\, \delta\lambda_{2\,b} +$

$54\, \mathbb{i}\, I\mu\, v^2\, Z\triangle^3\, \lambda\, \hbar\, \delta\lambda_{2\,b} + 108\, \mathbb{i}\, I\mu\, Z\triangle^2\, \mu^2\, \hbar\, \delta\lambda_{2\,b} + 12\, c\mu\, v^2\, Z\triangle^2\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} -$

$6\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 54\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} +$

$48\, I\mu^2\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 6\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 6\, I\mu^2\, n\, v^2\, Z\triangle^5\, \delta\lambda^2\, \hbar^2\, \delta\lambda_{2\,b} +$

$72\, \mathbb{i}\, I\mu\, \text{tfinn}\, Z\triangle^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 18\, \mathbb{i}\, I\mu\, n\, \text{tfinn}\, Z\triangle^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 72\, \mathbb{i}\, I\mu\, t\mu\, Z\triangle^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} +$

$18\, \mathbb{i}\, I\mu\, n\, t\mu\, Z\triangle^4\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 24\, c\mu\, v^2\, Z\triangle^2\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} - 12\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} +$

$108\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 96\, I\mu^2\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} +$

$12\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} + 12\, I\mu^2\, n\, v^2\, Z\triangle^5\, \delta\lambda\, \lambda\, \hbar^2\, \delta\lambda_{2\,b} +$

$12\, c\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} - 6\, \text{Ifinggn}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 6\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} +$

$54\, \text{Ifinhhn}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} - 54\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} +$

$6\, \text{Ifinggn}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} - 6\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^2\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} -$

$6\, \text{Ifinggp}\, I\mu\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 54\, \text{Ifinhhp}\, I\mu\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} +$

$48\, I\mu^2\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 6\, \text{Ifinggp}\, I\mu\, n\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} + 6\, I\mu^2\, n\, v^2\, Z\triangle^5\, \lambda^2\, \hbar^2\, \delta\lambda_{2\,b} -$

$72\, I\mu^2\, Z\triangle^4\, \lambda\, \mu^2\, \hbar^2\, \delta\lambda_{2\,b} - 18\, I\mu^2\, n\, Z\triangle^4\, \lambda\, \mu^2\, \hbar^2\, \delta\lambda_{2\,b} + 8\, \mathbb{i}\, c\mu\, I\mu\, v^2\, Z\triangle^4\, \delta\lambda^2\, \lambda\, \hbar^3\, \delta\lambda_{2\,b} +$

$$2\,\mathrm{i}\,c\mu\,I\mu\,n\,v^2\,Z\Delta^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\delta\lambda_{2\,b} + 16\,\mathrm{i}\,c\mu\,I\mu\,v^2\,Z\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,b} +$$
$$4\,\mathrm{i}\,c\mu\,I\mu\,n\,v^2\,Z\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,b} + 8\,\mathrm{i}\,c\mu\,I\mu\,v^2\,Z\Delta^4\,\lambda^3\,\hbar^3\,\delta\lambda_{2\,b} + 2\,\mathrm{i}\,c\mu\,I\mu\,n\,v^2\,Z\Delta^4\,\lambda^3\,\hbar^3\,\delta\lambda_{2\,b} +$$
$$18\,\mathrm{i}\,I\mu\,n\,\mathrm{tfinn}\,Z\Delta^4\,\hbar^2\,\delta\lambda_{2\,a}\,\delta\lambda_{2\,b} + 18\,\mathrm{i}\,I\mu\,n\,t\mu\,Z\Delta^4\,\hbar^2\,\delta\lambda_{2\,a}\,\delta\lambda_{2\,b} -$$
$$18\,I\mu^2\,n\,Z\Delta^4\,\mu^2\,\hbar^2\,\delta\lambda_{2\,a}\,\delta\lambda_{2\,b} + 2\,\mathrm{i}\,c\mu\,I\mu\,n\,v^2\,Z\Delta^4\,\delta\lambda^2\,\hbar^3\,\delta\lambda_{2\,a}\,\delta\lambda_{2\,b} +$$
$$4\,\mathrm{i}\,c\mu\,I\mu\,n\,v^2\,Z\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\delta\lambda_{2\,a}\,\delta\lambda_{2\,b} + 2\,\mathrm{i}\,c\mu\,I\mu\,n\,v^2\,Z\Delta^4\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,a}\,\delta\lambda_{2\,b} +$$
$$36\,\mathrm{i}\,I\mu\,v^2\,Z\Delta^3\,\hbar\,\delta\lambda_b\,\delta\lambda_{2\,b} + 36\,\mathrm{i}\,I\mu\,\mathrm{tfinn}\,Z\Delta^4\,\hbar^2\,\delta\lambda_{2\,b}^2 + 36\,\mathrm{i}\,I\mu\,t\mu\,Z\Delta^4\,\hbar^2\,\delta\lambda_{2\,b}^2 -$$
$$36\,I\mu^2\,Z\Delta^4\,\mu^2\,\hbar^2\,\delta\lambda_{2\,b}^2 + 4\,\mathrm{i}\,c\mu\,I\mu\,v^2\,Z\Delta^4\,\delta\lambda^2\,\hbar^3\,\delta\lambda_{2\,b}^2 + 8\,\mathrm{i}\,c\mu\,I\mu\,v^2\,Z\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\delta\lambda_{2\,b}^2 +$$
$$4\,\mathrm{i}\,c\mu\,I\mu\,v^2\,Z\Delta^4\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,b}^2 + 108\,\mathrm{i}\,\delta m_1^2\,\left(-3\,\mathrm{i} + I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,b}\right) +$$
$$18\,\mathrm{i}\,v^2\,Z\Delta\,\delta\lambda_a\,\left(-3\,\mathrm{i} + I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,b}\right)\Big)\Big) \Big/ \Big(18\,\left(-3\,\mathrm{i} + I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,b}\right)$$
$$\left(-6\,\mathrm{i} + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,b}\right)\Big)\Big)\Big\}$$

Check solutions

$$\left(p^2 - \mathrm{mg2} + \mathrm{i}\,\hbar\,\left(\frac{(\lambda)\,v}{3}\right)^2\,(\mathrm{Ifingp} - \mathrm{Ifing0}) - \left(Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\,\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\,Z\Delta^2\,(\mathrm{tg}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\mathrm{tn}) + \mathrm{i}\,\hbar\,\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\,Z\Delta^3\,\mathrm{Ing}\right)\right)\ /.$$

```
intrules /. mn2soln /. mg2soln /. msbarrules // Simplify
```

```
0
```

$$\left(p^2 - \mathrm{mn2} + \frac{\mathrm{i}\,\hbar}{2}\,\left(\frac{(\lambda)\,v}{3}\right)^2\,(n-1)\,(\mathrm{Ifinggp} - \mathrm{Ifinggn}) + \frac{\mathrm{i}\,\hbar}{2}\,(\lambda)^2\,v^2\,(\mathrm{Ifinhhp} - \mathrm{Ifinhhn}) - \left(Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{3\,\lambda + \delta\lambda_{1\,a} + 2\,\delta\lambda_{1\,b}}{6}\,v^2 - \frac{\hbar}{6}\,\left(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\,Z\Delta^2\,\mathrm{tn} - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(n-1)\,\mathrm{tg} + \frac{\mathrm{i}\,\hbar}{2}\,\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\,Z\Delta^3\,(n-1)\,\mathrm{Igg} + \frac{\mathrm{i}\,\hbar}{2}\,(\lambda+\delta\lambda)^2\,v^2\,Z\Delta^3\,\mathrm{Ihh}\right)\right)\ /.$$

```
intrules /. mn2soln /. mg2soln /. msbarrules // Simplify
```

$$-\frac{1}{2}\,\mathrm{i}\,(\mathrm{Ifinhhmn} - \mathrm{Ifinhhn})\,v^2\,\lambda^2\,\hbar$$

Gather kinematically distinct divergences for Goldstone EOM

$$\left(\left(p^2 - \mathrm{mg2} + \mathrm{i}\,\hbar\,\left(\frac{(\lambda)\,v}{3}\right)^2\,(\mathrm{Ifingp} - \mathrm{Ifing0}) - \left(p^2 - m^2 - \frac{\lambda}{6}\,v^2 - \frac{\hbar}{6}\,\left((n+1)\,\lambda\right)\,(\mathrm{tfing}) - \frac{\hbar}{6}\,(\lambda)\,(\mathrm{tfinn}) + \mathrm{i}\,\hbar\,\left(\frac{(\lambda)\,v}{3}\right)^2\,(\mathrm{Ifingp})\right)\right)\ /.\ \mathrm{intrules}\ /.\ \mathrm{mn2soln}\ /.\ \mathrm{mg2soln}\ /.\ \mathrm{msbarrules}\ //\ \mathrm{Simplify}\right)\ //$$

```
CoefficientList[#, {p, v, tfing, tfinn, Ifingp, Ifinggp, Ifinhhp}] & //
Flatten // Simplify // DeleteDuplicates
```

$$\Big\{\Big(\mathrm{i}\,\Big(6\,\delta m_1^2 + Z\Delta^2\,\hbar\,\Big(c0\,\Lambda^2 + \left(c1\,\mu^2 - \mathrm{i}\,c2\,(m^2 - \mu^2)\right)\,\mathrm{Log}\!\left[\frac{\Lambda^2}{\mu^2}\right]\Big)\,\left((2+n)\,\lambda + n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\Big)\Big)\Big/$$

$$\left(-6\,\mathbb{i}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$\left.\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right),\ 0,$$

$$\left(3\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}\right)\Big/\left(\left(\left(-3\,\mathbb{i}+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right)\right.\right.$$

$$\left(-6\,\mathbb{i}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$\left.\left.\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right)\right)+\lambda\,\hbar\left(\frac{1}{6}+\left(3\,\mathrm{Z}\Delta^2\right)\Big/\right.$$

$$\left(\left(-3\,\mathbb{i}+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right)\left(-6\,\mathbb{i}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.$$

$$\left.\left.\left.\left.\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right)\right)\right)\right),$$

$$\left(\hbar\left(\lambda\left(18\,(1+\mathrm{n})\left(-1+\mathrm{Z}\Delta^2\right)+3\,\mathbb{i}\,\mathrm{c2}\,\mathrm{Z}\Delta^2\left(-4-5\,\mathrm{n}-\mathrm{n}^2+4\,\mathrm{Z}\Delta^2+2\,\mathrm{n}\,\mathrm{Z}\Delta^2\right)\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.\right.$$

$$\left.\mathrm{c2}^2\left(2+3\,\mathrm{n}+\mathrm{n}^2\right)\mathrm{Z}\Delta^4\,\lambda^2\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right)+\mathrm{Z}\Delta^2\left(36+6\,\mathbb{i}\,\mathrm{c2}\left(-2+4\,\mathrm{Z}\Delta^2+\mathrm{n}\left(-2+\mathrm{Z}\Delta^2\right)\right)\right.$$

$$\left.\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}^2\left(4+5\,\mathrm{n}+\mathrm{n}^2\right)\mathrm{Z}\Delta^2\,\lambda^2\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right)\delta\lambda_{2\,\mathrm{b}}+$$

$$2\,\mathrm{c2}\,\mathrm{Z}\Delta^4\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(6\,\mathbb{i}+\mathrm{c2}\,(1+\mathrm{n})\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,\mathrm{b}}^2+\mathrm{Z}\Delta^2\,\delta\lambda_{2\,\mathrm{a}}$$

$$\left(18\,(-1+\mathrm{n})-3\,\mathbb{i}\,\mathrm{c2}\,\mathrm{n}\left(1+\mathrm{n}-2\,\mathrm{Z}\Delta^2\right)\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}^2\,\mathrm{n}\,(1+\mathrm{n})\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+\right.$$

$$\left.\left.\left.\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(6\,\mathbb{i}+\mathrm{c2}\,(1+\mathrm{n})\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,\mathrm{b}}\right)\right)\right)\Big/$$

$$\left(6\left(-3\,\mathbb{i}+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right)\left(-6\,\mathbb{i}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.$$

$$\left.\left.\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\mathrm{b}}\right)\right),$$

$$\left(-54\,\lambda+54\,\mathrm{Z}\Delta\,\lambda-36\,\mathbb{i}\,\mathrm{c2}\,\mathrm{Z}\Delta^3\,\delta\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-72\,\mathbb{i}\,\mathrm{c2}\,\mathrm{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-\right.$$

$$36\,\mathbb{i}\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-9\,\mathbb{i}\,\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-36\,\mathbb{i}\,\mathrm{c2}\,\mathrm{Z}\Delta^3\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$12\,\mathrm{a1}\,\mathrm{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+6\,\mathrm{a1}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$24\,\mathrm{a1}\,\mathrm{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+12\,\mathrm{a1}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$12\,\mathrm{a1}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-6\,\mathrm{c2}\,\mathtt{Ifing0}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$3\,\mathrm{c2}\,\mathtt{Ifinggn}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-27\,\mathrm{c2}\,\mathtt{Ifinhhn}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$6\,\mathrm{a1}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-6\,\mathrm{c2}\,\mathtt{Ifing0}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$

$$3\,\text{c2}\,\text{Ifinggn}\,n\,Z\triangle^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+12\,\text{a0}\,Z\triangle^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$6\,\text{a0}\,n\,Z\triangle^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-6\,\text{c2}^2\,Z\triangle^5\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$

$$3\,\text{c2}^2\,n\,Z\triangle^5\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+24\,\text{a0}\,Z\triangle^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$12\,\text{a0}\,n\,Z\triangle^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-12\,\text{c2}^2\,Z\triangle^5\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$

$$6\,\text{c2}^2\,n\,Z\triangle^5\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+12\,\text{a0}\,Z\triangle^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+6\,\text{a0}\,n\,Z\triangle^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$6\,\text{c2}^2\,Z\triangle^4\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+3\,\text{c2}^2\,n\,Z\triangle^4\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-6\,\text{c2}^2\,Z\triangle^5\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$

$$3\,\text{c2}^2\,n\,Z\triangle^5\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+4\,i\,\text{a1}\,\text{c2}\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$2\,i\,\text{a1}\,\text{c2}\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+8\,i\,\text{a1}\,\text{c2}\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$4\,i\,\text{a1}\,\text{c2}\,n\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+4\,i\,\text{a1}\,\text{c2}\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$

$$4\,i\,\text{c2}^2\,\text{Ifing0}\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+2\,i\,\text{a1}\,\text{c2}\,n\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$

$$2\,i\,\text{c2}^2\,\text{Ifing0}\,n\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+4\,i\,\text{a0}\,\text{c2}\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3+$$

$$2\,i\,\text{a0}\,\text{c2}\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3+8\,i\,\text{a0}\,\text{c2}\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3+$$

$$4\,i\,\text{a0}\,\text{c2}\,n\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3+4\,i\,\text{a0}\,\text{c2}\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3+$$

$$2\,i\,\text{a0}\,\text{c2}\,n\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3-18\,i\,\text{c2}\,Z\triangle^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{\text{b}}-$$

$$36\,i\,\text{c2}\,Z\triangle^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}}+18\,i\,\text{c2}\,Z\triangle^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}}+$$

$$12\,\text{a1}\,Z\triangle^2\,\delta\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}}+24\,\text{a1}\,Z\triangle^2\,\delta\lambda\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}}+$$

$$12\,\text{a1}\,Z\triangle^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}}-12\,\text{c2}\,\text{Ifing0}\,Z\triangle^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}}+$$

$$12\,\text{a0}\,Z\triangle^2\,\delta\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+12\,\text{c2}^2\,Z\triangle^5\,\delta\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+$$

$$24\,\text{a0}\,Z\triangle^2\,\delta\lambda\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+24\,\text{c2}^2\,Z\triangle^5\,\delta\lambda\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+$$

$$12\,\text{a0}\,Z\triangle^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+12\,\text{c2}^2\,Z\triangle^4\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+$$

$$3\,\text{c2}^2\,n\,Z\triangle^4\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+12\,\text{c2}^2\,Z\triangle^5\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+$$

$$8\,i\,\text{a1}\,\text{c2}\,Z\triangle^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+2\,i\,\text{a1}\,\text{c2}\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}+$$

$$16\,\mathrm{i}\,\text{a1}\,\text{c2}\,Z\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}+4\,\mathrm{i}\,\text{a1}\,\text{c2}\,n\,Z\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}+$$

$$8\,\mathrm{i}\,\text{a1}\,\text{c2}\,Z\Delta^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}-8\,\mathrm{i}\,\text{c2}^2\,\text{Ifing0}\,Z\Delta^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}+$$

$$2\,\mathrm{i}\,\text{a1}\,\text{c2}\,n\,Z\Delta^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}-2\,\mathrm{i}\,\text{c2}^2\,\text{Ifing0}\,n\,Z\Delta^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}+$$

$$8\,\mathrm{i}\,\text{a0}\,\text{c2}\,Z\Delta^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}+2\,\mathrm{i}\,\text{a0}\,\text{c2}\,n\,Z\Delta^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}+$$

$$16\,\mathrm{i}\,\text{a0}\,\text{c2}\,Z\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}+4\,\mathrm{i}\,\text{a0}\,\text{c2}\,n\,Z\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}+$$

$$8\,\mathrm{i}\,\text{a0}\,\text{c2}\,Z\Delta^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}+2\,\mathrm{i}\,\text{a0}\,\text{c2}\,n\,Z\Delta^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}+$$

$$6\,\text{c2}^2\,Z\Delta^4\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+4\,\mathrm{i}\,\text{a1}\,\text{c2}\,Z\Delta^4\,\delta\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+$$

$$8\,\mathrm{i}\,\text{a1}\,\text{c2}\,Z\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+4\,\mathrm{i}\,\text{a1}\,\text{c2}\,Z\Delta^4\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2-$$

$$4\,\mathrm{i}\,\text{c2}^2\,\text{Ifing0}\,Z\Delta^4\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+4\,\mathrm{i}\,\text{a0}\,\text{c2}\,Z\Delta^4\,\delta\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}^2+$$

$$8\,\mathrm{i}\,\text{a0}\,\text{c2}\,Z\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}^2+4\,\mathrm{i}\,\text{a0}\,\text{c2}\,Z\Delta^4\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}^2+$$

$$18\,Z\Delta\,\delta\lambda_a\left(3+\mathrm{i}\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\mathrm{i}\,\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)+Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]$$

$$\delta\lambda_{2\,a}\left(-9\,\mathrm{i}\,\text{c2}\,n\,\lambda-18\,\mathrm{i}\,\text{c2}\,Z\Delta\,\lambda+6\,\text{a1}\,n\,\delta\lambda^2\,\hbar+12\,\text{a1}\,n\,\delta\lambda\,\lambda\,\hbar+6\,\text{c2}\,\text{Ifing0}\,\lambda^2\,\hbar+\right.$$

$$3\,\text{c2}\,\text{Ifinggn}\,\lambda^2\,\hbar-27\,\text{c2}\,\text{Ifinhhn}\,\lambda^2\,\hbar+6\,\text{a1}\,n\,\lambda^2\,\hbar-6\,\text{c2}\,\text{Ifing0}\,n\,\lambda^2\,\hbar-$$

$$3\,\text{c2}\,\text{Ifinggn}\,n\,\lambda^2\,\hbar+6\,\text{a0}\,n\,\delta\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-18\,\text{c2}^2\,Z\Delta^3\,\delta\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$

$$3\,\text{c2}^2\,n\,Z\Delta^3\,\delta\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+12\,\text{a0}\,n\,\delta\lambda\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-36\,\text{c2}^2\,Z\Delta^3\,\delta\lambda\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$

$$6\,\text{c2}^2\,n\,Z\Delta^3\,\delta\lambda\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+6\,\text{a0}\,n\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+3\,\text{c2}^2\,n\,Z\Delta^2\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$

$$18\,\text{c2}^2\,Z\Delta^3\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-3\,\text{c2}^2\,n\,Z\Delta^3\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+2\,\mathrm{i}\,\text{a1}\,\text{c2}\,n\,Z\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$4\,\mathrm{i}\,\text{a1}\,\text{c2}\,n\,Z\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+2\,\mathrm{i}\,\text{a1}\,\text{c2}\,n\,Z\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-2\,\mathrm{i}\,\text{c2}^2\,\text{Ifing0}\,n\,Z\Delta^2$$

$$\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+2\,\mathrm{i}\,\text{a0}\,\text{c2}\,n\,Z\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+4\,\mathrm{i}\,\text{a0}\,\text{c2}\,n\,Z\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$2\,\mathrm{i}\,\text{a0}\,\text{c2}\,n\,Z\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-18\,\mathrm{i}\,\text{c2}\,Z\Delta\,\delta\lambda_b+\mathrm{i}\,\text{c2}\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]$$

$$\left.\left(2\,\text{a1}\,(\delta\lambda+\lambda)^2\,\hbar-\text{c2}\,\lambda\,\left(3\,\mathrm{i}+2\,\text{Ifing0}\,\lambda\,\hbar\right)+2\,\text{a0}\,(\delta\lambda+\lambda)^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,b}\right)\Big)\Big/$$

$$\left(18\left(-3\,\mathrm{i}+\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\left(-6\,\mathrm{i}+2\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.$$

$$\left.\left.\text{c2}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a}+2\,\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right),$$

$$-\left(\left(3\,\text{c2}\,Z\Delta^2\left(-\lambda^2+Z\Delta^3(\delta\lambda+\lambda)^2\right)\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\lambda+\delta\lambda_{2\text{a}})\right)\Big/\left(2\left(-3\,\mathbb{i}+\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{a}}+2\,\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\right)\right),$$

$$-\left(\left(\text{c2}\,(-1+n)\,Z\Delta^2\left(-\lambda^2+Z\Delta^3(\delta\lambda+\lambda)^2\right)\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\lambda+\delta\lambda_{2\text{a}})\right)\Big/\left(6\left(-3\,\mathbb{i}+\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{a}}+2\,\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\right)\right),$$

$$\left(\left(-\lambda^2+Z\Delta^3(\delta\lambda+\lambda)^2\right)\hbar\left(-6\,\mathbb{i}+3\,\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{a}}+2\,\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\right)\Big/\left(3\left(-3\,\mathbb{i}+\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{a}}+2\,\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\right),$$

$$-\left(\left(6\,\mathbb{i}\,(-1+Z\,Z\Delta)\right)\Big/\left(-6\,\mathbb{i}+2\,\text{c2}\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\lambda\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{a}}+2\,\text{c2}\,Z\Delta^2\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\text{b}}\right)\right)\}$$

```
cteq = ((mg2 - m^2 - λ/6 v^2 - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) //
    CoefficientList[#, {p, v, tfing, tfinn, Ifingp, Ifinggp, Ifinhhp}] & //
    Flatten // Simplify // DeleteDuplicates) == 0 // Thread
```

$$\Big\{\left(-6\,\mathbb{i}\,\delta m_1^2+Z\Delta^2\left(-\mathbb{i}\,t\mu+\text{I}\mu\left(-m^2+\mu^2\right)\right)\hbar\left((2+n)\lambda+n\,\delta\lambda_{2\text{a}}+2\,\delta\lambda_{2\text{b}}\right)\right)\Big/\left(-6\,\mathbb{i}+2\,\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2\text{a}}+2\,\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)=0,$$

$$\text{True},\ -\left(\left(3\,Z\Delta^2\hbar\,\delta\lambda_{2\text{a}}\right)\Big/\left(\left(-3\,\mathbb{i}+\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2\text{a}}+2\,\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)\right)\right)+\lambda\hbar\left(-\frac{1}{6}-\left(3\,Z\Delta^2\right)\Big/\left(\left(-3\,\mathbb{i}+\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2\text{a}}+2\,\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)\right)\right)=0,$$

$$-\Big(\Big(\hbar\Big(\lambda\left(\text{I}\mu\,n^2Z\Delta^2\lambda\hbar\left(-3\,\mathbb{i}+\text{I}\mu\,Z\Delta^2\lambda\hbar\right)+3\,n\left(-6+Z\Delta^2\left(6-5\,\mathbb{i}\,\text{I}\mu\,\lambda\hbar\right)+\text{I}\mu\,Z\Delta^4\lambda\hbar\left(2\,\mathbb{i}+\text{I}\mu\,\lambda\hbar\right)\right)+2\left(-9+Z\Delta^2\left(9-6\,\mathbb{i}\,\text{I}\mu\,\lambda\hbar\right)+\text{I}\mu\,Z\Delta^4\lambda\hbar\left(6\,\mathbb{i}+\text{I}\mu\,\lambda\hbar\right)\right)\right)+Z\Delta^2\left(36+6\,\mathbb{i}\,\text{I}\mu\left(-2+4\,Z\Delta^2+n\left(-2+Z\Delta^2\right)\right)\lambda\hbar+\text{I}\mu^2\left(4+5\,n+n^2\right)Z\Delta^2\lambda^2\hbar^2\right)\delta\lambda_{2\text{b}}+2\,\text{I}\mu\,Z\Delta^4\hbar\left(6\,\mathbb{i}+\text{I}\mu\,(1+n)\lambda\hbar\right)\delta\lambda_{2\text{b}}^2+Z\Delta^2\delta\lambda_{2\text{a}}\left(-18+\text{I}\mu\,n^2\lambda\hbar\left(-3\,\mathbb{i}+\text{I}\mu\,Z\Delta^2\lambda\hbar\right)+n\left(18+3\,\mathbb{i}\,\text{I}\mu\left(-1+2\,Z\Delta^2\right)\lambda\hbar+\text{I}\mu^2Z\Delta^2\lambda^2\hbar^2\right)+\text{I}\mu\,n\,Z\Delta^2\hbar\left(6\,\mathbb{i}+\text{I}\mu\,(1+n)\lambda\hbar\right)\delta\lambda_{2\text{b}}\right)\Big)\Big)\Big/\left(6\left(-3\,\mathbb{i}+\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{I}\mu\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\lambda\hbar+\text{I}\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2\text{a}}+2\,\text{I}\mu\,Z\Delta^2\hbar\,\delta\lambda_{2\text{b}}\right)\right)\Big)=0,$$

$$-\Big(\Big(-54\,\lambda+54\,\mathrm{Z}\Delta\,\lambda-36\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^3\,\delta\lambda^2\,\hbar-72\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar-36\,\mathrm{i}\,\mathrm{Ifing0}\,\lambda^2\,\hbar-$$
$$36\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar-9\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar-36\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^3\,\lambda^2\,\hbar+12\,\mathrm{c}\mu\,\mathrm{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2+$$
$$6\,\mathrm{c}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2-6\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^5\,\delta\lambda^2\,\lambda\,\hbar^2-3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^5\,\delta\lambda^2\,\lambda\,\hbar^2+24\,\mathrm{c}\mu\,\mathrm{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2+$$
$$12\,\mathrm{c}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2-12\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^5\,\delta\lambda\,\lambda^2\,\hbar^2-6\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^5\,\delta\lambda\,\lambda^2\,\hbar^2+12\,\mathrm{c}\mu\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2+$$
$$18\,\mathrm{Ifing0}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2+3\,\mathrm{Ifinggn}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2-27\,\mathrm{Ifinhhn}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2+$$
$$6\,\mathrm{c}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2-3\,\mathrm{Ifinggn}\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2+6\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^4\,\lambda^3\,\hbar^2+3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\lambda^3\,\hbar^2-$$
$$6\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^5\,\lambda^3\,\hbar^2-3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^5\,\lambda^3\,\hbar^2+4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\delta\lambda^2\,\lambda^2\,\hbar^3+2\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\delta\lambda^2\,\lambda^2\,\hbar^3+$$
$$8\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\delta\lambda\,\lambda^3\,\hbar^3+4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\delta\lambda\,\lambda^3\,\hbar^3+4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\lambda^4\,\hbar^3+$$
$$2\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\lambda^4\,\hbar^3-18\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^3\,\lambda\,\hbar\,\delta\lambda_{\mathrm{b}}-36\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\delta\lambda_{2\,\mathrm{b}}+18\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^3\,\lambda\,\hbar\,\delta\lambda_{2\,\mathrm{b}}+$$
$$12\,\mathrm{c}\mu\,\mathrm{Z}\Delta^2\,\delta\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+12\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^5\,\delta\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+24\,\mathrm{c}\mu\,\mathrm{Z}\Delta^2\,\delta\lambda\,\lambda\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+$$
$$24\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^5\,\delta\lambda\,\lambda\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+12\,\mathrm{c}\mu\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+12\,\mathrm{Ifing0}\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+$$
$$12\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^4\,\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+12\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^5\,\lambda^2\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}+$$
$$8\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}+2\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}+16\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}+$$
$$4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}+8\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\lambda^3\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}+2\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^4\,\lambda^3\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}+$$
$$6\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^4\,\lambda\,\hbar^2\,\delta\lambda_{2\,\mathrm{b}}^2+4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\delta\lambda^2\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}^2+8\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}^2+$$
$$4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{Z}\Delta^4\,\lambda^2\,\hbar^3\,\delta\lambda_{2\,\mathrm{b}}^2+18\,\mathrm{i}\,\mathrm{Z}\Delta\,\delta\lambda_{\mathrm{a}}\left(-3\,\mathrm{i}+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)-$$
$$\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}\Big(9\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{n}\,\lambda+18\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta\,\lambda-6\,\mathrm{c}\mu\,\mathrm{n}\,\delta\lambda^2\,\hbar+18\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^3\,\delta\lambda^2\,\hbar+3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^3\,\delta\lambda^2\,\hbar-$$
$$12\,\mathrm{c}\mu\,\mathrm{n}\,\delta\lambda\,\lambda\,\hbar+36\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar+6\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar-6\,\mathrm{Ifing0}\,\mathrm{I}\mu\,\lambda^2\,\hbar-3\,\mathrm{Ifinggn}$$
$$\mathrm{I}\mu\,\lambda^2\,\hbar+27\,\mathrm{Ifinhhn}\,\mathrm{I}\mu\,\lambda^2\,\hbar-6\,\mathrm{c}\mu\,\mathrm{n}\,\lambda^2\,\hbar+3\,\mathrm{Ifinggn}\,\mathrm{I}\mu\,\mathrm{n}\,\lambda^2\,\hbar-3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^2\,\hbar+$$
$$18\,\mathrm{I}\mu^2\,\mathrm{Z}\Delta^3\,\lambda^2\,\hbar+3\,\mathrm{I}\mu^2\,\mathrm{n}\,\mathrm{Z}\Delta^3\,\lambda^2\,\hbar-2\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2-4\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2-$$
$$2\,\mathrm{i}\,\mathrm{c}\mu\,\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda^3\,\hbar^2+18\,\mathrm{i}\,\mathrm{I}\mu\,\mathrm{Z}\Delta\,\delta\lambda_{\mathrm{b}}-\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\left(3\,\mathrm{I}\mu\,\lambda+2\,\mathrm{i}\,\mathrm{c}\mu\,(\delta\lambda+\lambda)^2\,\hbar\right)\delta\lambda_{2\,\mathrm{b}}\Big)\Big)\Big/$$
$$\Big(18\left(-3\,\mathrm{i}+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)\Big(-6\,\mathrm{i}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+$$
$$\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\Big)\Big)\Big)=0,$$

$$\left(3\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\left(-\lambda^2+\mathrm{Z}\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar^2\,(\lambda+\delta\lambda_{2\,\mathrm{a}})\right)\Big/\Big(2\left(-3\,\mathrm{i}+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)$$
$$\left(-6\,\mathrm{i}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)\Big)=0,$$

$$\left(\mathrm{I}\mu\,(-1+\mathrm{n})\,\mathrm{Z}\Delta^2\left(-\lambda^2+\mathrm{Z}\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar^2\,(\lambda+\delta\lambda_{2\,\mathrm{a}})\right)\Big/$$
$$\Big(6\left(-3\,\mathrm{i}+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)$$
$$\left(-6\,\mathrm{i}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)\Big)=0,$$

$$-\Big(\Big(\left(-\lambda^2+\mathrm{Z}\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar\left(-6\,\mathrm{i}+3\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)\Big)\Big/$$
$$\Big(3\left(-3\,\mathrm{i}+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)$$
$$\left(-6\,\mathrm{i}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)\Big)\Big)=0,$$

$$\left(6\,\mathrm{i}\,(-1+\mathrm{Z}\,\mathrm{Z}\Delta)\right)\Big/\left(-6\,\mathrm{i}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar+\mathrm{I}\mu\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{I}\mu\,\mathrm{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\mathrm{b}}\right)=$$
$$0\}$$

```
cteq = (cteq /. msbarrules // Simplify // DeleteDuplicates)
```

$$\Big\{-\Big(\Big(\mathrm{i}\left(6\,\delta\mathrm{m}_1^2+\mathrm{Z}\Delta^2\,\hbar\left(\mathrm{c0}\,\Lambda^2+\left(\mathrm{c1}\,\mu^2-\mathrm{i}\,\mathrm{c2}\left(\mathrm{m}^2-\mu^2\right)\right)\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\right)\left((2+\mathrm{n})\,\lambda+\mathrm{n}\,\delta\lambda_{2\,\mathrm{a}}+2\,\delta\lambda_{2\,\mathrm{b}}\right)\right)\Big)\Big/$$
$$\Big(-6\,\mathrm{i}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]+\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]+$$
$$\mathrm{c2}\,\mathrm{n}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,\mathrm{a}}+2\,\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,\mathrm{b}}\Big)\Big)=0,\ \mathrm{True},$$

$$\lambda\,\hbar\left(-\frac{1}{6}-\left(3\,\mathrm{Z}\Delta^2\right)\Big/\Big(\Big(-3\,\mathrm{i}+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]+\mathrm{c2}\,\mathrm{Z}\Delta^2\,\hbar\,\mathrm{Log}\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,\mathrm{b}}\Big)\right.$$

$$\left(-6\,\mathbb{i}+2\,\text{c2}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$\left.\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{a}}+2\,\text{c2}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\Bigg)\Bigg)==$$

$$\left(3\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}}\right)\Big/\left(\left(-3\,\mathbb{i}+\text{c2}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\right.$$

$$\left(-6\,\mathbb{i}+2\,\text{c2}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$\left.\left.\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{a}}+2\,\text{c2}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\right),$$

$$\left(\hbar\left(\lambda\left(18\,(1+\text{n})\,\left(-1+\text{Z}\Delta^2\right)+3\,\mathbb{i}\,\text{c2}\,\text{Z}\Delta^2\,\left(-4-5\,\text{n}-\text{n}^2+4\,\text{Z}\Delta^2+2\,\text{n}\,\text{Z}\Delta^2\right)\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.\right.$$

$$\left.\text{c2}^2\,\left(2+3\,\text{n}+\text{n}^2\right)\text{Z}\Delta^4\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right)+\text{Z}\Delta^2\,\left(36+6\,\mathbb{i}\,\text{c2}\,\left(-2+4\,\text{Z}\Delta^2+\text{n}\,\left(-2+\text{Z}\Delta^2\right)\right)\right.$$

$$\left.\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}^2\,\left(4+5\,\text{n}+\text{n}^2\right)\text{Z}\Delta^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right)\delta\lambda_{2\,\text{b}}+$$

$$2\,\text{c2}\,\text{Z}\Delta^4\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(6\,\mathbb{i}+\text{c2}\,(1+\text{n})\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,\text{b}}^2+\text{Z}\Delta^2\,\delta\lambda_{2\,\text{a}}$$

$$\left(18\,(-1+\text{n})-3\,\mathbb{i}\,\text{c2}\,\text{n}\,\left(1+\text{n}-2\,\text{Z}\Delta^2\right)\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}^2\,\text{n}\,(1+\text{n})\,\text{Z}\Delta^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+\right.$$

$$\left.\left.\left.\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(6\,\mathbb{i}+\text{c2}\,(1+\text{n})\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,\text{b}}\right)\right)\right)\Big/$$

$$\left(\left(-3\,\mathbb{i}+\text{c2}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\left(-6\,\mathbb{i}+2\,\text{c2}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.$$

$$\left.\left.\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{a}}+2\,\text{c2}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\right)==0,$$

$$-\left(\left(\mathbb{i}\,\left(54\,\mathbb{i}\,\lambda-54\,\mathbb{i}\,\text{Z}\Delta\,\lambda-36\,\text{Ifing0}\,\lambda^2\,\hbar-36\,\text{c2}\,\text{Z}\Delta^3\,\delta\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-72\,\text{c2}\,\text{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-\right.\right.\right.$$

$$36\,\text{c2}\,\text{Z}\Delta^2\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-9\,\text{c2}\,\text{n}\,\text{Z}\Delta^2\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-36\,\text{c2}\,\text{Z}\Delta^3\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$

$$12\,\mathbb{i}\,\text{a1}\,\text{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-6\,\mathbb{i}\,\text{a1}\,\text{n}\,\text{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-24\,\mathbb{i}\,\text{a1}\,\text{Z}\Delta^2\,\delta\lambda$$

$$\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-12\,\mathbb{i}\,\text{a1}\,\text{n}\,\text{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-12\,\mathbb{i}\,\text{a1}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-$$

$$18\,\mathbb{i}\,\text{c2}\,\text{Ifing0}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-3\,\mathbb{i}\,\text{c2}\,\text{Ifinggn}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$27\,\mathbb{i}\,\text{c2}\,\text{Ifinhhn}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-6\,\mathbb{i}\,\text{a1}\,\text{n}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+$$

$$3\,\mathbb{i}\,\text{c2}\,\text{Ifinggn}\,\text{n}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-12\,\mathbb{i}\,\text{a0}\,\text{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-$$

$$6\,\mathbb{i}\,\text{a0}\,\text{n}\,\text{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+6\,\mathbb{i}\,\text{c2}^2\,\text{Z}\Delta^5\,\delta\lambda^2\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+3\,\mathbb{i}\,\text{c2}^2\,\text{n}\,\text{Z}\Delta^5\,\delta\lambda^2\,\lambda$$

$$\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-24\,\mathbb{i}\,\text{a0}\,\text{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2-12\,\mathbb{i}\,\text{a0}\,\text{n}\,\text{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2+$$

$$12\,\mathbb{i}\,c2^2\,Z\triangle^5\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 6\,\mathbb{i}\,c2^2\,n\,Z\triangle^5\,\delta\lambda\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 -$$

$$12\,\mathbb{i}\,a0\,Z\triangle^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 - 6\,\mathbb{i}\,a0\,n\,Z\triangle^2\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 - 6\,\mathbb{i}\,c2^2\,Z\triangle^4\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 -$$

$$3\,\mathbb{i}\,c2^2\,n\,Z\triangle^4\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 6\,\mathbb{i}\,c2^2\,Z\triangle^5\,\lambda^3\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 3\,\mathbb{i}\,c2^2\,n\,Z\triangle^5\,\lambda^3\,\hbar^2$$

$$\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 4\,a1\,c2\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 2\,a1\,c2\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 +$$

$$8\,a1\,c2\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 4\,a1\,c2\,n\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 4\,a1\,c2\,Z\triangle^4$$

$$\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 2\,a1\,c2\,n\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + 4\,a0\,c2\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3 +$$

$$2\,a0\,c2\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3 + 8\,a0\,c2\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3 +$$

$$4\,a0\,c2\,n\,Z\triangle^4\,\delta\lambda\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3 + 4\,a0\,c2\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3 +$$

$$2\,a0\,c2\,n\,Z\triangle^4\,\lambda^4\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3 - 18\,c2\,Z\triangle^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{\text{b}} - 36\,c2\,Z\triangle^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}} +$$

$$18\,c2\,Z\triangle^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}} - 12\,\mathbb{i}\,a1\,Z\triangle^2\,\delta\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}} -$$

$$24\,\mathbb{i}\,a1\,Z\triangle^2\,\delta\lambda\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}} - 12\,\mathbb{i}\,a1\,Z\triangle^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}} -$$

$$12\,\mathbb{i}\,c2\,\text{Ifing0}\,Z\triangle^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,\text{b}} - 12\,\mathbb{i}\,a0\,Z\triangle^2\,\delta\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} -$$

$$12\,\mathbb{i}\,c2^2\,Z\triangle^5\,\delta\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} - 24\,\mathbb{i}\,a0\,Z\triangle^2\,\delta\lambda\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} -$$

$$24\,\mathbb{i}\,c2^2\,Z\triangle^5\,\delta\lambda\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} - 12\,\mathbb{i}\,a0\,Z\triangle^2\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} -$$

$$12\,\mathbb{i}\,c2^2\,Z\triangle^4\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} - 3\,\mathbb{i}\,c2^2\,n\,Z\triangle^4\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} -$$

$$12\,\mathbb{i}\,c2^2\,Z\triangle^5\,\lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} + 8\,a1\,c2\,Z\triangle^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} +$$

$$2\,a1\,c2\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} + 16\,a1\,c2\,Z\triangle^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} +$$

$$4\,a1\,c2\,n\,Z\triangle^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} + 8\,a1\,c2\,Z\triangle^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} +$$

$$2\,a1\,c2\,n\,Z\triangle^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}} + 8\,a0\,c2\,Z\triangle^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\,\delta\lambda_{2\,\text{b}} +$$

$$2\,a0\,c2\,n\,Z\triangle^4\,\delta\lambda^2\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\,\delta\lambda_{2\,\text{b}} + 16\,a0\,c2\,Z\triangle^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\,\delta\lambda_{2\,\text{b}} +$$

$$4\,a0\,c2\,n\,Z\triangle^4\,\delta\lambda\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\,\delta\lambda_{2\,\text{b}} + 8\,a0\,c2\,Z\triangle^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\,\delta\lambda_{2\,\text{b}} +$$

$$2\,a0\,c2\,n\,Z\triangle^4\,\lambda^3\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\,\delta\lambda_{2\,\text{b}} - 6\,\mathbb{i}\,c2^2\,Z\triangle^4\,\lambda\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,\delta\lambda_{2\,\text{b}}^2 +$$

$$4\,\text{a1}\,\text{c2}\,Z\Delta^4\,\delta\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+8\,\text{a1}\,\text{c2}\,Z\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+$$

$$4\,\text{a1}\,\text{c2}\,Z\Delta^4\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\delta\lambda_{2\,b}^2+4\,\text{a0}\,\text{c2}\,Z\Delta^4\,\delta\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}^2+$$

$$8\,\text{a0}\,\text{c2}\,Z\Delta^4\,\delta\lambda\,\lambda\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}^2+4\,\text{a0}\,\text{c2}\,Z\Delta^4\,\lambda^2\,\hbar^3\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^3\delta\lambda_{2\,b}^2+$$

$$18\,Z\Delta\,\delta\lambda_a\left(-3\,i+\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)+$$

$$Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a}\left(i\left(-3\left(2\,\text{a1}\,n\,(\delta\lambda+\lambda)^2\,\hbar+\text{c2}\,\lambda\left(-6\,i\,Z\Delta+(2\,\text{Ifing0}+\text{Ifinggn}-\right.\right.\right.\right.$$

$$\left.\left.\left.9\,\text{Ifinhhn})\,\lambda\,\hbar-n\left(3\,i+\text{Ifinggn}\,\lambda\,\hbar\right)\right)\right)+\hbar\left(-6\,\text{a0}\,n\,(\delta\lambda+\lambda)^2+\right.$$

$$\left.\text{c2}\,Z\Delta^2\left(3\,\text{c2}\left(-n\,\lambda^2+(6+n)\,Z\Delta\,(\delta\lambda+\lambda)^2\right)-2\,i\,\text{a1}\,n\,\lambda\,(\delta\lambda+\lambda)^2\,\hbar\right)\right)$$

$$\left.\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]-2\,i\,\text{a0}\,\text{c2}\,n\,Z\Delta^2\,\lambda\,(\delta\lambda+\lambda)^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right)-18\,\text{c2}\,Z\Delta\,\delta\lambda_b+\text{c2}\,n\,Z\Delta^2$$

$$\left.\left.\left.\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(-3\,i\,\text{c2}\,\lambda+2\,\text{a1}\,(\delta\lambda+\lambda)^2\,\hbar+2\,\text{a0}\,(\delta\lambda+\lambda)^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,b}\right)\right)\right)\Big/$$

$$\left(\left(-3\,i+\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\left(-6\,i+2\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.\right.$$

$$\left.\left.\text{c2}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a}+2\,\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right)=0,$$

$$\left(\text{c2}\,Z\Delta\left(-\lambda^2+Z\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\lambda+\delta\lambda_{2\,a})\right)\Big/$$

$$\left(\left(-3\,i+\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right.$$

$$\left(-6\,i+2\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$\left.\left.\text{c2}\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a}+2\,\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right)=0,$$

$$\left(\text{c2}\,(-1+n)\,Z\Delta\left(-\lambda^2+Z\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\lambda+\delta\lambda_{2\,a})\right)\Big/$$

$$\left(\left(-3\,i+\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right.$$

$$\left(-6\,i+2\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$\left.\left.\text{c2}\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a}+2\,\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right)=0,$$

$$\left(\left(-\lambda^2+Z\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar\left(-6\,i+3\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a}+\right.\right.$$

$$\left.\left.2\,\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right)\Big/$$

$$\left(\left(-3\,i+\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right.$$

$$\left(-6\,i+2\,\text{c2}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c2}\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$

$$c2\ n\ Z\Delta^2\ \hbar\ \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\ \delta\lambda_{2\,a} + 2\ c2\ Z\Delta^2\ \hbar\ \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\ \delta\lambda_{2\,b}\Big)\Big) == 0,$$

$$\left(\mathrm{i}\ (-1 + Z\ Z\Delta)\right) \Big/ \left(-6\ \mathrm{i} + 2\ c2\ Z\Delta^2\ \lambda\ \hbar\ \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c2\ n\ Z\Delta^2\ \lambda\ \hbar\ \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c2\ n\ Z\Delta^2\ \hbar\ \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\ \delta\lambda_{2\,a} + 2\ c2\ Z\Delta^2\ \hbar\ \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\ \delta\lambda_{2\,b}\right) == 0\Big\}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

```
cts = Solve[cteq, {δm₁, δλ1a, δλ1b, δλ2a, δλ2b, δλ, Z, ZΔ}] // DeleteDuplicates;
$Aborted
```

```
{δm₁², δλ1a, δλ1b, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
  (((((mn2 - (λ v²/3) - m² - λ/6 v² - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn) /. mg2soln) /. Solve[
          neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
        DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
     CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

Final Counterterms

```
({δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

```
counterterms = Thread[{δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

The should be momentum independent :

```
({δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```

Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to thesis Chapter 4.

Author: Michael J. Brown

Mathematica notebook to compute couter-terms for two loop truncations of the SI-3PIEA.

Hartree-Fock

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators Δ_G^{-1} and Δ_N^{-1},
mg2 is the Goldstone mass squared m_G^2,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, δm_1^2 is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1a}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

$$\text{geom} = p^2 - \text{mg2} == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta \frac{\lambda + \delta\lambda_{1a}}{6} v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda + \left(n-1\right)\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right) Z\Delta^2\,(\text{t}\infty\text{g} + \text{tfing}) - \frac{\hbar}{6}(\lambda + \delta\lambda_{2a})\, Z\Delta^2\,(\text{t}\infty\text{n} + \text{tfinn})$$

$$-\text{mg2} + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6} v^2\, Z\Delta\,(\lambda + \delta\lambda_a) - \frac{1}{6}(\text{tfinn} + \text{t}\infty\text{n})\, Z\Delta^2\, \hbar\,(\lambda + \delta\lambda_{2a}) - \frac{1}{6}(\text{tfing} + \text{t}\infty\text{g})\, Z\Delta^2\, \hbar\,((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})$$

Higgs equation of motion

```
neom = p^2 - mn2 == -λ v^2/3 ZΔ + p^2 - mg2
```

$$-\text{mn2} + p^2 == -\text{mg2} + p^2 - \frac{1}{3} v^2 Z\Delta\,\lambda$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

```
regularisedtadpoles = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}
```

$$\left\{t\infty g \to c0\,\Lambda^2 + c1\,\text{mg2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ t\infty n \to c0\,\Lambda^2 + c1\,\text{mn2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

```
mg2soln =
 mg2 /. (geom /. regularisedtadpoles /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

$$\Big(-18\,m^2 - 18\,p^2 + 18\,p^2\,Z\,Z\Delta - 3\,v^2\,Z\Delta\,\lambda - 3\,\text{tfing}\,Z\Delta^2\,\lambda\,\hbar - 3\,n\,\text{tfing}\,Z\Delta^2\,\lambda\,\hbar - 3\,\text{tfinn}\,Z\Delta^2\,\lambda\,\hbar - 6\,c0\,Z\Delta^2\,\lambda\,\Lambda^2\,\hbar - 3\,c0\,n\,Z\Delta^2\,\lambda\,\Lambda^2\,\hbar - c1\,v^2\,Z\Delta^3\,\lambda^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] - 18\,\delta m_1^2 - 3\,v^2\,Z\Delta\,\delta\lambda_a + 3\,\text{tfing}\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,a} - 3\,n\,\text{tfing}\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,a} - 3\,\text{tfinn}\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,a} - 3\,c0\,n\,Z\Delta^2\,\Lambda^2\,\hbar\,\delta\lambda_{2\,a} - c1\,v^2\,Z\Delta^3\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,a} - 6\,\text{tfing}\,Z\Delta^2\,\hbar\,\delta\lambda_{2\,b} - 6\,c0\,Z\Delta^2\,\Lambda^2\,\hbar\,\delta\lambda_{2\,b}\Big) \Big/ \left(3\left(-6 + 2\,c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,n\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,n\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,a} + 2\,c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)\right)$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

```
cteq =
 ((CoefficientList[mg2soln + (-m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn)), {p, v,
        tfing, tfinn}] // Flatten) //
      DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

$$\left\{-\left(\left(6\,\delta m_1^2+Z\Delta^2\hbar\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda+n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\Big/\left(-6+c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\right)=0,\ -\frac{\lambda\,\hbar}{6}-\left(Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2a})\right)\Big/\left(-6+c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)=0,\right.$$

$$\left(\hbar\left((1+n)\,\lambda\left(6-6\,Z\Delta^2-c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)+Z\Delta^2\left(-\left(6\,(-1+n)+c1\,n\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2a}-2\left(6+c1\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2b}\right)\right)\right)\Big/\left(6\left(-6+c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\right)=0,$$

$$\mathrm{True},\ -\left(\left(6\,Z\Delta\,\delta\lambda_a+\lambda\left(6\,(-1+Z\Delta)+c1\,Z\Delta^2\,(2+n+2\,Z\Delta)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left((n+2\,Z\Delta)\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\right)\Big/\left(6\left(-6+c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\right)\right)=0,$$

$$\left.(-6+6\,Z\,Z\Delta)\Big/\left(-6+c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)=0\right\}$$

Solve for counterterms

```
cts = {δm1^2, δλ1a, δλ2a, δλ2b, Z, ZΔ} /. Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] //
   FullSimplify // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ \frac{\lambda\left(6-6\,Z\Delta-c1\,(4+n)\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{Z\Delta\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)},\ \lambda\left(-1+\frac{6}{Z\Delta^2\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \lambda\left(-1+\frac{6}{Z\Delta^2\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

ZΔ is redundant in this truncation, can remove it :

```
cts /. ZΔ → 1 // FullSimplify
```

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ -\frac{c1\,(4+n)\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ \lambda\left(-1+\frac{6}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{6}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ 1,\ 1\right\}\right\}$$

$$\delta\lambda_{2\,a} == \frac{n+2}{n+4}\,\delta\lambda_{1\,a}\ /.\ \mathrm{Solve}[cteq,\{\delta m_1,\delta\lambda_{1\,a},\delta\lambda_{2\,a},\delta\lambda_{2\,b},Z,Z_\Delta\}]\ /.\ \{Z\Delta\to 1\}\ //\ \mathrm{FullSimplify}\ //\ \mathrm{DeleteDuplicates}$$

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

{True}

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$
Ifingp is the finite sunset integral $I^{\mathrm{fin}}_{\mathrm{NG}}(p)$,
Ifing0 is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_G)$,
Ifingn is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\mathrm{geom} = p^2 - mg2 + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2\left(\mathrm{Ifingp}-\mathrm{Ifing0}\right) == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda+\delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda+\left(n-1\right)\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,(tg) - \frac{\hbar}{6}\,(\lambda+\delta\lambda_{2\,a})\,Z\Delta^2\,(tn) + i\,\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\mathrm{Ing}$$

$$-mg2+p^2+\frac{1}{9}\,i\,(-\mathrm{Ifing0}+\mathrm{Ifingp})\,v^2\,\lambda^2\,\hbar == -m^2+p^2\,Z\,Z\Delta+\frac{1}{9}\,i\,\mathrm{Ing}\,v^2\,Z\Delta^3\,(\delta\lambda+\lambda)^2\,\hbar-\delta m_1^2-\frac{1}{6}\,v^2\,Z\Delta\,(\lambda+\delta\lambda_a)-\frac{1}{6}\,tn\,Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})-\frac{1}{6}\,tg\,Z\Delta^2\,\hbar\,((1+n)\,\lambda+(-1+n)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})$$

$$\texttt{neom} = p^2 - \texttt{mn2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifingn}) == \frac{-Z\triangle\,(\lambda+\delta\lambda)\,v^2}{3} + p^2 - \texttt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifing0})$$

$$-\texttt{mn2} + p^2 + \frac{1}{9}\,i\,(-\texttt{Ifingn} + \texttt{Ifingp})\,v^2\,\lambda^2\,\hbar == -\texttt{mg2} + p^2 - \frac{1}{3}\,v^2\,Z\triangle\,(\delta\lambda + \lambda) + \frac{1}{9}\,i\,(-\texttt{Ifing0} + \texttt{Ifingp})\,v^2\,\lambda^2\,\hbar$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\texttt{intrules} = \left\{\texttt{Ing} \to \texttt{I}\mu + \texttt{Ifingp} + \texttt{Ifing0},\ \texttt{tg} \to \texttt{t}\mu - i\,(\texttt{mg2} - \mu^2)\,\texttt{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\texttt{c}\mu + \texttt{tfing},\ \texttt{tn} \to \texttt{t}\mu - i\,(\texttt{mn2} - \mu^2)\,\texttt{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\texttt{c}\mu + \texttt{tfinn}\right\}$$

$$\left\{\texttt{Ing} \to \texttt{Ifing0} + \texttt{Ifingp} + \texttt{I}\mu,\ \texttt{tg} \to \texttt{tfing} + \texttt{t}\mu - i\,\texttt{I}\mu\,(\texttt{mg2} - \mu^2) + \frac{1}{9}\,\texttt{c}\mu\,v^2\,(\delta\lambda+\lambda)^2\,\hbar,\ \texttt{tn} \to \texttt{tfinn} + \texttt{t}\mu - i\,\texttt{I}\mu\,(\texttt{mn2} - \mu^2) + \frac{1}{9}\,\texttt{c}\mu\,v^2\,(\delta\lambda+\lambda)^2\,\hbar\right\}$$

$$\texttt{regularisedtadpoles} = \left\{\texttt{I}\mu \to \texttt{c2}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \texttt{t}\mu \to \texttt{c0}\,\Lambda^2 + \texttt{c1}\,\mu^2\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \texttt{c}\mu \to \texttt{a0}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \texttt{a1}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

$$\left\{\texttt{I}\mu \to \texttt{c2}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \texttt{t}\mu \to \texttt{c0}\,\Lambda^2 + \texttt{c1}\,\mu^2\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \texttt{c}\mu \to \texttt{a1}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \texttt{a0}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
mg2soln = ((geom /. intrules(*/.regularisedtadpoles*) /. Solve[neom, mn2][[1]]) //
    Solve[#, mg2] &)[[1]]
```

$$\Big\{mg2 \to \Big(-m^2 - p^2 + p^2\, Z\, Z\Delta - \frac{1}{9}\, i\, (-\text{Ifing0} + \text{Ifingp})\, v^2\, \lambda^2\, \hbar + \frac{1}{9}\, i\, (\text{Ifing0} + \text{Ifingp} + \text{I}\mu)\, v^2\, Z\Delta^3\, (\delta\lambda + \lambda)^2\, \hbar - \delta m_1^2 - \frac{1}{6}\, v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}\, \text{tfinn}\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, \text{t}\mu\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, \text{I}\mu\, v^2\, Z\Delta^3\, \delta\lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, \text{I}\mu\, v^2\, Z\Delta^3\, \lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2\, \mu^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, \text{c}\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2\, \hbar^2\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, \text{I}\mu\, Z\Delta^2\, \hbar\, \big(\text{Ifing0}\, v^2\, \lambda^2\, \hbar - \text{Ifingn}\, v^2\, \lambda^2\, \hbar\big)\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, \text{tfing}\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{6}\, \text{t}\mu\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2\, \mu^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{54}\, \text{c}\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2\, \hbar^2\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\Big) \Big/ \Big(-1 - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\Big)\Big\}$$

Gather kinematically distinct divergences for Goldstone EOM

```
cteq = ((mg2 - m^2 - λ/6 v^2 - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) //
      CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

$$\Big\{\left(-6\,\mathrm{i}\,\delta m_1^2+Z\Delta^2\left(-\mathrm{i}\,t\mu+I\mu\left(-m^2+\mu^2\right)\right)\hbar\left((2+n)\lambda+n\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\Big/$$
$$\left(-6\,\mathrm{i}+2\,I\mu\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a}+2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)=0,$$
$$\text{True},\ \frac{1}{6}\hbar\left(-\lambda-\left(6\,\mathrm{i}\,Z\Delta^2\left(\lambda+\delta\lambda_{2a}\right)\right)\Big/\right.$$
$$\left.\left(-6\,\mathrm{i}+2\,I\mu\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a}+2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right)=0,$$
$$\frac{1}{6}\hbar\left(-(1+n)\lambda-\left(6\,\mathrm{i}\,Z\Delta^2\left(\lambda+n\lambda+(-1+n)\,\delta\lambda_{2a}+2\,\delta\lambda_{2b}\right)\right)\Big/\right.$$
$$\left.\left(-6\,\mathrm{i}+2\,I\mu\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a}+2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right)=0,$$
$$-\Big(\Big(\mathrm{i}\,\Big(-18\lambda+18\,Z\Delta\,\lambda-12\,\mathrm{i}\,\mathtt{Ifing0}\,Z\Delta^3\delta\lambda^2\hbar-12\,\mathrm{i}\,I\mu\,Z\Delta^3\delta\lambda^2\hbar-24\,\mathrm{i}\,\mathtt{Ifing0}\,Z\Delta^3\delta\lambda\,\lambda\hbar-$$
$$30\,\mathrm{i}\,I\mu\,Z\Delta^3\delta\lambda\,\lambda\hbar-12\,\mathrm{i}\,\mathtt{Ifing0}\,\lambda^2\hbar-6\,\mathrm{i}\,I\mu\,Z\Delta^2\lambda^2\hbar-3\,\mathrm{i}\,I\mu\,n\,Z\Delta^2\lambda^2\hbar-$$
$$12\,\mathrm{i}\,\mathtt{Ifing0}\,Z\Delta^3\lambda^2\hbar-18\,\mathrm{i}\,I\mu\,Z\Delta^3\lambda^2\hbar+4\,c\mu\,Z\Delta^2\delta\lambda^2\lambda\hbar^2+2\,c\mu\,n\,Z\Delta^2\delta\lambda^2\lambda\hbar^2+$$
$$8\,c\mu\,Z\Delta^2\delta\lambda\,\lambda^2\hbar^2+4\,c\mu\,n\,Z\Delta^2\delta\lambda\,\lambda^2\hbar^2+4\,c\mu\,Z\Delta^2\lambda^3\hbar^2+2\,\mathtt{Ifing0}\,I\mu\,Z\Delta^2\lambda^3\hbar^2-$$
$$2\,\mathtt{Ifingn}\,I\mu\,Z\Delta^2\lambda^3\hbar^2+2\,c\mu\,n\,Z\Delta^2\lambda^3\hbar^2+18\,Z\Delta\,\delta\lambda_a+Z\Delta^2\hbar\,\Big(2\,c\mu\,n\,(\delta\lambda+\lambda)^2\hbar+$$
$$I\mu\left(-6\,\mathrm{i}\,Z\Delta\,(\delta\lambda+\lambda)+\lambda\left(-3\,\mathrm{i}\,n+2\,(\mathtt{Ifing0}-\mathtt{Ifingn})\,\lambda\hbar\right)\right)\Big)\,\delta\lambda_{2a}-$$
$$6\,\mathrm{i}\,I\mu\,Z\Delta^2\lambda\hbar\,\delta\lambda_{2b}+4\,c\mu\,Z\Delta^2\delta\lambda^2\hbar^2\delta\lambda_{2b}+8\,c\mu\,Z\Delta^2\delta\lambda\,\lambda\hbar^2\delta\lambda_{2b}+4\,c\mu\,Z\Delta^2\lambda^2\hbar^2\delta\lambda_{2b}\Big)\Big)\Big/$$
$$\left(18\left(-6\,\mathrm{i}+2\,I\mu\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a}+2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right)\Big)=$$
$$0,\ -\left(\left(2\left(-\lambda^2+Z\Delta^3(\delta\lambda+\lambda)^2\right)\hbar\right)\Big/\right.$$
$$\left.\left(3\left(-6\,\mathrm{i}+2\,I\mu\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a}+2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right)\right)=0,$$
$$\left(6\,\mathrm{i}\,(-1+Z\,Z\Delta)\right)\Big/\left(-6\,\mathrm{i}+2\,I\mu\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\lambda\hbar+I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a}+2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)=$$
$$0\Big\}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

```
cts =
  Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ}] // FullSimplify // DeleteDuplicates;
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

```
{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\left(i\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,i+I\mu\,(2+n)\,\lambda\,\hbar},\right.\right.$$
$$\left(\lambda\left(6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)+2\,Z\Delta^3\left(9\,i+\lambda\,\hbar\left(-6\,(2\,\text{Ifing0}+I\mu)+i\,I\mu\,(\text{Ifingn}+I\mu\,(2+n)+\text{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\right)\Big/$$
$$\left(3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)\right),\ \lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\left.\lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \left(-1-\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\right\},$$
$$\left\{-\frac{(2+n)\,\lambda\,\left(i\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,i+I\mu\,(2+n)\,\lambda\,\hbar},\right.$$
$$\left(\lambda\left(-6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)+2\,Z\Delta^3\left(9\,i+\lambda\,\hbar\left(-6\,(2\,\text{Ifing0}+I\mu)+i\,I\mu\,(\text{Ifingn}+I\mu\,(2+n)+\text{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\right)\Big/$$
$$\left(3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)\right),\ \lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\left.\left.\lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \left(-1+\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
 (((((mn2 - (λ v^2/3) - m^2 - λ/6 v^2 - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) /. Solve[
          neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
       DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
    CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
   Simplify // DeleteDuplicates) == 0 // Thread
```

$$\left\{\text{True},\ \frac{i\,\lambda\left(3\,i+\sqrt{Z\Delta}\,\left(3\,i+\text{Ifing0}\,\lambda\,\hbar-\text{Ifingn}\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}==0,\ \frac{\lambda\left(3+i\,\sqrt{Z\Delta}\,\left(3\,i+\text{Ifing0}\,\lambda\,\hbar-\text{Ifingn}\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}==0\right\}$$

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

$$\left\{\left\{Z\Delta\to-\frac{9}{\left(3\,i+\text{Ifing0}\,\lambda\,\hbar-\text{Ifingn}\,\lambda\,\hbar\right)^2}\right\}\right\}$$

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

```
True
```

Final Counterterms

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

```
counterterms = Thread[{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

$$\Big\{\delta m_1^2 \to -\frac{(2+n)\,\lambda\,\left(i\, t\mu + I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,i + I\mu\,(2+n)\,\lambda\,\hbar},$$

$$\delta\lambda_a \to \Bigg(\lambda\,\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^8 \Bigg(-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2 + 1458\,I\mu\,\lambda\,\hbar \left(-\frac{1}{\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^2}\right)^{5/2} - \frac{19683\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)}{\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^8} - \left(1458\,\left(9\,i + \lambda\,\hbar\,\left(-6\,(2\,\text{Ifing0} + I\mu) + i\,I\mu\,(\text{Ifingn} + I\mu\,(2+n) + \text{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right) \Big/ \left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^6\Bigg)\Bigg) \Big/ \left(19683\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)\right),$$

$$\delta\lambda_{2\,a} \to \lambda\left(-1 + \frac{2\,i\,\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^4}{27\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$

$$\delta\lambda_{2\,b} \to \lambda\left(-1 + \frac{2\,i\,\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^4}{27\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$

$$\delta\lambda \to \lambda\left(-1 - \frac{1}{27\left(-\frac{1}{(3\,i+\text{Ifing0}\,\lambda\,\hbar-\text{Ifingn}\,\lambda\,\hbar)^2}\right)^{3/2}}\right),$$

$$Z \to -\frac{1}{9}\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^2,$$

$$Z\Delta \to -\frac{9}{\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^2}\Big\}$$

The should be momentum independent :

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```

```
True
```

```
True
```

Renormalization of Soft Symmetry Improved 2PIEA gap equations in the Hartree-Fock approximation

Supplement to thesis Chapter 5 "Soft Symmetry Improvement"

Mathematica notebook to compute couter-terms for the Hartree-Fock truncation of the SSI-2PIEA

Author: Michael Brown

Hartree-Fock

In[41]:=
```
ClearAll[veom, geom, neom, regularisedtadpoles, mg2soln, mn2soln,
  cteq, cteq2, cteqs, ctsolns, cts, ctegs3, rnveom, veomCtEqs, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators Δ_G^{-1} and Δ_N^{-1},
mg2 is the Goldstone mass squared m_G^2,
mn2 is the Higgs mass squared m_H^2,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, δm_0^2, δm_1^2 are its counter-terms,
λ is the (renormalized) four point coupling,
$\delta\lambda_0$, $\delta\lambda_{1a}$, $\delta\lambda_{1b}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
ξ is the stiffness parameter,
ϵ is the solution of the Goldstone zero mode equation,
ssi = $\frac{1}{V\beta m_G^2}\left(\frac{1}{\epsilon}-1\right)$ is the soft symmetry improvement term in the propagator eoms,
ssi2 = $\frac{1}{\xi}(n-1)\,2\left(m_G^2\,\epsilon\right)^2$ is the other soft symmetry improvement term in the vev eom,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

Vev equation of motion

In[42]:=
$$\text{veom} = Z\Delta^{-1}\left(m^2 + \delta m_0{}^2\right) v + \frac{\lambda + \delta\lambda_0}{6} v^3 + \frac{\hbar}{6} Z\Delta \left(n-1\right) (\lambda + \delta\lambda_{1\,a})\, v\, (\text{t}\infty\text{g} + \text{tfing} + \text{ssi}) + \frac{\hbar}{6} Z\Delta \left(3\,\lambda + \delta\lambda_{1\,a} + 2\,\delta\lambda_{1\,b}\right) v\, (\text{t}\infty\text{n} + \text{tfinn}) + v\,\text{ssi2}$$

Out[42]=
$$\text{ssi2}\, v + \frac{v\left(m^2 + \delta m_0^2\right)}{Z\Delta} + \frac{1}{6} v^3 (\lambda + \delta\lambda_0) + \frac{1}{6}(-1+n)(\text{ssi} + \text{tfing} + \text{t}\infty\text{g})\, v\, Z\Delta\, \hbar\, (\lambda + \delta\lambda_a) + \frac{1}{6}(\text{tfinn} + \text{t}\infty\text{n})\, v\, Z\Delta\, \hbar\, (3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b)$$

Goldstone equation of motion

In[43]:=
$$\text{geom} = p^2 - \text{mg2} == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta \frac{\lambda + \delta\lambda_{1\,a}}{6} v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda + \left(n-1\right)\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\, (\text{t}\infty\text{g} + \text{tfing} + \text{ssi}) - \frac{\hbar}{6}(\lambda + \delta\lambda_{2\,a})\, Z\Delta^2\, (\text{t}\infty\text{n} + \text{tfinn})$$

Out[43]=
$$-\text{mg2} + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6} v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}(\text{tfinn} + \text{t}\infty\text{n})\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}(\text{ssi} + \text{tfing} + \text{t}\infty\text{g})\, Z\Delta^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})$$

Higgs equation of motion

In[44]:=
$$\text{neom} = p^2 - \text{mn2} == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta\, v^2 \frac{\left(3\,\lambda + \delta\lambda_{1\,a} + 2\,\delta\lambda_{1\,b}\right)}{6} - \frac{\hbar}{6}(\lambda + \delta\lambda_{2\,a})\left(n-1\right) Z\Delta^2\, (\text{t}\infty\text{g} + \text{tfing} + \text{ssi}) - \frac{\hbar}{6}\left(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\, (\text{t}\infty\text{n} + \text{tfinn})$$

Out[44]=
$$-\text{mn2} + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6}(-1+n)(\text{ssi} + \text{tfing} + \text{t}\infty\text{g})\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6} v^2\, Z\Delta\, (3\,\lambda + \delta\lambda_a + 2\,\delta\lambda_b) - \frac{1}{6}(\text{tfinn} + \text{t}\infty\text{n})\, Z\Delta^2\, \hbar\, (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

In[45]:=
```
regularisedtadpoles = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}
```

Out[45]=
$$\left\{\text{t}\infty\text{g} \to \text{c0}\,\Lambda^2 + \text{c1}\,\text{mg2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\infty\text{n} \to \text{c0}\,\Lambda^2 + \text{c1}\,\text{mn2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

In[46]:=
```
mg2soln = mg2 /. (geom /. regularisedtadpoles /.
      Solve[neom /. regularisedtadpoles, mn2][[1]] // Solve[#, mg2][[1]] &)
```

Out[46]=

$$
\begin{aligned}
&\Bigg(-m^2 - p^2 + p^2\, Z\, Z\triangle - \delta m_1^2 - \frac{1}{6} v^2\, Z\triangle\, (\lambda + \delta\lambda_{a}) - \frac{1}{6}\, \mathrm{tfinn}\, Z\triangle^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \\
&\quad \frac{1}{6}\, \mathrm{c0}\, Z\triangle^2\, \Lambda^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, \mathrm{ssi}\, Z\triangle^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \\
&\quad \frac{1}{6}\, \mathrm{tfing}\, Z\triangle^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \\
&\quad \frac{1}{6}\, \mathrm{c0}\, Z\triangle^2\, \Lambda^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) + \\
&\quad \frac{\mathrm{c1}\, m^2\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})}{6\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c1}\, p^2\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})}{6\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} - \\
&\quad \frac{\mathrm{c1}\, p^2\, Z\, Z\triangle^3\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})}{6\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] \delta m_1^2\, (\lambda + \delta\lambda_{2\,a})}{6\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c1}\, (-1+n)\, \mathrm{ssi}\, Z\triangle^4\, \hbar^2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})^2}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c1}\, (-1+n)\, \mathrm{tfing}\, Z\triangle^4\, \hbar^2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})^2}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c0}\, \mathrm{c1}\, (-1+n)\, Z\triangle^4\, \Lambda^2\, \hbar^2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})^2}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c1}\, v^2\, Z\triangle^3\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})\, (3\,\lambda + \delta\lambda_{a} + 2\,\delta\lambda_{b})}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c1}\, \mathrm{tfinn}\, Z\triangle^4\, \hbar^2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})\, (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)} + \\
&\quad \frac{\mathrm{c0}\, \mathrm{c1}\, Z\triangle^4\, \Lambda^2\, \hbar^2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (\lambda + \delta\lambda_{2\,a})\, (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)}\Bigg) \Bigg/ \\
&\Bigg(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \\
&\quad \frac{\mathrm{c1}^2\, (-1+n)\, Z\triangle^4\, \hbar^2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 (\lambda + \delta\lambda_{2\,a})^2}{36\left(-1 + \frac{1}{6}\,\mathrm{c1}\, Z\triangle^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] (3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)}\Bigg)
\end{aligned}
$$

In[47]:= **mn2soln = mn2 /. (neom /. regularisedtadpoles /. mg2 → mg2soln // Solve[#, mn2][[1]] &)**

Out[47]=

$$\frac{1}{-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})}$$

$$\left(-m^2-p^2+p^2\,Z\,Z\Delta-\delta m_1^2-\frac{1}{6}\,v^2\,Z\Delta\,(3\,\lambda+\delta\lambda_a+2\,\delta\lambda_b)-\frac{1}{6}\,\mathrm{tfinn}\,Z\Delta^2\,\hbar\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})-\right.$$

$$\frac{1}{6}\,\mathrm{c0}\,Z\Delta^2\,\Lambda^2\,\hbar\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})-\frac{1}{6}\,(-1+n)\,Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})\,\Bigg(\mathrm{ssi}+\mathrm{tfing}+\mathrm{c0}\,\Lambda^2+$$

$$\Bigg(\mathrm{c1}\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\Bigg(-m^2-p^2+p^2\,Z\,Z\Delta-\delta m_1^2-\frac{1}{6}\,v^2\,Z\Delta\,(\lambda+\delta\lambda_a)-\frac{1}{6}\,\mathrm{tfinn}\,Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})-$$

$$\frac{1}{6}\,\mathrm{c0}\,Z\Delta^2\,\Lambda^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})-\frac{1}{6}\,\mathrm{ssi}\,Z\Delta^2\,\hbar\,((1+n)\,\lambda+(-1+n)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})-$$

$$\frac{1}{6}\,\mathrm{tfing}\,Z\Delta^2\,\hbar\,((1+n)\,\lambda+(-1+n)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})-\frac{1}{6}\,\mathrm{c0}\,Z\Delta^2\,\Lambda^2\,\hbar\,((1+n)\,\lambda+$$

$$(-1+n)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})+\frac{\mathrm{c1}\,m^2\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})}{6\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\frac{\mathrm{c1}\,p^2\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})}{6\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}-$$

$$\frac{\mathrm{c1}\,p^2\,Z\,Z\Delta^3\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})}{6\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\frac{\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta m_1^2\,(\lambda+\delta\lambda_{2\,a})}{6\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\frac{\mathrm{c1}\,(-1+n)\,\mathrm{ssi}\,Z\Delta^4\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})^2}{36\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\frac{\mathrm{c1}\,(-1+n)\,\mathrm{tfing}\,Z\Delta^4\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})^2}{36\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\frac{\mathrm{c0}\,\mathrm{c1}\,(-1+n)\,Z\Delta^4\,\Lambda^2\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})^2}{36\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\frac{\mathrm{c1}\,v^2\,Z\Delta^3\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})\,(3\,\lambda+\delta\lambda_a+2\,\delta\lambda_b)}{36\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}+$$

$$\left(\mathrm{c1}\,\mathrm{tfinn}\,Z\Delta^4\,\hbar^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\Big/$$

$$\left(36\left(-1+\frac{1}{6}\,\mathrm{c1}\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\right)+$$

$$\left(\text{c0 c1 Z}\Delta^4\,\Lambda^2\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(\lambda+\delta\lambda_{2\,a})\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\Big/$$

$$\left(36\left(-1+\frac{1}{6}\,\text{c1 Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\right)\Bigg)\Bigg)\Bigg/$$

$$\left(-1+\frac{1}{6}\,\text{c1 Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,((1+\text{n})\,\lambda+(-1+\text{n})\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})-\right.$$

$$\left.\frac{\text{c1}^2\,(-1+\text{n})\,\text{Z}\Delta^4\,\hbar^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\,(\lambda+\delta\lambda_{2\,a})^2}{36\left(-1+\frac{1}{6}\,\text{c1 Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(3\,\lambda+\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)}\right)\Bigg)\Bigg)$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

In[48]:=

$$\textbf{cteq} = \left(\left(\textbf{CoefficientList}\left[\textbf{mg2soln}+\left(-\textbf{m}^2-\frac{\lambda}{6}\,\textbf{v}^2-\frac{\hbar}{6}\,((\textbf{n}+\textbf{1})\,\lambda)\,(\textbf{tfing}+\textbf{ssi})-\frac{\hbar}{6}\,(\lambda)\,(\textbf{tfinn})\right),\right.\right.\right.$$

$$\left.\left.\{\textbf{p},\textbf{v},\textbf{tfing},\textbf{tfinn}\}\right]\,//\,\textbf{Flatten}\right)\,//$$

$$\left.\textbf{DeleteDuplicates}\,//\,\textbf{Simplify}\,//\,\textbf{FullSimplify}\right) == \textbf{0}\,//\,\textbf{Thread}$$

Out[48]=

$$\left\{-\left(\left(\lambda\,\hbar\left(-18\,(1+\text{n})\,\text{ssi}\,(-1+\text{Z}\Delta^2)-18\,\text{c0}\,(2+\text{n})\,\text{Z}\Delta^2\,\Lambda^2+\right.\right.\right.\right.$$

$$\text{c1 Z}\Delta^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(-18\,\text{m}^2\,(2+\text{n})+3\,\lambda\,\left(-(1+\text{n})\,(4+\text{n})\,\text{ssi}+2\,(2+\text{n})\,\text{ssi Z}\Delta^2+\right.\right.$$

$$\left.\left.\left.2\,\text{c0}\,(2+\text{n})\,\text{Z}\Delta^2\,\Lambda^2\right)\hbar+\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\left(6\,\text{m}^2+(1+\text{n})\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\right)+$$

$$36\,\delta\text{m}_1^2\left(-3+\text{c1 Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1 Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)+$$

$$\text{Z}\Delta^2\,\hbar\left(\delta\lambda_{2\,b}\left(-36\,(\text{ssi}+\text{c0}\,\Lambda^2)+\right.\right.$$

$$\text{c1 Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(-36\,\text{m}^2+6\,\lambda\,\left(-2\,(1+\text{n})\,\text{ssi}+(4+\text{n})\,\text{ssi Z}\Delta^2+\text{c0}\,(4+\text{n})\,\text{Z}\Delta^2\,\Lambda^2\right)\hbar+\right.$$

$$\left.\text{c1}\,(4+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\left(6\,\text{m}^2+(1+\text{n})\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)+2\,\text{c1 Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]$$

$$\left.\left(6\,(\text{ssi}+\text{c0}\,\Lambda^2)+\text{c1}\,\left(6\,\text{m}^2+(1+\text{n})\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,b}\right)+\delta\lambda_{2\,a}$$

$$\left(-18\,((-1+\text{n})\,\text{ssi}+\text{c0 n}\,\Lambda^2)+\text{c1 n Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(-18\,\text{m}^2-3\,\lambda\,\left(\text{ssi}+\text{n ssi}-2\,\text{ssi Z}\Delta^2-\right.\right.\right.$$

$$\left.\left.2\,\text{c0 Z}\Delta^2\,\Lambda^2\right)\hbar+\text{c1 Z}\Delta^2\,\lambda\,\hbar\,\left(6\,\text{m}^2+(1+\text{n})\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)+\text{c1 n Z}\Delta^2\,\hbar$$

$$\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(6\left(\text{ssi}+\text{c0}\,\Lambda^2\right)+\text{c1}\left(6\,\text{m}^2+(1+\text{n})\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,\text{b}}\Big)\Big)\Big)\Big/$$
$$\left(6\left(-3+\text{c1}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\right.$$
$$\left.\left(-6+\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\text{n}\,\delta\lambda_{2\,\text{a}}+2\,\delta\lambda_{2\,\text{b}})\right)\right)\Big)=0,$$

$$-\frac{\lambda\,\hbar}{6}+\left(3\,\text{Z}\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,\text{a}})\right)\Big/\left(\left(-3+\text{c1}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)\right.$$
$$\left.\left(-6+\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\text{n}\,\delta\lambda_{2\,\text{a}}+2\,\delta\lambda_{2\,\text{b}})\right)\right)=0,$$

$$-\frac{1}{6\,\text{c1}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(6+\text{c1}\,(1+\text{n})\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$
$$\frac{18}{\text{n}\left(-3+\text{c1}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)}+$$
$$(36\,(-1+\text{n}))\Big/$$
$$\left.\left(\text{n}\left(-6+\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\text{n}\,\delta\lambda_{2\,\text{a}}+2\,\delta\lambda_{2\,\text{b}})\right)\right)\right)=0,$$

$$\text{True},\ -\frac{\lambda}{6}+\frac{\text{Z}\Delta\,(\lambda+\delta\lambda_{\text{b}})}{\text{n}\left(-3+\text{c1}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)}-$$
$$(\text{Z}\Delta\,((2+\text{n})\,\lambda+\text{n}\,\delta\lambda_{\text{a}}+2\,\delta\lambda_{\text{b}}))\Big/$$
$$\left(\text{n}\left(-6+\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\text{n}\,\delta\lambda_{2\,\text{a}}+2\,\delta\lambda_{2\,\text{b}})\right)\right)=0,$$

$$(-6+6\,\text{Z}\,\text{Z}\Delta)\Big/\left(-6+\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\right.$$
$$\left.\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right](\text{n}\,\delta\lambda_{2\,\text{a}}+2\,\delta\lambda_{2\,\text{b}})\right)=0\Big\}$$

In[49]:=
```
cteq2 =
  ((CoefficientList[mn2soln + (-m^2 - λ/2 v^2 - ℏ/6 ((n - 1) λ) (tfing + ssi) - ℏ/2 (λ) (tfinn)),
        {p, v, tfing, tfinn}] // Flatten) //
      DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

Out[49]=
$$\Big\{-\Big(\Big(\lambda\,\hbar\,\Big(-18\,(-1+\text{n})\,\text{ssi}\,\left(-1+\text{Z}\Delta^2\right)-18\,\text{c0}\,(2+\text{n})\,\text{Z}\Delta^2\,\Lambda^2+$$
$$\text{c1}\,\text{Z}\Delta^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\Big(-18\,\text{m}^2\,(2+\text{n})+3\,\lambda\,\left(-(-1+\text{n})\,(4+\text{n})\,\text{ssi}+2\,\text{c0}\,(2+\text{n})\,\text{Z}\Delta^2\,\Lambda^2\right)\hbar+$$
$$\text{c1}\,(2+\text{n})\,\text{Z}\Delta^2\,\lambda\,\hbar\,\left(6\,\text{m}^2+(-1+\text{n})\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\Big)\Big)+$$
$$36\,\delta\text{m}_1^2\left(-3+\text{c1}\,\text{Z}\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,\text{Z}\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,\text{b}}\right)+\text{Z}\Delta^2\,\hbar$$

$$\Big(\delta\lambda_{2\,b}\,\Big(-36\,c0\,\Lambda^2 + c1\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\Big(6\,\Big(-6\,m^2 + \lambda\,\Big(-2\,(-1+n)\,ssi + c0\,(4+n)\,Z\triangle^2\,\Lambda^2\Big)\,\hbar\Big) + c1\,(4+n)\,Z\triangle^2\,\lambda\,\hbar\,\Big(6\,m^2 + (-1+n)\,ssi\,\lambda\,\hbar\Big)\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\Big) + 2\,c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\Big(6\,c0\,\Lambda^2 + c1\,\Big(6\,m^2 + (-1+n)\,ssi\,\lambda\,\hbar\Big)\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\Big)\,\delta\lambda_{2\,b}\Big) + \delta\lambda_{2\,a}\,\Big(-18\,\Big((-1+n)\,ssi + c0\,n\,\Lambda^2\Big) + c1\,n\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\Big(3\,\Big(-6\,m^2 + \lambda\,\Big(ssi - n\,ssi + 2\,c0\,Z\triangle^2\,\Lambda^2\Big)\,\hbar\Big) + c1\,Z\triangle^2\,\lambda\,\hbar\,\Big(6\,m^2 + (-1+n)\,ssi\,\lambda\,\hbar\Big)\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\Big) + c1\,n\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\Big(6\,c0\,\Lambda^2 + c1\,\Big(6\,m^2 + (-1+n)\,ssi\,\lambda\,\hbar\Big)\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\Big)\,\delta\lambda_{2\,b}\Big)\Big)\Big) \Big/ \Big(6\,\Big(-3 + c1\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)\,\Big(-6 + c1\,(2+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big)\Big) == 0,$$

$$-\frac{1}{2\,c1\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]}\Big(2 + c1\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + \frac{6\,(-1+n)}{n\,\Big(-3 + c1\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)} + 12 \Big/ \Big(n\,\Big(-6 + c1\,(2+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big)\Big) == 0,$$

$$\Big((-1+n)\,\hbar\,\Big(-Z\triangle^2\,\delta\lambda_{2\,a}\,\Big(-18 + c1\,n\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\Big(-3 + c1\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\Big) + c1^2\,n\,Z\triangle^2\,\lambda\,\hbar^2\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]^2\,\delta\lambda_{2\,b}\Big) + \lambda\,\Big(18\,\Big(-1 + Z\triangle^2\Big) + c1\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\Big(3\,(4+n) - c1\,(2+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\Big) - c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\,\Big(-12 + c1\,(4+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + 2\,c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)\Big)\Big)\Big) \Big/ \Big(6\,\Big(-3 + c1\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)\,\Big(-6 + c1\,(2+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big) == 0,$$

$$True,\ -\frac{\lambda}{2} - \frac{(-1+n)\,Z\triangle\,(\lambda + \delta\lambda_b)}{n\,\Big(-3 + c1\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,\delta\lambda_{2\,b}\Big)} - (Z\triangle\,((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b)) \Big/ \Big(n\,\Big(-6 + c1\,(2+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] + c1\,Z\triangle^2\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big) == 0,$$

$$(-6 + 6\,Z\,Z\triangle) \Big/ \Big(-6 + c1\,(2+n)\,Z\triangle^2\,\lambda\,\hbar\,Log\Big[\frac{\Lambda^2}{\mu^2}\Big] +$$

$$c1\, Z\Delta^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\, (n\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\Big) == 0\}$$

Solve for counterterms

In[50]:= **cteqs = {cteq, cteq2} // Flatten // FullSimplify // DeleteDuplicates**

Out[50]=

$$\Bigg\{\frac{1}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\Bigg(6\,\text{ssi} + 6\,c0\,\Lambda^2 + 6\,c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,\text{ssi}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,n\,\text{ssi}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{18\,\text{ssi}}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)} + \left(36\left((-1+n)\,\text{ssi} + c0\,n\,\Lambda^2 + c1\,m^2\,n\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,n\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta m_1^2\right)\right)\Big/ \left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right)\Bigg) == 0,$$

$$\hbar\left(\lambda - \left(18\,Z\Delta^2\,(\lambda + \delta\lambda_{2\,a})\right)\Big/\left(\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right)\right) == 0,$$

$$\frac{1}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\Bigg(6 + c1\,(1+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \frac{18}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)} + (36\,(-1+n))\Big/ \left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right)\Bigg) == 0,$$

$$\text{True},\ \lambda + (6\,Z\Delta\,((2+n)\,\lambda + n\,\delta\lambda_a + 2\,\delta\lambda_b))\Big/ \left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right) == \frac{6\,Z\Delta\,(\lambda + \delta\lambda_b)}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)},$$

$$(-1 + Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right) == 0,$$

$$\frac{1}{c1\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\Bigg(6\,c0\,\Lambda^2 + 6\,c1\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] - c1\,\text{ssi}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,n\,\text{ssi}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] - \frac{18\,(-1+n)\,\text{ssi}}{n\left(-3 + c1\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)} + \left(36\left((-1+n)\,\text{ssi} + c0\,n\,\Lambda^2 + c1\,m^2\,n\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,n\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta m_1^2\right)\right)\Big/ \left(n\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right)\Bigg) == 0,$$

$$\frac{1}{\text{c1 Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\left(2+\text{c1}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\frac{6\,(-1+n)}{n\left(-3+\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)}+\right.$$
$$\left.12\Big/\left(n\left(-6+\text{c1}\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\right)\right)==0,$$

$$(-1+n)\,\hbar\left(\lambda-\left(18\,Z\Delta^2\,(\lambda+\delta\lambda_{2\,a})\right)\Big/\left(\left(-3+\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)\right.\right.$$
$$\left.\left.\left(-6+\text{c1}\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\right)\right)==0,$$

$$\lambda+\frac{2\,(-1+n)\,Z\Delta\,(\lambda+\delta\lambda_b)}{n\left(-3+\text{c1}\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,\delta\lambda_{2\,b}\right)}+\left(2\,Z\Delta\,((2+n)\,\lambda+n\,\delta\lambda_a+2\,\delta\lambda_b)\right)\Big/$$
$$\left(n\left(-6+\text{c1}\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\text{c1}\,Z\Delta^2\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\,(n\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})\right)\right)==0\Big\}$$

In[51]:= **ctsolns =**
Solve[cteqs, {δm₁, δλ₁ₐ, δλ₂ₐ, δλ₁ᵦ, δλ₂ᵦ, Z, ZΔ}] // FullSimplify // DeleteDuplicates

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[51]=
$$\Big\{\Big\{\delta m_1\to-\frac{i\sqrt{2+n}\,\sqrt{\lambda}\,\sqrt{\hbar}\,\sqrt{\text{c0}\,\Lambda^2+\text{c1}\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}}{\sqrt{6+\text{c1}\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}},$$
$$\delta\lambda_a\to\lambda\left(-1+\frac{6\,(2+n)}{n\,Z\Delta\left(6+\text{c1}\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}-\frac{6}{3\,n\,Z\Delta+\text{c1}\,n\,Z\Delta\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),$$
$$\delta\lambda_{2\,a}\to\lambda\left(-1+\frac{18}{Z\Delta^2\left(3+\text{c1}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left(6+\text{c1}\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\delta\lambda_b\to\lambda\left(-1+\frac{3}{3\,Z\Delta+\text{c1}\,Z\Delta\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \delta\lambda_{2\,b}\to\lambda\left(-1+\frac{3}{Z\Delta^2\left(3+\text{c1}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ Z\to\frac{1}{Z\Delta}\Big\},$$

$$\Big\{\delta m_1\to\frac{i\sqrt{2+n}\,\sqrt{\lambda}\,\sqrt{\hbar}\,\sqrt{\text{c0}\,\Lambda^2+\text{c1}\,m^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}}{\sqrt{6+\text{c1}\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}},$$
$$\delta\lambda_a\to\lambda\left(-1+\frac{6\,(2+n)}{n\,Z\Delta\left(6+\text{c1}\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}-\frac{6}{3\,n\,Z\Delta+\text{c1}\,n\,Z\Delta\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),$$
$$\delta\lambda_{2\,a}\to\lambda\left(-1+\frac{18}{Z\Delta^2\left(3+\text{c1}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left(6+\text{c1}\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\delta\lambda_b\to\lambda\left(-1+\frac{3}{3\,Z\Delta+\text{c1}\,Z\Delta\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \delta\lambda_{2\,b}\to\lambda\left(-1+\frac{3}{Z\Delta^2\left(3+\text{c1}\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ Z\to\frac{1}{Z\Delta}\Big\}\Big\}$$

In[52]:= **cts = {δm₁², δλ₁ₐ, δλ₂ₐ, δλ₁ᵦ, δλ₂ᵦ, Z, ZΔ} /. ctsolns // FullSimplify // DeleteDuplicates**

Out[52]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$
$$\lambda\left(-1+\frac{6\,(2+n)}{n\,Z\Delta\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}-\frac{6}{3\,n\,Z\Delta+c1\,n\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),$$
$$\lambda\left(-1+\frac{18}{Z\Delta^2\,\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\left.\left.\lambda\left(-1+\frac{3}{3\,Z\Delta+c1\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{3}{Z\Delta^2\,\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

ZΔ is redundant in this truncation, can remove it :

In[53]:= **cts /. ZΔ → 1 // FullSimplify**

Out[53]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$
$$\lambda\left(-1-\frac{6}{3\,n+c1\,n\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}+\frac{6\,(2+n)}{n\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\lambda\left(-1+\frac{18}{\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\,\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),$$
$$\left.\left.\lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ 1,\ 1\right\}\right\}$$

In[54]:= **mg2soln /. ctsolns /. ZΔ → 1 // FullSimplify // DeleteDuplicates**

Out[54]=
$$\left\{\frac{1}{6}\left(6\,m^2+\lambda\,\left(v^2+\left((1+n)\,(ssi+tfing)+tfinn\right)\,\hbar\right)\right)\right\}$$

In[55]:= **mn2 /.**
((neom /. regularisedtadpoles /. mg2 → mg2soln /. ctsolns /. ZΔ → 1 // FullSimplify //
DeleteDuplicates) // Solve[#, mn2] &) // FullSimplify

Out[55]=
$$\left\{\frac{1}{6}\left(6\,m^2+3\,v^2\,\lambda+\left((-1+n)\,(ssi+tfing)+3\,tfinn\right)\,\lambda\,\hbar\right)\right\}$$

In[56]:= **rnveom =**

$$\text{veom} \;/.\; \left\{\text{mg2} \to m^2 + \frac{\lambda}{6} v^2 + \frac{\hbar}{6}\left(\left(n+1\right)\lambda\right)(\text{tfing}+\text{ssi}) + \frac{\hbar}{6}(\lambda)(\text{tfinn}),\ \text{mn2} \to m^2 + \frac{\lambda}{2} v^2 + \frac{\hbar}{6}\left(\left(n-1\right)\lambda\right)(\text{tfing}+\text{ssi}) + \frac{\hbar}{2}(\lambda)(\text{tfinn})\right\} \;//\; \text{Simplify} \;//\; \text{DeleteDuplicates}$$

Out[56]=
$$\frac{1}{6} v \left(6\,\text{ssi2} + \frac{6\left(m^2 + \delta m_0^2\right)}{Z\triangle} + v^2\left(\lambda + \delta\lambda_0\right) + (-1+n)(\text{ssi}+\text{tfing}+\text{t}\infty\text{g})\, Z\triangle\, \hbar\left(\lambda + \delta\lambda_a\right) + (\text{tfinn}+\text{t}\infty\text{n})\, Z\triangle\, \hbar\left(3\lambda + \delta\lambda_a + 2\,\delta\lambda_b\right)\right)$$

In[57]:= **veomCtEqs =**

$$\left(\left(\left(\text{CoefficientList}\left[\left(\frac{1}{v}\,\text{rnveom} - \left(m^2 + \frac{\lambda}{6} v^2 + \frac{\hbar}{6}\left(\left(n-1\right)\lambda\right)(\text{tfing}+\text{ssi}) + \frac{\hbar}{2}(\lambda)(\text{tfinn}) + \text{ssi2}\right)\right) \;/.\; \text{regularisedtadpoles} \;/.\; \left\{\text{mg2} \to m^2 + \frac{\lambda}{6} v^2 + \frac{\hbar}{6}\left(\left(n+1\right)\lambda\right)(\text{tfing}+\text{ssi}) + \frac{\hbar}{6}(\lambda)(\text{tfinn}),\ \text{mn2} \to m^2 + \frac{\lambda}{2} v^2 + \frac{\hbar}{6}\left(\left(n-1\right)\lambda\right)(\text{tfing}+\text{ssi}) + \frac{\hbar}{2}(\lambda)(\text{tfinn})\right\} \;//\; \text{Simplify} \;//\; \text{Expand} \;//\; \text{FullSimplify},\ \{v,\ \text{tfing},\ \text{tfinn}\}\right] \;//\; \text{Simplify} \;//\; \text{Flatten}\right) \;//\; \text{DeleteDuplicates} \;//\; \text{Simplify} \;//\; \text{FullSimplify} \;//\; \text{DeleteDuplicates}\right) == 0 \;//\; \text{Thread}\right)$$

Out[57]=
$$\Big\{\frac{1}{36\, Z\triangle}\Big(-36\, m^2(-1+Z\triangle) + 6\, Z\triangle\,\lambda\left((-1+n)\,\text{ssi}\,(-1+Z\triangle) + c0\,(2+n)\,Z\triangle\,\Lambda^2\right)\hbar + c1\, Z\triangle^2\,\lambda\,\hbar\left(6\, m^2(2+n) + (-1+n)(4+n)\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + 36\,\delta m_0^2 + Z\triangle^2\,\hbar\left(\left(6(-1+n)\,\text{ssi} + 6\, c0\, n\,\Lambda^2 + c1\left(6\, m^2 n + \left(-2+n+n^2\right)\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_a + 2\left(6\, c0\,\Lambda^2 + c1\left(6\, m^2 + (-1+n)\,\text{ssi}\,\lambda\,\hbar\right)\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_b\right)\Big) == 0,$$

$$\frac{1}{36}\hbar\left(\lambda\left(18(-1+Z\triangle) + c1\,(8+n)\,Z\triangle\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) + Z\triangle\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_a + 6\, Z\triangle\left(2 + c1\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_b\right) == 0,$$

$$\frac{1}{36}(-1+n)\,\hbar\left(Z\triangle\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_a + \lambda\left(6(-1+Z\triangle) + c1\,(4+n)\,Z\triangle\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + 2\, c1\, Z\triangle\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_b\right)\right) == 0,$$

$$\text{True},\ \frac{1}{36}\left(6\,\delta\lambda_0 + c1\, Z\triangle\,\lambda\,\hbar\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left((8+n)\,\lambda + (2+n)\,\delta\lambda_a + 6\,\delta\lambda_b\right)\right) == 0\Big\}$$

In[58]:= `ctegs3 = (veomCtEqs /. ctsolns // Simplify // DeleteDuplicates // FullSimplify)[[1]]`

Out[58]=
$$\left\{\left(-6\,m^2\,(-1+Z\triangle)+c0\,(2+n)\,Z\triangle\,\lambda\,\Lambda^2\,\hbar+c1\,m^2\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]+\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta m_0^2\right)\Big/\left(Z\triangle\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\right)=0,\ \mathrm{True},\ \mathrm{True},\ \mathrm{True},\ 3\,\lambda\left(1+\frac{2-2\,n}{3\,n+c1\,n\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}-\frac{2\,(2+n)}{n\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right)+\delta\lambda_0=0\right\}$$

In[59]:= `{δm0^2, δλ0} /. Solve[ctegs3, {δm0, δλ0}] /. Z△ → 1 // DeleteDuplicates // Simplify`

Out[59]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ -\frac{3\,c1\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(8+n+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{\left(3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left(6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right\}\right\}$$

In[60]:= `{δm1^2 == δm0^2, δλ1a == δλ2a, δλ1b == δλ2b} /. ctsolns /. Solve[ctegs3, {δm0, δλ0}] /. Z△ → 1 // FullSimplify // Flatten // DeleteDuplicates`

Out[60]= {True}

In[61]:= `{δλ1a/δλ1b} /. ctsolns /. Solve[ctegs3, {δm0, δλ0}] /. Z△ → 1 // FullSimplify // Flatten // DeleteDuplicates`

Out[61]=
$$\left\{1+\frac{3\,(2+n)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right\}$$

In[62]:= `δλ1b == δλ2b /. ctsolns /. Z△ → 1 // FullSimplify // DeleteDuplicates`

Out[62]= {True}

In[63]:= `δλ1b /. ctsolns /. Z△ → 1 // FullSimplify // DeleteDuplicates`

Out[63]=
$$\left\{\lambda\left(-1+\frac{3}{3+c1\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right)\right\}$$

In[64]:= `{δλ0 == 1 δλ1a + 2 δλ1b} /. ctsolns /. Solve[ctegs3, {δm0, δλ0}] /. Z△ → 1 // FullSimplify // Flatten // DeleteDuplicates`

Out[64]= {True}

In[65]:=
$$\left\{\delta m_0^2=-\frac{\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\lambda\,\hbar}{3}\left(\frac{\delta\lambda_{1a}}{\delta\lambda_{1b}}-1\right)\right\}$$
`/. ctsolns /. Solve[ctegs3, {δm0, δλ0}] /. Z△ → 1 // FullSimplify // Flatten // DeleteDuplicates`

Out[65]= {True}