A Note on Demographic Shocks in a Multi-Sector Growth Model

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Forthcoming in *Economics Bulletin*

Abstract

We introduce demographic shocks in a multi-sector endogenous growth model, á-la Uzawa-Lucas. We show that an analytical solution of the stochastic problem can be found, under the restriction that the capital share equals both the inverse of the intertemporal elasticity of substitution and the degree of altruism. We show that uncertainty lowers the optimal levels of consumption and the physical capital stock, while they do not affect the share of human capital employed in production.
Keywords: Demographic Shocks, Economic Growth, Closed-form Solution
JEL Classification: O40, O41, J13

1 Introduction

Demographic shocks consist of changes in population growth rates or immigration policies, and their main effect is altering the size of labor force (population), without automatically affecting the physical or human capital stock. They are thought to have important effects on macroeconomic variables such as growth rates and investment decisions. A benchmark model to study the effects of demographic shocks is the Uzawa (1965) and Lucas (1988) model. Robertson (2002) studies the transitional effects of demographic shocks in a model á-la Uzawa-Lucas in which unskilled labor is a separate factor of production. However, he analyzes such shocks simply through a comparative statics exercise.

The aim of this paper is twofold. First, in order to explicitly analyze the implications of demographic shocks on the economy, we assume population is hit by random shocks driven by a geometric Brownian motion, as in Smith (2007). We study an augmented two-sector model of endogenous-growth, in which unskilled labor is a separate factor of production, showing that a closed-form solution can be found under the condition that the altruism parameter equals both the inverse of the intertemporal elasticity of substitution and the physical capital share. In such a case, population shocks lower the optimal levels of consumption and the physical capital stock, while they do not affect the share of human capital devoted to production. Second, we contribute to the literature on closed-form solutions of continuous time growth model, by providing a different solution with respect to that given by Bucci et al. (2011) for a two-sector economy case.

The paper proceeds as follows: Section 2 presents the model and the main results, by considering also a particular case in which unskilled labor is not a separate production input, while Section 3 as usual concludes.

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2 The Model

The model is a Uzawa-Lucas model of optimal growth where the representative agent seeks to maximize his welfare subject to the capital and demographic constraints, choosing consumption, c_t, and the share of human capital employed in production, u_t. The welfare is the expected infinite discounted sum of the product of the instantaneous utility function (assumed to be iso-elastic, $u(c(t)) = \frac{c(t)^{1-\theta}-1}{1-\theta}$, where $\theta > 0$ is the inverse of the intertemporal elasticity of substitution) and the population size, $N(t)$ (weighted by the agent's degree of altruism). Physical capital, $K(t)$, accumulation is given by the difference between net (of depreciation, δ_K) production of the unique final good and consumption activity: $\dot{K}(t) = AK(t)^\alpha[u(t)H(t)]^\beta N(t)^{1-\alpha-\beta} - \delta_K K(t) - c(t)N(t)$. The law of motion of human capital, $H(t)$, is instead given by net (of depreciation, δ_H) production of new human capital: $\dot{H}(t) = B[1-u(t)]H(t) - \delta_H H(t)$. Demographic dynamics is instead stochastic and is driven by a geometric Brownian motion, as in Smith (2007): $dN(t) = \mu N(t)dt + N(t)\sigma dW(t)$, where μ is the drift and $\sigma \geq 0$ the variance parameter, while $dW(t)$ is the increment of a Wiener process such that $E[dW(t)] = 0$ and $\text{var}(dW(t)) = dt$.

The social planner maximizes the social welfare function by choosing $c(t)$ and $u(t)$ in order to maximize agents lifetime utility function subject to physical and human capital accumulation constraints, demographic dynamics and initial conditions:

$$\begin{aligned}
\max_{c(t),u(t)} \quad & W = E\left[\int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} N(t)^{1-\epsilon} e^{-\rho t} dt\right] \\
s.t. \quad & \dot{K}(t) = AK(t)^\alpha[u(t)H(t)]^\beta N(t)^{1-\alpha-\beta} - \delta_K K(t) - c(t)N(t) \\
& \dot{H}(t) = B[1-u(t)]H(t) - \delta_H H(t) \\
& dN(t) = \mu N(t)dt + N(t)\sigma dW(t) \\
& K(0),\ H(0),\ N(0) \text{ given.}
\end{aligned}$$

The term $1-\epsilon$ determines the welfare function. Two extreme cases, $\epsilon = 0$ and $\epsilon = 1$, representing respectively the case in which welfare is of the Benthamite and Millian type (see for example Palivos and Yip (1993)), are mostly discussed in the literature. As in Nerlove et al. (1982) and Strulik (2005), we instead assume that a continuum of intermediate cases exists, and therefore $1-\epsilon \in [0,1]$ controls for the degree of altruism towards future generations[1]. The altruism is maximal (minimal) if $\epsilon = 0$ ($\epsilon = 1$) while for medium values the altruism is instead said to be impure. Notice that unless the case $\epsilon = 1$, demographic shocks have two different effects: both the physical sector and the objective function are directly affected.

First of all, notice that the previous problem is totally equivalent to the following:

$$\begin{aligned}
\max_{C(t),u(t)} \quad & W = E\left[\int_0^\infty \left(\frac{C(t)^{1-\theta}}{1-\theta} N(t)^{\theta-\epsilon} - \frac{1}{1-\theta} N(t)^{1-\epsilon}\right) e^{-\rho t} dt\right] && (1) \\
s.t. \quad & \dot{K}(t) = AK(t)^\alpha[u(t)H(t)]^\beta N(t)^{1-\alpha-\beta} - \delta_K K(t) - C(t) && (2) \\
& \dot{H}(t) = B[1-u(t)]H(t) - \delta_H H(t) && (3) \\
& dN(t) = \mu N(t)dt + N(t)\sigma dW(t) && (4) \\
& K(0),\ H(0),\ N(0) \text{ given,} && (5)
\end{aligned}$$

where $C(t) = c(t)N(t)$ represents aggregate consumption.

Define $J(K(t), H(t), N(t))$ as the maximum expected value associated with the stochastic optimization problem described above. The Hamilton-Jacobi-Bellman (HJB) equation is:

$$\rho J = \max_{C(t),u(t)} \left\{ \frac{C(t)^{1-\theta}N(t)^{\theta-\epsilon}}{1-\theta} - \frac{N(t)^{1-\epsilon}}{1-\theta} + J_K \dot{K}(t) + J_H \dot{H}(t) + J_N \mu N(t) + \right.$$

[1] See also Bucci and La Torre (2009) who analyze a Uzawa-Lucas type of model where not only education but also investment in physical capital determine human capital accumulation.

$$+\frac{J_{NN}\sigma^2 N(t)^2}{2}\Bigg\}, \tag{6}$$

where the differential equations for $K(t)$ and $H(t)$ are defined in (2) and (3) and subscripts denote partial derivatives of J with respect to the relevant variables of interest. Differentiating (6) with respect to the control variables gives:

$$\begin{aligned} C(t) &= J_K^{-\frac{1}{\theta}} N(t)^{\frac{\theta-\epsilon}{\theta}}, & (7)\\ u(t) &= \left(\frac{\beta A J_K}{B J_H}\right)^{\frac{1}{1-\beta}} K(t)^{\frac{\alpha}{1-\beta}} H(t)^{-1} N(t)^{\frac{1-\alpha-\beta}{1-\beta}}, & (8)\end{aligned}$$

which substituted back into (6) yield:

$$\begin{aligned} \rho J &= \left(\frac{\theta}{1-\theta}\right) J_K^{-\frac{1-\theta}{\theta}} N(t)^{\frac{\theta-\epsilon}{\theta}} - \frac{1}{1-\theta} N(t)^{1-\epsilon} - \delta_K J_K K(t) + (B-\delta_H) J_H H(t) + \\ &\quad +(1-\beta)\beta^{\frac{\beta}{1-\beta}} [A J_K]^{\frac{1}{1-\beta}} [B J_H]^{-\frac{\beta}{1-\beta}} K(t)^{\frac{\alpha}{1-\beta}} N(t)^{\frac{1-\alpha-\beta}{1-\beta}} + \mu J_N N(t) + \frac{1}{2}\sigma^2 J_{NN} N(t)^2. \end{aligned} \tag{9}$$

By applying the "guess and verify" method to the previous equation, it is possible to show that a closed form solution to the problem exists under a particular combination of parameter values.

Proposition 1. *Assume that* $\theta = \epsilon = \alpha$ *and* $B(1-u) - \delta_H - \rho + (1-\alpha-\beta)\left[\mu - \frac{1}{2}\sigma^2(\alpha+\beta)\right] < 0$, *where* u *is defined later in (12); then (9) has a solution given by:*

$$J(K(t), H(t), N(t)) = T_1 K(t)^{1-\alpha} + T_2 H(t)^{\beta} N(t)^{1-\alpha-\beta} + T_3 N(t)^{1-\alpha}, \tag{10}$$

where:

$$\begin{aligned} T_1 &= \frac{\alpha^\alpha}{(1-\alpha)[\rho + (1-\alpha)\delta_K]^\alpha}, \quad T_3 = \frac{1}{(1-\alpha)[\mu(1-\alpha) - \frac{1}{2}\sigma^2(1-\alpha)\alpha - \rho]}, \\ T_2 &= \frac{(1-\beta)^{1-\beta} A[(1-\alpha)T_1]}{B^\beta\{\rho - \beta(B-\delta_H) - (1-\alpha-\beta)[\mu - \frac{1}{2}\sigma^2(\alpha+\beta)]\}^{1-\beta}}. \end{aligned}$$

Moreover, the optimal rules for the level of consumption and share of human capital devoted to production are given by:

$$\begin{aligned} C(t) &= \Omega K(t), & (11)\\ u(t) &= u = \left[\frac{A(1-\alpha)T_1}{B T_2}\right]^{\frac{1}{1-\beta}}, & (12)\end{aligned}$$

while the optimal paths of human and physical capital are respectively:

$$\begin{aligned} H(t) &= H(0) e^{[B(1-u)-\delta_H]t}, & (13)\\ K(t) &= e^{-(\delta_K+\Omega)t}\left[K(0)^{1-\alpha} + (1-\alpha)\Psi \int_0^t e^{\{(1-\alpha)(\delta_K+\Omega)+\beta[B(1-u)-\delta_H]\}s} N(s)^{1-\alpha-\beta} ds\right]^{\frac{1}{1-\alpha}}, & (14)\end{aligned}$$

where $\Omega = \frac{\rho + (1-\alpha)\delta_K}{\alpha}$ *and* $\Psi = A u^\beta H_0^\beta$.

Notice that this parameter restriction is almost standard in this kind of models: Smith (2006, 2007) uses the restriction $\theta = \alpha$ to analytically solve both a deterministic and stochastic version of a (Ramsey type) one sector model[2], while Marsiglio and La Torre (2012) show that the further condition $\epsilon = \theta$ is needed in a two sector framework if population growth is not constant[3]. Bucci et al. (2011) look for a closed-form solution of a Uzawa-Lucas type growth model similar to ours, in which technology is the random factor, rather than demography. We note that our guessed value function, being non-linear in human capital, allows us not to impose any restriction to the value of the rate of time preference. Equation (11) provides a result similar to that of the AK-model since for all t, there exists a linear relationship between the optimal level of consumption and capital, as in Smith (2007). Equation (12) shows that the optimal share of human capital employed in physical production is constant, as in Bethman (2007). Proposition 1 shows that the optimal levels of the state variable $K(t)$ is function of $N(t)$, a random variable, while $H(t)$ is independent of it. By making reference Jensen's inequality for a concave function of $N(t)$ we are therefore able to contrast the results of the deterministic version of the model, which is indicated with a subscript D, with those of the stochastic version.

Proposition 2. *Assume that $\theta = \epsilon = \alpha$; then we have for all $t = 0, \ldots, \infty$:*

$$E[K(t)^{1-\alpha}] \leq [K_D(t)]^{1-\alpha}, \quad E[C(t)^{1-\alpha}] \leq [C_D(t)]^{1-\alpha},$$
$$E[H(t)] = H_D(t), \quad E[u(t)] = u_D(t).$$

Shocks on population lower on average the optimal level of consumption and the stock of physical capital, while they do not affect the rate of investment in physical capital and the human capital stock.

2.1 The Case $\beta = 1 - \alpha$

If $\beta = 1 - \alpha$, unskilled labor is not an input in the production of the consumption good and as a result the optimal paths of physical and human capital are independent of population. The effects of population shocks are instead present in per-capita variables, $k(t) = \frac{K(t)}{N(t)}$ and $h(t) = \frac{H(t)}{N(t)}$. In fact, using Ito's lemma, the law of motion of per-capita physical and human capital are respectively given by:

$$\frac{dk(t)}{k(t)} = \left[Au^{1-\alpha}\left(\frac{k(t)}{h(t)}\right)^{\alpha-1} - \delta_K - \Omega - \mu + \sigma^2\right] dt - \sigma dW(t) \tag{15}$$

$$\frac{dh(t)}{h(t)} = \left[B(1-u) - \delta_H - \mu + \sigma^2\right] dt - \sigma dW(t). \tag{16}$$

In order to understand the role of population shocks, we need to take expectations of per-capita physical and human capital. Using the fact that $E[X(t)] = X(0)e^{(\sigma^2-\mu)t}$, where $X(t) = \frac{1}{N(t)}$, and since $E[k(t)] = E[\frac{K(t)}{N(t)}] = E[K(t)X(t)]$ it is straightforward finding the expected value of $k(t)$:

$$E[k(t)] = e^{(\sigma^2-\mu-\delta_K-\Omega)t}\left[k(0)^{1-\alpha} + \frac{Au^{1-\alpha}h(0)^{1-\alpha}\left(e^{\{(1-\alpha)[\delta_K+\Omega+B(1-u)-\delta_H]\}t} - 1\right)}{\delta_K + \Omega + B(1-u) - \delta_H}\right]^{\frac{1}{1-\alpha}}. \tag{17}$$

The same reasoning applies fot $h(t)$:

$$E[h(t)] = h(0)e^{[B(1-u)-\delta_H+\sigma^2-\mu]t}. \tag{18}$$

[2] The condition $\theta = \alpha$ has been firstly proposed by Xie (1991) to obtain the explicit dynamics of a deterministic Ramsey-type model. Xie (1994) uses the same parameter restriction in order to analyze explicitly the transitional dynamics in the Lucas model.

[3] It is possible to find an explicit solution of the Bellman equation even for the extreme values of the altruism parameter. However, both cases have already been analyzed by Smith (2007), when dealing with stochastic technology (it corresponds to $\epsilon = 1$ by reinterpreting technology as population) and stochastic population ($\epsilon = 0$). We expect that in a two-sector framework the results do not differ much from his. For this reason, we prefer presenting the solution for the impure altruism case, being probably more original and interesting.

Setting $\sigma = 0$ in (17) and (18) yields the levels of the state per-capita variable in the deterministic version of the model. It is straightforward proving the following result:

Proposition 3. *Suppose that $\theta = \epsilon = \alpha$; if $\beta = 1 - \alpha$, then we have for all $t = 0, \ldots, \infty$:*

$$E[k(t)] \geq k_D(t), \quad E[h(t)] \geq h_D(t).$$

According to this result, if unskilled labor is not a production factor, then uncertainty on labor force will increase on average the stock of per-capita human and physical capital, as discussed in Marsiglio and La Torre (2012).

3 Conclusion

The sudden variations in the migration flows, fertility and mortality rates make the evolution of population highly uncertain. For this reason, analyzing the implications of demographic shocks on the economy is particularly important in order to understand their effects on the optimal policy rules. In this paper we introduce random shocks driven by a geometric Brownian motion to the level of population in the Uzawa-Lucas model. We show that if the degree of altruism equals both the inverse of the intertemporal elasticity of substitution and the physical capital share, a closed-form solution of the stochastic optimization problem can be found. Moreover, shocks on population lower the optimal level of consumption and the stock of physical capital, while they do not affect the human capital stock. If instead unskilled labor is not a separate factor of production, uncertainty on labor force will increase on average the stock of per-capita human and physical capital. For further research, it would be interesting to study whether and how differences in the altruism parameter (that means adopting the Benthamite rather than the Millian criterion) can affect the optimal level of consumption in a stochastic framework.

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