

INTRODUCTORY  
TRANSPORT THEORY  
FOR CHARGED  
PARTICLES IN GASES

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## **Dedication**

To Carola - this one is especially for you  
and  
to Kailash Kumar, teacher and friend

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# Preface

Many areas of basic physics research depend upon a good physical understanding of charged particle motion in gases, a statement which is as true now as it was in the early part of the last century, when modern physics was taking shape – see, for example, the Introduction to *The diffusion and drift of electrons in gases* (Wiley, 1974), by L.G.H. Huxley and R.W. Crompton for comments on the seminal experiments of J.J. Thomson, J.S. Townsend, and others. The same goes for technological applications, perhaps even more so. Thus, for example, there has been a huge investment of resources in technologies associated with low temperature plasmas, in the microchip industry in particular, and the contribution to the world economy amounts to many billions of dollars annually. Importantly, it is generally acknowledged that the full potential of such technology can be realised only when the basic physics associated with charged particle transport theory has been mastered, and inevitably this means solving Boltzmann’s partial differential-integral kinetic equation, or carrying out an equivalent computational simulation.

Although Boltzmann had already presented his famous equation for the velocity distribution function in 1872, it took some time for its significance in the description of charged particle in gases to become fully appreciated, and unfortunately misunderstandings persist to the present day. For example, the famous Franck-Hertz experiment of 1914, which confirmed the Bohr quantization postulates, is still often discussed crudely in terms of single electrons traversing a gas, all with the same velocity, using the same language as one would describe a single-scattering, monoenergetic beam experiment. The actual physical picture is quite different: electrons in fact undergo



many scatterings, and have a wide distribution of velocities. In general, a correct physical description of the real, macroscopic world, requires the microscopic, single-scattering quantities to be *averaged* over a distribution function, which is to be *calculated* from solution of Boltzmann's equation. This is the realm of the kinetic theory of gases, and is the theme of this book.

Although mainly simple, idealized systems of electrons and ions in gases are considered here, it is important to realize that much of the analysis carries over more or less directly to a much wider range of physical phenomena; for example, to positron annihilation in gases, muon catalyzed fusion in hydrogen and its isotopes, plasma processing technology, multiwire drift tube chambers used in high energy particle detectors, hot atom chemistry, spectral line broadening by perturbers in a foreign gas, and neutron transport. There is a one-to-one correspondence with charge carrier transport in semiconductors, where phonons arising from lattice vibrations play the role of a gas, and with the dispersion of pollutants in the turbulent atmospheric boundary layer. The range of applications is truly enormous.

This book derives from a graduate course given by the author at the Homer L. Dodge Department of Physics, the University of Oklahoma, in the Fall semester of 2005. Rather than writing a comprehensive treatise, the aim has been to provide an introduction to and overview of some of the theoretical methods used to investigate transport properties, all based on the premise that physical understanding, rather than unsparing rigour, is paramount.

My sincere thanks go to Michael Morrison for many stimulating discussions and for his constant encouragement and interest. To departmental Chair Ryan Doezema, and indeed to all Faculty and Staff, thank you for your help and friendship, and for making my stay in Norman so enjoyable and productive.

Robert Robson,  
Canberra,  
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