Fractals and Self-Similarity in Economics: the Case of a Stochastic Two-Sector Growth Model

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Abstract

We study a stochastic, discrete-time, two-sector optimal growth model in which the production of the homogeneous consumption good uses a Cobb-Douglas technology, combining physical capital and an endogenously determined share of human capital. Education is intensive in human capital as in Lucas (1988), but the marginal returns of the share of human capital employed in education are decreasing, as suggested by Rebelo (1991). Assuming that the exogenous shocks are i.i.d. and affect both physical and human capital, we build specific configurations for the primitives of the model so that the optimal dynamics for the state variables can be converted, through an appropriate log-transformation, into an Iterated Function System converging to an invariant distribution supported on a generalized Sierpinski gasket.

Keywords: fractals, iterated function system, self-similarity, Sierpinski gasket, stochastic growth

JEL classification: C61, O41

1 Introduction

Mandelbrot (1982) in his seminal work presented the first description of *self-similar* sets, namely sets that may be expressed as unions of rescaled copies of themselves. He called these sets *fractals*, because their (fractional) Hausdorff-Besicovitch dimensions exceeded their (integer-valued) topological dimensions. The *Cantor set*, the *von Koch snowflake curve* and the *Sierpinski gasket* are some of the most famous examples of such sets. Hutchinson (1981) and, shortly thereafter, Barnsley and Demko (1985) and Barnsley (1989) showed how systems of contractive maps with associated probabilities, referred to as *Iterated Function Systems* (IFS), can be used to construct fractal, self-similar sets and measures supported on such sets. These sets and measures are attractive fixed points of fractal transform operators.

After these pioneering papers, applications of IFS theory in several fields have been widely developed, eventually landing, at the end of the last century, also into Economics. As a matter of fact,

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economists are intrinsically reluctant to accept the idea that economic dynamics may generate fractals. A first breakthrough has been introduced by Boldrin and Montrucchio (1986), who showed that complicated (chaotic) optimal dynamics can occur in deterministic concave intertemporal optimization models when the discount factor is small enough. This result opened a new chapter in mainstream Economics, starting a huge literature aimed at studying complexity and chaos in almost all economic fields. Prominent, but by no means exhaustive,[1] references are Montrucchio (1994), Nishimura and Yano (1995), Brock and Hommes (1997) and, more recently, Gardini *et al.* (2009), who exploited the IFS framework to construct a deterministic OLG-model converging to a fractal attractor.

A decade later complex behavior started to be investigated in stochastic concave intertemporal optimization models as well. Montrucchio and Privileggi (1999) borrowed from the literature on fractal images generation (specifically, from the 'Collage Theorem' by Hutchinson, 1981; Barnsley, 1989; Vrscay, 1991) to show that standard stochastic concave optimal growth models may exhibit optimal trajectories which are random processes converging to singular invariant distributions supported on fractal sets regardless of the discount factor. Such economies have optimal dynamics defined by IFS with *linear maps*. Mitra *et al.* (2004) investigated a simple one-sector growth model with two random shocks whose optimal path is defined by a linear IFS which, for some values of parameters, converges to a singular distribution supported on a Cantor set. They also characterized singularity versus absolute continuity of the invariant probability in terms of (almost) all parameters' values. Mitra and Privileggi (2004, 2006) further generalized that model and eventually (2009) provided an estimate of the Lipschitz constant for the (nonlinear) maps of the IFS defining the optimal policy in a class of stochastic one-sector optimal growth models in the Brock and Mirman (1972) tradition. This result yields sufficient conditions for the model to converge to a singular distribution supported on a generalized Cantor set directly in terms of the parameters' values.

In this paper we consider a neoclassic stochastic, discrete-time, two-sector growth model in which production of a unique homogeneous good depends on both physical and human capital through a Cobb-Douglas technology, while education requires only human capital, as suggested by Lucas (1988). However, we modify the Lucas (1988) framework by postulating that the marginal returns of the human capital employed in education are decreasing, thus embedding Rebelo (1991) assumption. Production in both sectors is multiplicatively affected by random i.i.d. shocks taking on a finite number of values. Our main contribution is to provide sufficient conditions on the parameters of the model – namely, on the exponents of the Cobb-Douglas production function and of the human capital production function, and on the values of random shocks – such that the IFS corresponding to the optimal policy function converges to a unique invariant distribution supported on a (generalized) Sierpinski gasket. Hence, this result can be seen as a further extension of the approach pursued by Mitra and Privileggi (2004, 2006, 2009) for the one-sector growth model to a multi-sector growth model under uncertainty.

In Section 2 the main results from the IFS theory are briefly recalled. In Section 3 the model is stated and the optimal dynamics are explicitly computed. Section 4 contains the central contribution of this paper: a linear IFS conjugate to the true optimal dynamics is constructed and sufficient conditions for its attractor to be a Sierpinski gasket supporting the unique invariant distribution of the economy are provided directly in terms of parameters of the model. Finally, in Section 5 a few examples of economies converging to differently shaped Sierpinski gaskets are described, while Section 6 reports some concluding remarks. All proofs are gathered in the Appendix.

[1]For a recent and quite comprehensive survey on complex dynamics arising in non-competitive economies see Bischi *et al.* (2010) and the references listed there.

2 Iterated Function Systems

Iterated Function Systems allow to formalize the notion of self-similarity or scale invariance of some mathematical object. Hutchinson (1981) and Barnsley and Demko (1985) showed how systems of contractive maps with associated probabilities can be used to construct self-similar sets and measures. In the IFS literature, these are called IFS with probabilities (IFSP) and are based on the action of a contractive Markov operator on the complete metric space of all Borel probability measures endowed with the Monge-Kantorovich metric. Applications of these methods can be found in image compression, approximation theory, signal analysis, denoising, and density estimation (see, *e.g.*, Mendivil and Vrscay, 2002a,b; Iacus and La Torre, 2005a,b; La Torre *et al.*, 2006; Kunze *et al.*, 2007; La Torre and Mendivil, 2008, 2009; La Torre *et al.*, 2009; La Torre and Vrscay, 2009; Freiberg *et al.*, 2011). In what follows, let (X, d) be a complete metric space and $w = \{w_1, \ldots, w_N\}$ be a family of injective contraction maps $w_i : X \to X$, to be referred to as an N-map IFS. Let $c_i \in (0, 1)$ denote the contraction factor of w_i and define $c = \max_{i\in\{1,\ldots,N\}} c_i$. Note that $c \in (0, 1)$. Associated with the IFS mappings $w_1, \ldots, w_N$ there is a set-valued mapping $\hat{w} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined over the space $\mathcal{K}(X)$ of all non-empty compact sets in X as:

$$\hat{w}(S) = \bigcup_{i=1}^{N} w_i(S), \qquad S \in \mathcal{K}(X), \tag{1}$$

where $w_i(S) = \{w_i(x) : x \in S\}$ is the image of S under w_i, for $i = 1, \ldots, N$. A set $S_w \subset X$ is said to be an *invariant set* of w if it is compact and it is invariant under (1), that is, it satisfies $\hat{w}(S_w) = S_w$. If in addition, the contractive mappings w_i are assumed to be similitudes, *i.e.*, if we assume that there exist numbers $c_i \in (0, 1)$ such that

$$d(w_i(x), w_i(y)) = c_i d(x, y), \qquad x, y \in X, \quad i = 1, \ldots, N,$$

the invariant set S_w is said to be *self–similar*. In $\mathcal{K}(X)$ it is possible to define the so-called Hausdorff distance d_H between compact sets which reads as:

$$d_H(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{x\in B}\inf_{y\in A} d(x, y)\right\},$$

and it can be proved that $(\mathcal{K}(X), d_H)$ is a complete metric space (Hutchinson, 1981).

Theorem 1 (Hutchinson, 1981) *$\hat{w}$ is a contraction mapping on $(\mathcal{K}(X), d_H)$; specifically:*

$$d_H(\hat{w}(A), \hat{w}(B)) \le c d_H(A, B), \qquad \forall A, B \in \mathcal{K}(X).$$

We have the following corollary from the Banach fixed point theorem.

Corollary 1 *There exists a unique compact set $A \in \mathcal{K}(X)$, such that $\hat{w}(A) = A$, which is called the* attractor *of the IFS w. Moreover, for any $S \in \mathcal{K}(X)$, $d_H(\hat{w}^n(S), A) \to 0$ as $n \to \infty$.*

The latter property provides a construction method of approximating a fractal. The equation $\hat{w}(A) = A$ obviously implies that A is *self-tiling*, *i.e.*, A is the union of (distorted) copies of itself.

Let $\mathcal{M}(X)$ be the space of probability measures defined on the σ-algebra $\mathcal{B}(X)$ of Borel measurable subsets of X and define for some $a \in X$ the set:

$$\mathcal{M}_1(X) = \left\{\mu \in \mathcal{M}(X) : \int_X d(a, x)\, d\mu(x) < \infty\right\}.$$

Note that the definition of $\mathcal{M}_1(X)$ does not depend on the choice of a (if the integral is finite for a certain $a \in X$ then it is finite for all $a \in X$). For $\mu, \nu \in \mathcal{M}_1(X)$, we define the Monge-Kantorovich distance as follows:

$$d_M(\mu, \nu) = \sup \left\{ \int_X f d(\mu - \nu) : f \in \mathcal{L}ip_1(X) \right\},$$

where $\mathcal{L}ip_1$ is the set of all Lipschitz functions with Lipschitz constant equal to 1. It can be proved that $(\mathcal{M}_1(X), d_M)$ is a complete metric space under the Monge-Kantorovich metric provided that X is a separable complete metric space. Furthermore, if X is compact, then $\mathcal{M}(X) = \mathcal{M}_1(X)$ and both are compact metric spaces under the Monge-Kantorovich distance (Barnsley *et al.*, 2008).

Let $p = (p_1, p_2, \dots, p_N)$, $0 < p_i < 1$, $1 \leq i \leq N$, be a partition of unity associated with the IFS mappings w_i, so that $\sum_{i=1}^{N} p_i = 1$. Associated with this IFS with probabilities (IFSP) (w, p) is the so-called Markov operator, $M : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$, defined as:

$$(M\mu)(S) = \sum_{i=1}^{N} p_i \mu\left(w_i^{-1}(S)\right), \qquad \forall S \in \mathcal{B}(X),$$

where $w_i^{-1}(S) = \{y \in X : w_i(y) \in S\}$.

Theorem 2 (Barnsley *et al.*, 2008) *M is a contraction mapping on $(\mathcal{M}_1(X), d_M)$; specifically:*

$$d_M(M\mu, M\nu) \leq \left(\sum_i p_i c_i\right) d_M(\mu, \nu), \qquad \forall \mu, \nu \in \mathcal{M}_1(X).$$

Corollary 2 *There exists a unique probability measure $\bar{\mu} \in \mathcal{M}_1(X)$, called* invariant measure *of the IFSP (w, p), such that $M\bar{\mu} = \bar{\mu}$. Moreover, for any $\mu \in \mathcal{M}_1(X)$, $d_M(M^n\mu, \bar{\mu}) \to 0$ as $n \to \infty$.*

Note that for any μ-integrable function $u : X \to \mathbb{R}$, it holds that:

$$\int_X u(x)\, d\mu(x) = \sum_{i=1}^{N} p_i \int_X u[w_i(x)]\, d\mu(x).$$

Let $C^0(X)$ denote the Banach space of continuous functions on X endowed with the uniform metric d_∞. Associated with the IFSP (w, p) define the following operator $T : C^0(X) \to C^0(X)$:

$$Tu = \sum_{i=1}^{N} p_i (u \circ w_i), \qquad \forall u \in C^0(X).$$

For a given $\nu \in \mathcal{M}_1(X)$ define the linear functional $F_\nu : C^0(X) \to \mathbb{R}$ as:

$$F_\nu(u) = \langle u, \nu \rangle = \int_X u(x)\, d\nu(x).$$

Then $\langle Tf, \nu \rangle = \langle f, M\nu \rangle$, *i.e.*, T is the adjoint operator of M. The operator T is a contraction on the complete metric space $(C^0(X), d_\infty)$ with contraction factor $p = \max_{i \in \{1,\dots,N\}} p_i < 1$. Thus we have:

$$\int_X u(x)\, d\mu(x) = \lim_{n \to +\infty} \int_X T^n f(x)\, d\mu_n(x)$$

where $\mu_n = M^n\lambda \to \mu$ in the Monge-Kantorovich distance and λ is the Lebesgue measure on X.

It is worth mentioning the concept of V*-variable fractals* recently introduced by Barnsley *et al*. (2008) allowing for the description of new families of random fractals, which are intermediate between deterministic and random fractals, including recursive as well as homogeneous random fractals. More precisely, given a (not necessarily finite) family of IFSP's, such fractals are the result of random applications of the related set valued mappings and measure valued Markov operators. The parameter V describes the degree of "variability" of the realizations. Roughly speaking, this means that at each construction step we have at most V different fundamental shapes.

3 The Model

We study an optimal growth model under uncertainty in which the social planner seeks to maximize the representative household's infinite discounted sum of instantaneous utility functions – which are assumed to be logarithmic – subject to the laws of motion of physical, k_t, and human, h_t, capital. At each time t, the planner chooses consumption, c_t, and the share of human capital, u_t, to allocate into production of a unique homogeneous consumption good which uses a Cobb-Douglas technology that combines physical and human capital. Education is assumed to be intensive in human capital, as in Lucas (1988), but the marginal returns of the share of human capital employed in education are decreasing, in accordance with Rebelo (1991).

The final good and the education sectors are affected by exogenous perturbations, z_t and η_t respectively, which enter multiplicatively both production functions; they are independent and identically distributed, and take on finite values: $z \in \{q_1, q_2, 1\}$ and $\eta \in \{r, 1\}$, with $0 < q_1 < q_2 < 1$ and $0 < r < 1$. We assume that only three pairs of shock values can occur with positive probability, $(z, \eta) \in \{(q_1, r), (q_2, 1), (1, 1)\}$, each with (constant) probability p_1, p_2 and p_3 respectively, where $p_i \in (0, 1)$, $i = 1, 2, 3$, and $\sum_{i=1}^{3} p_i = 1$. Such three shock configurations may be interpreted as 1) a deep financial crisis typically having wide effects on the economy as a whole and thus involving both production and education sectors,[2] corresponding to $(z, \eta) = (q_1, r)$, 2) a sudden surge in raw materials' (*e.g.*, oil) prices affecting only the production sector but not education, corresponding to $(z, \eta) = (q_2, 1)$, and 3) a scenario with no shocks in which the whole economy evolves along its full capacity, corresponding to $(z, \eta) = (1, 1)$.

The social planner problem can thus be summarized as:

$$V(k_0, h_0, z_0, \eta_0) = \max_{\{c_t, u_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \tag{2}$$

$$\text{s.t.} \begin{cases} k_{t+1} = z_t k_t^{\alpha} (u_t h_t)^{1-\alpha} - c_t \\ h_{t+1} = \eta_t [(1 - u_t) h_t]^{\phi} \\ k_0 > 0,\ h_0 > 0,\ z_0 \in \{q_1, q_2, 1\},\ \eta_0 \in \{r, 1\} \text{ are given,} \end{cases} \tag{3}$$

where $\mathbb{E}_0$ denotes expectation at time $t = 0$, $0 < \beta < 1$ is the discount factor, k_t and h_t denote physical and human capital at time t, $0 < \alpha < 1$ and $0 < \phi < 1$.

The Bellman equation associated to (2) reads as:

$$V(k_t, h_t, z_t, \eta_t) = \max_{c_t, u_t} [\ln c_t + \beta \mathbb{E}_t V(k_{t+1}, h_{t+1}, z_{t+1}, \eta_{t+1})]. \tag{4}$$

Thanks to the log-Cobb-Douglas specification of the model, both the value function $V(\cdot, \cdot, \cdot, \cdot)$ and the optimal policy of (2) can be explicitly computed by applying the "guess and verify" method[3] to the Bellman equation (4).

[2]Consider, for example, the global financial crisis triggered in 2009: both the productive and education sector have been strongly damaged by the falling prices in the stock market.

[3]A similar approach has been pursued by Bethmann (2007) in a Lucas-Uzawa model of endogenous growth.

Proposition 1

i) *The solution* $V(k,h,z,\eta)$ *of the Bellman equation in (4) is given by:*

$$V(k,h,z,\eta)=\theta+\theta_k\ln k+\theta_h\ln h+\theta_z\ln z+\theta_\eta\ln\eta, \tag{5}$$

where the constants θ_k, θ_h, θ_z *and* θ_η *are defined as follows:*

$$\theta_k=\frac{\alpha}{1-\alpha\beta},\quad \theta_h=\frac{1-\alpha}{(1-\alpha\beta)(1-\beta\phi)},\quad \theta_z=\frac{1}{1-\alpha\beta},\quad \theta_\eta=\frac{(1-\alpha)\beta}{(1-\alpha\beta)(1-\beta\phi)},$$

and the constant term θ *is given by:*

$$\begin{aligned}\theta=\frac{1}{1-\beta}\Bigg[&\ln(1-\alpha\beta)+\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)+\frac{1-\alpha}{1-\alpha\beta}\ln(1-\beta\phi)\\&+\frac{(1-\alpha)\beta\phi}{(1-\alpha\beta)(1-\beta\phi)}\ln(\beta\phi)+\frac{\beta}{(1-\alpha\beta)}\mathbb{E}\ln z+\frac{(1-\alpha)\beta^2}{(1-\alpha\beta)(1-\beta\phi)}\mathbb{E}\ln\eta\Bigg].\end{aligned}\tag{6}$$

ii) *The optimal policy rules for consumption and share of human capital allocated to physical production are respectively given by:*

$$c_t=(1-\alpha\beta)(1-\beta\phi)^{1-\alpha}z_tk_t^\alpha h_t^{1-\alpha} \tag{7}$$
$$u_t=1-\beta\phi, \tag{8}$$

while physical and human capital follow the (optimal) dynamics defined by:

$$\begin{cases}k_{t+1}=\alpha\beta(1-\beta\phi)^{1-\alpha}z_tk_t^\alpha h_t^{1-\alpha}\\ h_{t+1}=(\beta\phi)^\phi\eta_t h_t^\phi.\end{cases}\tag{9}$$

The proof is reported in the Appendix.

An argument parallel to that described on pp. 273-277 in Stokey and Lucas (1989) establishes that the function $V(k,h,z,\eta)$ defined in (5) is actually the value function of problem (2).

4 Conjugate Linear IFSP

The optimal dynamics for the physical and human capital in (9) have the form of products of powers, suggesting that a logarithmic transformation of both variables k_t and h_t may yield an equivalent conjugate system which is linear in the transformed variables. Specifically, a suitable transformation of (9) may lead to a contractive IFSP converging to a unique invariant distribution supported on some fractal attractor in accordance with Corollaries 1 and 2 of Section 2. The following proposition shows that, for specific sets of values for parameters α, ϕ, q_1, q_2 an r, a linear system conjugate to (9) exists defining a IFSP that converges to an invariant distribution supported on a (generalized) Sierpinski gasket with vertices $(0,0)$, $(1/2,1)$ and $(1,0)$.

Proposition 2 *Assume that* $\alpha\neq\phi$ *and let*

$$r=\exp\left[\frac{\alpha-\phi}{1-\alpha}(2\ln q_2-\ln q_1)\right]. \tag{10}$$

Then the one-to-one logarithmic transformation $(k_t, h_t) \to (x_t, y_t)$ *defined by:*

$$\begin{cases} x_t = \rho_a \ln k_t + \rho_b \ln h_t + \rho_c \\ y_t = \rho_d \ln h_t + \rho_e, \end{cases} \tag{11}$$

with

$$\rho_a = -\frac{1-\alpha}{2\ln q_2}, \qquad \rho_b = \frac{(1-\alpha)^2}{2(\phi-\alpha)\ln q_2}, \tag{12}$$

$$\rho_c = 1 + \frac{1}{2\ln q_2}\left\{\ln\left[\alpha\beta(1-\beta\phi)^{1-\alpha}\right] + \frac{1-\alpha}{\alpha-\phi}\ln\left[(\beta\phi)^{\phi}\right]\right\}, \tag{13}$$

$$\rho_d = \frac{(1-\alpha)(1-\phi)}{(\phi-\alpha)(2\ln q_2 - \ln q_1)}, \qquad \rho_e = 1 + \frac{(1-\alpha)\ln\left[(\beta\phi)^{\phi}\right]}{(\alpha-\phi)(2\ln q_2 - \ln q_1)}, \tag{14}$$

defines a contractive linear IFSP which is equivalent to the nonlinear dynamics in (9) and is composed of the three maps $w_1, w_2, w_3 : \mathbb{R}^2 \to \mathbb{R}^2$ *given by:*

$$\begin{cases} (x_{t+1}, y_{t+1}) = w_1(x_t, y_t) = (\alpha x_t, \phi y_t) & \text{with probability } p_1 \\ (x_{t+1}, y_{t+1}) = w_2(x_t, y_t) = (\alpha x_t + (1-\alpha)/2, \phi y_t + (1-\phi)) & \text{with probability } p_2 \\ (x_{t+1}, y_{t+1}) = w_2(x_t, y_t) = (\alpha x_t + (1-\alpha), \phi y_t) & \text{with probability } p_2. \end{cases} \tag{15}$$

The IFSP defined by (15) converges to an invariant distribution supported on a (generalized) Sierpinski gasket with vertices $(0,0)$, $(1/2,1)$ *and* $(1,0)$.

The proof is reported in the Appendix.

Rewriting the IFSP in (15) as

$$\begin{cases} x_{t+1} = \alpha x_t + \gamma_t \\ y_{t+1} = \phi y_t + \vartheta_t, \end{cases} \tag{16}$$

it is immediately seen that the three values $(0,0)$, $((1-\alpha)/2, (1-\phi))$ and $((1-\alpha), 0)$ taken on by the (conjugate) random vector (γ_t, ϑ_t) correspond respectively to the three scenarios (q_1, r), $(q_2, 1)$ and $(1, 1)$ for the original random values (z, η) discussed in Section 3.

The mild restriction $\alpha \neq \phi$ required in Proposition 2 precludes the possibility of generating the standard Sierpinski gasket with vertices $(0,0)$, $(1/2,1)$ and $(1,0)$ through (15), as its construction postulates that $\alpha = \phi = 1/2$ must hold. In this sense, we say that the attractor of (15) is a *generalized Sierpinski gasket*. As it is clear from the proof, condition (10) turns out to be the key restriction needed to construct the dynamics (15) equivalent to (9).

5 Examples of Sierpinski Gasket-like Attractors

We consider four different parametrizations of the physical production and human capital production parameters, α and ϕ. Note that any triple $0 < q_1 < q_2 < 1$ and $0 < r < 1$ satisfying condition (10) in Proposition 2 does the job; thus we do not set values for these parameters. Similarly, probabilities p_1, p_2 and p_3 can be any numbers between 0 and 1 summing up to 1. In the first two scenarios, we tackle a framework very close to the benchmark case $\alpha = \phi = 1/2$, corresponding to the standard Sierpinski gasket with vertices $(0,0)$, $(1/2,1)$, $(1,0)$ as the unique attractor of the IFSP (15). As Proposition 2 requires $\alpha \neq \phi$, we set $\alpha = 0.5$ and $\phi = 0.49$. Figure 1(a) shows the first 8 iterations[4]

[4]The Maple 13 code for approximating the attractor of our economy under repeated iterations of the map (1) is available from the authors upon request.

of the map (1) when the maps w_1, w_2, w_3 are given by (15) starting from the triangle of vertices $(0,0)$, $(1/2,1)$, $(1,0)$ as initial set S_0. While $\alpha = 1/2$ implies that the two lower triangles of each prefractal[5] have one vertex in common [*e.g.*, point $(1/2, 0)$ after one iteration], the assumption that $\phi < 1/2$ implies that the top vertices of the two lower triangles are disjoint from the bottom vertices of the top triangle. Clearly, whenever $\alpha \geq 1/2$ and $\phi \geq 1/2$ with at least one strict inequality, all triangles in each prefractal overlap, as shown in Figure 1(b) for $\alpha = 0.5$ and $\phi = 0.52$.

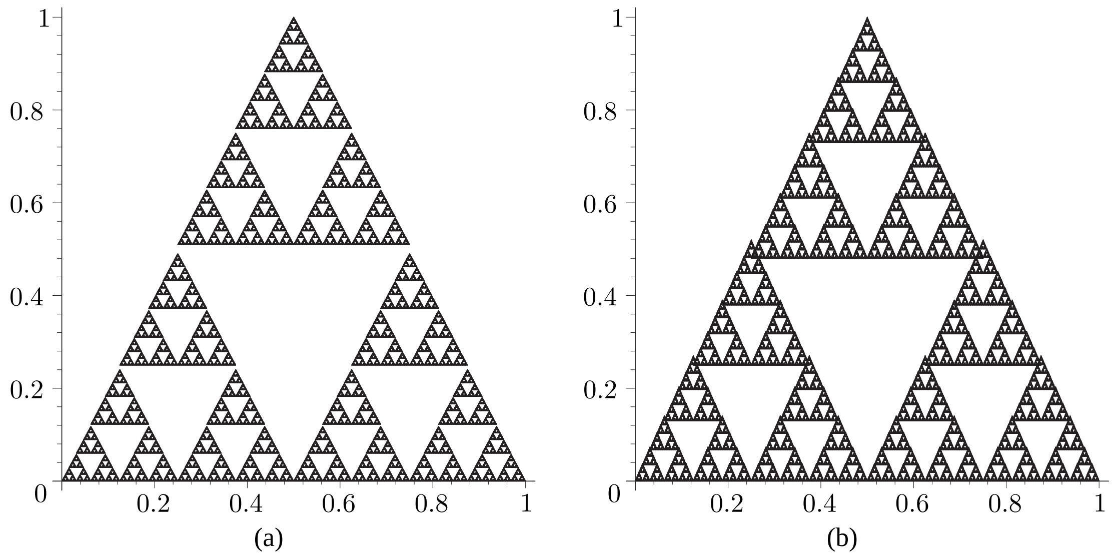

FIGURE 1: first 8 iterations of the map (1) for (a) $\alpha = 0.5$, $\phi = 0.49$, and (b) $\alpha = 0.5$, $\phi = 0.52$.

The last two cases consider a more realistic economy in which the capital share parameter is set to be $\alpha = 0.333$. In the economic literature the capital share parameter in the output of the physical sector, α, measuring its marginal returns on capital, has been traditionally considered the to be close to one third (Mankiw *et al.*,1992; Barro and Sala-i-Martin, 2004). A clear measure of the marginal returns of human capital in education has never been found in the empirical literature, since the human capital share in education is usually set to 1 in order to generate endogenous growth (Lucas, 1988). However, as argued by Rebelo (1991), we can reasonably assume that marginal returns of human capital are decreasing too. Probably, the most empirically relevant case is the one in which the education sector is relatively intensive in human capital, that is $\phi \leq 1 - \alpha$ (Barro and Sala-i-Martin, 2004); therefore, in these two scenarios we assume a reasonable $\phi = 0.5$ and a limiting case $\phi = 1 - \alpha = 0.667$. Figures 2(a) and 2(b) plot the first 7 iterations (which are enough in this case) of the map (1), again starting from the triangle of vertices $(0,0)$, $(1/2,1)$ and $(1,0)$ as initial set S_0, for $\alpha = 0.333$, $\phi = 0.5$ and for $\alpha = 0.333$, $\phi = 0.667$ respectively.

6 Conclusions

In this paper we built a neoclassic, stochastic, discrete-time, two-sector optimal growth model in which the production of a homogeneous consumption good depends on physical and human capital. Our model exhibits two peculiar features: 1) the log-Cobb-Douglas structure of preferences plus production allows for a closed form solution of the Bellman equation, thus allowing for the explicit computation of the optimal dynamics of the state variables (Proposition 1), and 2) through a simple

[5]The sets obtained after each iteration of the map (1) are called *prefractals*.

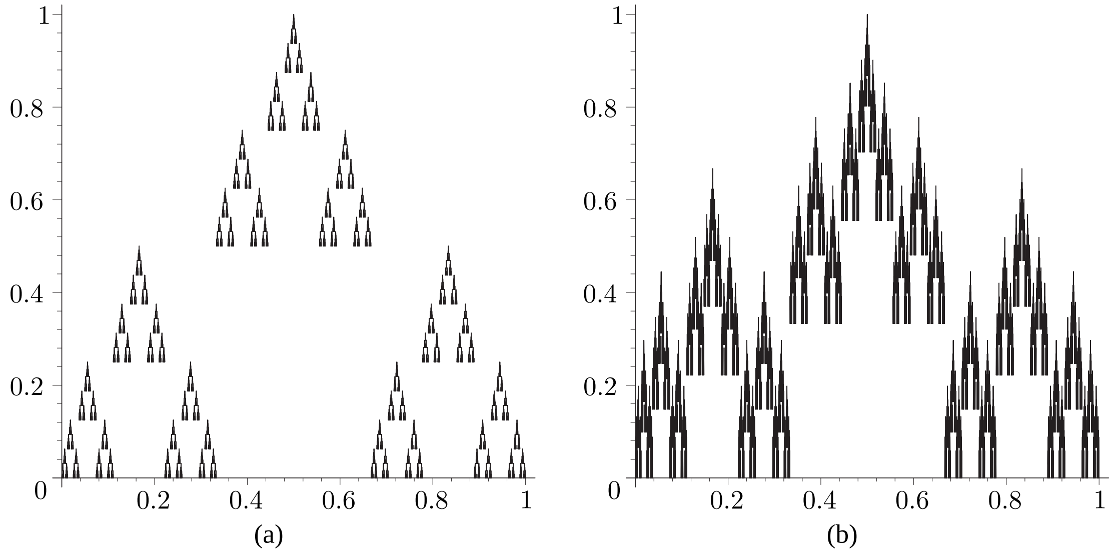

FIGURE 2: first 7 iterations of the map (1) for (a) $\alpha = 0.333$, $\phi = 0.5$, and (b) $\alpha = 0.333$, $\phi = 0.667$.

log-transformation of such dynamics we are able to show that for a sufficiently rich set of parameters' configurations this economy converges to an invariant distribution supported on a generalized Sierpinski gasket (Proposition 2). The only binding restriction is actually given by condition (10) which relates the value r of the shock affecting the education sector to the two values q_1 and q_2 of the shock affecting the production sector. However, we believe that our approach is sufficiently general as there is total freedom of choice on the values of two out of three exogenous shock parameters, leaving only the third dependent to the first two.

After investigating the (approximation of) the attractors of some economies in Figures 1 and 2, one may ask how the degree of overlapping among the prefractals may affect singularity properties of the invariant distribution. More precisely, it would be interesting to establish under what conditions on the model's parameters the invariant distribution turns out to be singular – or absolute continuous – with respect to Lebesgue measure. This exercise is left for future research.

Appendix

Proof of Proposition 1. Assuming the form as in (5) for the value function and dropping the time subscript, the Bellman equation (4) can be rewritten as:

$$\theta + \theta_k \ln k + \theta_h \ln h + \theta_z \ln z + \theta_\eta \ln \eta = \max_{c,u} \left\{ \ln c + \beta\theta + \beta\theta_k \ln[zk^\alpha (uh)^{1-\alpha} - c] \right. \qquad (17)$$

$$\left. + \beta\theta_h \ln \left[\eta (1-u)^\phi h^\phi\right] + \beta\theta_z \mathbb{E} \ln z + \beta\theta_\eta \mathbb{E} \ln \eta \right\}.$$

FOC on the RHS with respect to c and u yield respectively:

$$\frac{1}{c} = \frac{\beta\theta_k}{zk^\alpha (uh)^{1-\alpha} - c} \qquad (18)$$

$$\frac{\beta\phi\theta_h}{1-u} = \frac{\beta\theta_k (1-\alpha) zk^\alpha (uh)^{-\alpha} h}{zk^\alpha (uh)^{1-\alpha} - c}, \qquad (19)$$

while the envelope conditions read as:

$$\frac{\theta_k}{k} = \frac{\alpha\beta\theta_k z k^{\alpha-1}(uh)^{1-\alpha}}{zk^{\alpha}\left(uh\right)^{1-\alpha} - c} \tag{20}$$

$$\frac{\theta_h}{h} = \frac{\left(1-\alpha\right)\beta\theta_k z k^{\alpha}\left(uh\right)^{-\alpha} u}{zk^{\alpha}\left(uh\right)^{1-\alpha} - c} + \frac{\beta\phi\theta_h}{h}. \tag{21}$$

From (18) we get:

$$c = \frac{1}{1+\beta\theta_k} zk^{\alpha}\left(uh\right)^{1-\alpha}, \tag{22}$$

which, when plugged into (20), after some algebra leads to:

$$\theta_k = \frac{\alpha}{1-\alpha\beta}. \tag{23}$$

Using (22) and (23) into (21), again after some algebra yields:

$$\theta_h = \frac{1-\alpha}{\left(1-\alpha\beta\right)\left(1-\beta\phi\right)}.$$

From (19) and (21) we obtain $u = 1-\beta\phi$, which is the optimal human capital share as in (8), while joining (22) and (23) one immediately gets $c = \left(1-\alpha\beta\right)\left(1-\beta\phi\right)^{1-\alpha} zk^{\alpha}h^{1-\alpha}$, which is the optimal consumption as in (7). The optimal dynamics (9) are obtained by substituting (7) and (8) into the dynamic constraints (3).

Finally, in order to calculate the remaining constants θ, θ_z and θ_η we substitute θ_k, θ_h, c and u as computed above into (17), so that the terms in $\ln k$ and $\ln h$ cancel out and we are left with:

$$\begin{aligned}\theta + \theta_z \ln z + \theta_\eta \ln \eta = {} & \ln\left(1-\alpha\beta\right) + \tfrac{1-\alpha}{1-\alpha\beta}\ln\left(1-\beta\phi\right) + \beta\theta + \tfrac{\alpha\beta}{1-\alpha\beta}\ln\left(\alpha\beta\right) + \tfrac{(1-\alpha)\beta\phi}{(1-\alpha\beta)(1-\beta\phi)}\ln\left(\beta\phi\right) \\ & + \tfrac{1}{1-\alpha\beta}\ln z + \tfrac{(1-\alpha)\beta}{(1-\alpha\beta)(1-\beta\phi)}\ln\eta + \beta\theta_z\mathbb{E}\ln z + \beta\theta_\eta\mathbb{E}\ln\eta.\end{aligned}$$

For this equation to hold both the terms in $\ln z$ and $\ln\eta$ must vanish, which requires:

$$\theta_z = \frac{1}{1-\alpha\beta} \qquad \text{and} \qquad \theta_\eta = \frac{\left(1-\alpha\right)\beta}{\left(1-\alpha\beta\right)\left(1-\beta\phi\right)},$$

while θ turns out to be given by (6). ■

Proof of Proposition 2. Using (11), (16) can be rewritten as:

$$\begin{cases} \rho_a \ln k_{t+1} + \rho_b \ln h_{t+1} + \rho_c = \alpha\rho_a \ln k_t + \alpha\rho_b \ln h_t + \alpha\rho_c + \gamma_t \\ \rho_d \ln h_{t+1} + \rho_e = \phi\rho_d \ln h_t + \phi\rho_e + \vartheta_t. \end{cases} \tag{24}$$

Let us focus on the first equation in (24). Substituting k_{t+1} and h_{t+1} as in the first equation of (9), rearranging terms and after dropping the common terms $\alpha\rho_a \ln k_t$ such equation becomes:

$$\begin{aligned}&\rho_a \ln\left[\alpha\beta\left(1-\beta\phi\right)^{1-\alpha}\right] + \rho_b \ln\left[\left(\beta\phi\right)^{\phi}\right] + \left(1-\alpha\right)\rho_c \\ &\qquad + \left[\left(1-\alpha\right)\rho_a + \left(\phi-\alpha\right)\rho_b\right]\ln h_t = \gamma_t - \rho_a \ln z_t - \rho_b \ln \eta_t.\end{aligned} \tag{25}$$

In order to let the constant ρ_c be independent of h_t in the equation above, we need that $\left(1-\alpha\right)\rho_a + \left(\phi-\alpha\right)\rho_b = 0$, so that the last term in the LHS cancels out and, under the assumption that $\alpha \neq \phi$, we have:

$$\rho_b = \frac{1-\alpha}{\alpha-\phi}\rho_a. \tag{26}$$

Using (26), equation (25) boils down to:

$$\left\{\ln\left[\alpha\beta\left(1-\beta\phi\right)^{1-\alpha}\right]+\frac{1-\alpha}{\alpha-\phi}\ln\left[\left(\beta\phi\right)^{\phi}\right]\right\}\rho_a+\left(1-\alpha\right)\rho_c=\gamma_t-\left[\ln z_t+\frac{1-\alpha}{\alpha-\phi}\ln\eta_t\right]\rho_a. \quad (27)$$

As the LHS in (27) is constant, we can use the three values $\gamma_t = 0$, $\gamma_t = (1-\alpha)/2$ and $\gamma_t = (1-\alpha)$, corresponding respectively to $(z_t, \eta_t) = (q_1, r)$, $(z_t, \eta_t) = (q_2, 1)$ and $(z_t, \eta_t) = (1, 1)$ for the original shocks, and write:

$$-\left[\ln q_1+\frac{1-\alpha}{\alpha-\phi}\ln r\right]\rho_a=\frac{1-\alpha}{2}-\rho_a\ln q_2=1-\alpha.$$

From the second equation, using (26) we easily get ρ_a and ρ_b as in (12). Note, however, that the first equation on the left must hold as well, which, consistently with $\rho_a = -(1-\alpha)/(2\ln q_2)$, is equivalent to condition (10). As a matter of fact, condition (10) is the key assumption to let equation (27) – or, equivalently, equation (25) – be independent of h_t. Substituting $\gamma_t = 1-\alpha$ [corresponding to $(z_t, \eta_t) = (1, 1)$] and ρ_a as in (12) into equation (27) easily yields ρ_c as in (13).

As far as the second equation in (24) is concerned, substituting h_{t+1} as in the second equation of (9), rearranging terms and after dropping the common terms $\phi\rho_d \ln h_t$ such equation becomes:

$$\rho_d\ln\left[\left(\beta\phi\right)^{\phi}\right]+\left(1-\phi\right)\rho_e=\vartheta_t-\rho_d\ln\eta_t. \quad (28)$$

As the LHS is constant, we can use the two values $\vartheta_t = 0$ and $\vartheta_t = (1-\phi)$, corresponding respectively to $\eta_t = r$ and $\eta_t = 1$ for the original shocks on human capital, and write:

$$-\rho_d\ln r=1-\phi,$$

which immediately yields $\rho_d = -(1-\phi)/\ln r$, while $\rho_e = 1+\ln\left[(\beta\phi)^{\phi}\right]/\ln r$ is obtained by plugging the expression of ρ_d into (28). Finally, substituting $\ln r$ according to (10) yield ρ_d and ρ_e as in (14).

As $0 < \alpha < 1$ and $0 < \phi < 1$, the IFSP (15) – or, equivalently, (16) – is a contraction mapping; hence, Corollaries 1 and 2 apply and this is sufficient to show that the conjugate dynamics of system (9) describing the optimal evolution of the state variable in our economy have a unique invariant distribution supported on a generalized Sierpinski gasket to which the economy converges in the long run. ■

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