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THINKING, REASONING AND WORKING MATHEMATICALLY: A TEACHER'S RESPONSE TO CURRICULUM CHANGE

Thesis submitted by

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For

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ABSTRACT

Research consistently points out that the quality of mathematics teaching and learning needs improvement if students' numeracy outcomes are to be improved. Curriculum documents, such as the *Years 1-10 Mathematics Syllabus* (QSA, 2004) and *Queensland Curriculum and Assessment Reporting Framework* (DETA, 2005), recommend teachers adopt an inquiry based 'process approach' where there is an increased focus on student learning. The students should be encouraged to think and reason, communicate and reflect with and about mathematical ideas to construct and validate them in ways that make sense to themselves and others. This approach is intended to improve numeracy outcomes by focussing on the productive development of mathematical understandings, practices and dispositions. However, while past research has examined teachers' understanding of the key messages of reform, research about the practical implementation is limited. Thus a gap exists between the intended and practised curriculum in mathematics classrooms and this gap needs to be more fully explained and understood for numeracy outcomes to improve.

This 'descriptive case study' (Shank, 2006) focussed on that gap to better understand the practical implementation of curriculum ideals. The study was conducted in a year six classroom of a small private school in North Queensland. It investigated, in detail, one teacher's attempt to implement curriculum change to reveal how and why certain experiences challenge, inspire or motivate the teacher's facilitation and students' uptake of learning processes comprising thinking, reasoning and working mathematically. The researched change involved the teacher's adoption of certain mathematics practices that would arguably result in more effective instructional strategies and investigative learning processes. Student pre- and post-questionnaires were used to determine changes to disposition and willingness to engage in mathematics learning. Pre- and post-questionnaires were also used to explicate changes to the teacher's pedagogical beliefs or understandings as a result of implementing the curriculum change. Further data were obtained from semi-structured interviews and detailed classroom observations and all data were analysed through a qualitative content data analysis (Lankshear & Knobel, 2005). Whilst this study is limited to a sample size of one teacher, the rich, thick data revealed that change is complex; it is worthwhile, yet slow and abounds with challenges. The teacher's practice changed and the classroom atmosphere altered to enhance more collaborative mathematical inquiry. Student engagement and disposition started to improve in relation to reform ideals. Further research is needed to document, collaborate and perform cross case analyses to highlight exemplary practices and to examine the effect of reform oriented teaching on student learning outcomes and achievement. The results of this study will inform policymakers and researchers regarding future research directives and acquaint other teachers with some of the successes and challenges of implementing new policy directives.

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RESEARCH BACKGROUND

"Children are the world's most valuable resource and its best hope for the future."
(John F. Kennedy)

1.1 Introduction

As the opening quote suggests, children are our future, which is a notion recognised by policy documents. For instance, the Ministerial Council on Education, Employment, Training and Youth Affairs [MCEETYA] (2008) places young Australians at the centre of the *Melbourne Declaration on Educational Goals*. The Council believes that rapidly changing societal demands now necessitate the improvement of educational outcomes. Global integration and technological mobility have meant that the world is changing at a rapid pace. Hence, the Council (MCEETYA, 2008, p. 7) suggested that because Australia's future as a socially and economically prosperous nation is dependent on the abilities of young Australians, there is increased need for students to leave school with the capabilities to live "fulfilling, productive and responsible lives." Furthermore, the Council asserted that students must leave school as 'successful learners' and 'logical thinkers' who can make sense of their world and "acquire the skills to make informed learning and employment decisions throughout their lives" (p. 8). Numeracy, along with literacy, is foundational to the development of successful learning, according to the Council.

An important goal internationally and nationally is to improve numeracy outcomes for all students. Research consistently points out that the quality of mathematics teaching and learning needs improvement to ensure all students become proficient users of mathematics (Ball, 2003; Board of Teacher Registration [BTR], 2005; Council for the Australia Federation [CAF], 2007; Masters, 2009; Stephens, 2000). Proficient users of mathematics can apply and use mathematical knowledge with understanding, which implies that they are numerate or mathematically literate. International organisations, such as the Organisation for Economic Co-operation and Development [OECD] (2006), refer to the term ‘mathematical literacy,’ whereas ‘numeracy’ is the term used in Australia.

Mathematical literacy and numeracy are two different terms sharing similar meanings. A person who is mathematically literate has developed the ability to do and use mathematics in functional ways, for a variety of purposes and in a variety of contexts, according to the OECD (2006). Similarly, numeracy is defined in the National Curriculum Board’s (2009, p. 5) paper, *Shape of the Australian Curriculum: Mathematics*, as: “the capacity, confidence and disposition to use mathematics to meet the demands of learning, school, home, work, community and civic life.” Both perspectives on numeracy and mathematical literacy emphasise the key role of being able to do, use and apply mathematics. In short, to become numerate, or mathematically literate, requires knowing mathematics and knowing how to think about, work with and apply mathematical ideas in flexible ways in various contexts.

Curriculum documents are currently changing the focus of mathematics education to ensure students develop numeracy skills. This research aimed to gain insight into one practitioner’s attempt to implement curriculum change that was informed by the Queensland *Years 1-10 Mathematics Syllabus* (Queensland Studies Authority [QSA], 2004) and theoretically framed by the work of Richard Skemp (1986). Skemp’s notion of constructing mathematical understandings as a related network of ideas underpins this research, as explained in chapter two. The research empirically inquired into the teaching and learning of mathematical practices to more fully understand the challenges of implementation, as recommended by the RAND Mathematics Study Panel (Ball, 2003). It must be noted that the focus of this research was the process of curriculum implementation proposed to improve numeracy outcomes, not the numeracy outcomes themselves. The production of teaching knowledge and the improvement of practice are dependent on insights generated from what is being tried in practice (Ball, 2003). Hence, the aim of this “descriptive case study” (Shank, 2006, p. 127) was to document through a rich description, and analyse and evaluate, what is being tried in practice as curriculum change is implemented.

1.2 Changing Times

Whilst the knowledge and understandings students will need to successfully meet societal demands of contemporary times is uncertain, what is certain is that students need to become successful learners to confidently meet the challenges of a new era (MCEETYA, 2008). Successful learners, according to MCEETYA, are flexible and logical thinkers. They draw on essential skills, knowledge and understandings to solve problems, they use technology productively, they evaluate evidence to make informed decisions, and they work both independently and collaboratively. Making sure students leave school equipped to continue successful numeracy learning calls for a change in the learning processes in mathematics education.

Teaching and learning mathematics for numeracy now involves a shift beyond learning and memorising algorithmic procedures (Anghileri, 2000; Bobis, Mulligan, Lowrie & Taplin, 1999; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver & Human, 2000). To be numerate, students need to develop flexible ways of thinking about mathematics, and be confident in their ability to work mathematically (Ball, 2003; Hiebert et al., 2000; MCEETYA, 2008). Curriculum documents (QSA, 2007) have emphasised that students are now required to develop ways of knowing and understanding and ways of thinking and reasoning about mathematics so that they can make sense of things for themselves. Learning with understanding is paramount, which means that shallow learning of mathematical procedures will no longer suffice. To think flexibly about mathematics, students will need to learn mathematics in ways which help them to understand 'how' and 'why' mathematical ideas work and how ideas relate to each other (Hiebert et al., 2000).

For students to understand the how and why of mathematics, now more than ever, they need to take an active role in the learning process. Successful learning is dependent on active engagement in the learning (MCEETYA, 2008). However, active engagement is dependent on the development of a positive disposition towards mathematics and learning, according to the Queensland Studies Authority (2007). Thus the QSA are suggesting that active engagement will only occur when students feel confident in their ability to work with and learn mathematics. In other words, students need learning experiences that develop their confidence and interest, and thereby, improve their motivation and willingness to persevere. Yet in Australia, despite policy initiatives, students' mathematical abilities, performance, confidence and interest levels have been declining for some time, as discussed below.

1.3 Numeracy Skills are Declining

Data generated worldwide examine students' use and application of their mathematical understandings. In 2006, between 4 500 and 10 000 fifteen-year-old students in 56 countries were assessed by the Programme for International Student Assessment (PISA) (OECD, 2006) to determine the extent to which the knowledge and skills learned at school adequately prepared them for adult life. PISA studies adopted a broader reform approach to assessment, going beyond school-based activities towards solving everyday challenges and tasks. The students were required to reason quantitatively and to represent mathematical relationships or dependencies (OECD, 2006). The objective of this assessment approach was to examine and determine students' ability to think for themselves about mathematical situations, using mathematical understandings purposefully (OECD, 2006). The PISA objectives align with the current emphasis in syllabus documents (QSA, 2004) to ensure students can think about and make sense of mathematics for themselves. A significant element of the National Reform Agenda in Australian schools is to maintain a high rating in PISA studies (CAF, 2007; MCEETYA, 2008). In fact, MCEETYA's (2008, p. 5) vision for Australian students over the next decade is to become "second to none amongst the world's best school systems."

However, at present, Australian students have not been faring well in comparison to their counterparts. There was no improvement of mathematical literacy (numeracy) skills shown by Australian students in the PISA studies between 2000 and 2003 (CAF, 2007); whereas, in Canada, Finland and Hong Kong-China, the students' mathematical performance improved significantly in 2003. Then in 2007, the *National Report on Schooling in Australia* (MCEETYA, 2007), reported that at each progressive year level, the numeracy skills of Australian students' declined. Six percent of year three students, eleven percent of year five students and twenty percent of year seven students do not meet the national numeracy benchmarks. This progressive decline in numeracy skills implies that students are not developing mathematical understandings that enable them to advance in their learning, or to competently or confidently think about mathematics and its applications.

Alarmingly, Queensland students have not ranked well in benchmark tests over the past 30 years. Masters (2009) pointed out that in Queensland there has been a long-term decline, since the 1970s, of students' performance levels in mathematics. In addition, evidence from the 2008 National Achievement Program – Literacy and Numeracy (NAPLAN, MCEETYA, 2008a) and the 2007 Trends in International Mathematics and Science Study (TIMSS), indicated that Queensland students under-perform in numeracy, literacy and science. In 2007 and 2008 the average performance of Queensland Year three (age eight), four (age nine) and five (age ten) students was significantly lower than all other Australian states and territories (Masters, 2009).

Whilst the decline in numeracy skills has been at the forefront of curriculum recommendations, it has also attracted media attention. For instance, the editorial of *The Australian* newspaper ("Time to spell it out," 2009, p. 1) reported that students who leave school innumerate have been subjected to a form of "child abuse." The article explained that students need to know simple tasks such as calculating the interest on a bank loan, reading a bank statement and choosing a mobile phone plan. It continued by asserting that unless students have the basics of both literacy and numeracy they will not be adequately prepared to participate in society. A similar article was printed in *The Australian* newspaper in 2005 by Donnelly (2005), the then director of Education Strategies. He asserted that if students lack basic numeracy skills at each year level, they will be ill-equipped for future learning. These concerns about developing numeracy skills that enable future learning and sense-making of bank statements and mobile phone plans, as the articles suggested, require the development of insightful mathematical thinking currently projected by curriculum documents (QSA, 2004; 2007).

However, new curriculum ideals seem to be misunderstood, and consequently, have been blamed for the decline in numeracy development. For example, in another front page article of *The Australian* newspaper, Maiden (2005, p.1) reported: "AUSTRALIA's schools are infected with 'new age' curriculums that are leaving students behind their international peers in mathematics, science and English." Yet, one wonders about the basis of such a claim because, as Stephens (2000) pointed out, there has been limited research about the actual implementation of curriculum goals. Despite the criticism of new curriculum goals in the print media, political leaders are beginning to recognise that research is needed to more fully understand the current situation. For instance, in *The Weekend Australian* (July 12-13, 2008, p. 29) the Deputy Prime Minister, Julia Gillard wrote: “What we need now is focussed collaboration between governments, schools and communities to find out what does work and to share the evidence.” Hence, the focus of this research was to investigate the process of curriculum implementation to determine the teacher’s present and evolving understandings of the new goals, and to share evidence about what works, as Gillard recommended.

1.4 Rationale of the Study

It is time to start looking at successful teaching practices to identify critical features of the mathematics classroom that contribute to quality teaching and learning because criticising teaching methods or curriculum ideals seems counterproductive. Masters (2009) made the point that only if changes are made to teaching practices will achievement levels in numeracy improve. Even in relatively short periods of time, international research has revealed that targeted interventions can have positive impacts on students’ outcomes (Barber & Mourshed, 2007). However, it must be noted that the goal of this research was not to test whether the recommended features of teaching and learning outlined in policy will improve numeracy outcomes. Instead, the goal was to understand the curriculum implementation process and the circumstantial effects on student disposition and engagement. The quantification of numeracy outcomes would have required an entirely different research approach.

Making changes to teaching practices is complex, and hence, the work of the RAND Mathematics Study Panel (Ball, 2003) informed this research. The rationale for this decision was based on three contributing factors. First, the Panel shares Australia’s goal to support the improvement of mathematics education and increase levels of mathematical proficiency. Second, the Panel is composed of a broad spectrum of the mathematics research community including teachers, mathematics educators, mathematicians, psychologists and policy makers (Ball, 2003). The third reason related to the fact that the Panel’s beliefs aligned with Queensland curriculum’s (QSA, 2004; 2007) emphasis on mathematical thinking and reasoning. The Panel explained that “competent use of mathematics depends on the ways in which people approach, think about, and work with mathematical tools and ideas” (Ball, 2003, p. xviii). They proposed three specific ‘mathematics practices’ of representation, justification and generalisation, which this study anticipated may facilitate thinking and reasoning processes. The Panel hypothesised that these practices are not systematically cultivated in schools, and therefore, it is probable that students will have difficulties when learning mathematics. This research investigated the effectiveness of the practices to enhance mathematical thinking and reasoning processes.

Significantly, the RAND Panel (Ball, 2003) proposed that empirical enquiry should be carried out to gain insight into the wisdom of practitioners and to contribute to the production of knowledge about learning and teaching mathematics. Hence, paying heed to their advice, this research empirically inquired into one teacher’s attempt to implement curriculum change over two school terms in a year six classroom of a private school. The research goal was to understand, first, features of the learning context that facilitate or constrain the process of curriculum implementation, and second, changes to students’ dispositions and interest levels towards mathematics, and their willingness to engage in the learning. The aim of this study was not to find the ‘best’ approach, pedagogically speaking, but to identify features (knowledge, practices and dispositions) that may improve the quality of the learning process in mathematics teaching and learning.

1.5 Effective Mathematics Teaching and Quality Learning

At present there is not a “definitive formula for effective teaching” (Walshaw & Anthony, 2008, p. 538). However, there have been some defining features of the classroom that assist effective teaching and quality learning in mathematics. First is the notion that students construct their own mathematical understandings. In Queensland, the *Years 1-10 Mathematics Syllabus* (QSA, 2004) and the *Education Queensland Scoping and Sequencing Essentials: Years 1-9* (Department of Education, Training and the Arts [DETA], 2008) emphasise that quality teaching and learning involves a move from transmission modes of teaching towards constructivist modes. Under a constructivist epistemology, students alter their own cognitive schema rather than acquire mathematical knowledge presented by another (Commonwealth of Australia, 2004).

Constructivist theories of learning propose that learning is enhanced when the teacher can provide cognitive scaffolding that enables the students to make sense of things for themselves. Thus, teachers are now being encouraged to set instructional tasks that promote capacity in thinking and reasoning (DETA, 2008; QSA, 2004; 2007). The aim is for students to puzzle over mathematical ideas in ways that they can make sense of things for themselves and where they have the opportunity to initiate inquiry. However, such inquiry can be assisted by the teacher who asks probing questions to guide students' thinking and help them to connect new ideas to existing ideas (Battista, 1999; Van de Walle & Lovin, 2006). The more questions asked by the teacher, the more effective the teaching and learning as Everston, Anderson, Anderson and Brophy's (1980) research revealed.

Henceforth, the second feature of effective teaching and quality learning involves altering the culture of the classroom to support communal inquiry (Palmer, 1999; Van de Walle & Lovin, 2006). The classroom context needs to be framed around the development of a culture of inquiry where ideas are shared between the teacher and learners (Hiebert et al., 2000). Importantly though, if teacher-student and student-student dialogue are required to support knowledge construction, then the atmosphere of the classroom needs to be supportive (Palmer, 1999) and classroom roles may need to alter. For instance, an authoritarian teacher may need to relinquish authority of correctness, and instead, encourage the students to validate the truth of ideas for themselves (Hiebert et al., 2000). Thus a culture of neutrality and trust needs to be established where errors are considered an opportunity for growth (Hiebert et al., 2000; Palmer, 1999) and personal sense-making of mathematical ideas.

However, a questioning, inquiry-based approach to mathematics teaching and learning requires knowing how to think mathematically. The question arises though as to how a teacher might scaffold thinking and reasoning learning processes to develop robust mathematical understandings. It was envisaged in this research that the RAND Panel's (Ball, 2003) mathematics practices might assist the teacher in her attempt to help the students' thinking and sense-making about mathematical ideas and relationships. It was anticipated that when students engaged in practices that involved justifying, representing, and generalising in mathematics, they would develop robust understandings about how and why mathematical ideas work, and how these ideas relate to each other (Ball, 2003; Kilpatrick, Swafford & Findell, 2006). In fact, the RAND Panel (Ball, 2003, p. 34) hypothesised that an investment in "these 'process' dimensions of mathematics could have a high pay off for improving the ability of the nation's schools to help all students develop mathematical proficiency."

Recognition of the connections between mathematical ideas is another important feature of quality mathematics teaching and learning. When students investigate ideas in ways that enable them to communicate and make sense of the ideas for themselves, they link ideas in their minds in web-like structures (Van de Walle & Lovin, 2006). Then, as students use existing ideas to give meaning to new situations, they make new connections; the more connections that are made, the deeper the mathematical ideas are understood (Van de Walle & Lovin, 2006). Making explicit connections is an important feature of the learning process (Skemp, 1986). Some highly effective teachers of numeracy in England were classified as 'connectionist teachers' because they placed an emphasis on explicitly highlighting connections between mathematical ideas (Askew, Brown, Rhodes, Johnson & William, 1997; Doig, 2001). These teachers also helped the students to make connections between mathematics and the 'real-world' (Askew et al., 1997). Connectionist teachers valued students' methods of working and used students' understandings to guide learning. They also viewed misunderstandings as an integral part of the learning process, thereby valuing the students' idiosyncratic sense-making processes.

The final feature concerns those particular pedagogical strategies that support the construction of positive mathematical identities. The QSA (2004) believes that positive dispositions are developed when students are engaged in mathematical investigations, either collaboratively or individually. They believe that when students are given opportunities to communicate their thinking and reasoning about mathematical ideas, they make sense of things for themselves and others. The syllabus (QSA, 2004, p. 1) also added that "positive dispositions towards mathematics learning and active engagement with mathematical tasks are integral to thinking, reasoning and working mathematically." Hence, one feeds the other; that is, sense making through thinking and reasoning supports the development of a positive disposition and a willingness to engage in learning, and conversely, a positive disposition supports further engagement in thinking and reasoning processes.

However, at the core of a positive disposition towards mathematics and mathematics learning is the notion of sense-making in mathematics, again, reiterating the importance of establishing a supportive environment. As Kilpatrick and colleagues (2006, p. 131) asserted, when students make sense of the mathematical ideas for themselves they recognise the benefits of perseverance, and thereby, “experience the rewards of sense making in mathematics.” Hence, the activity of sense-making itself has the potential to intrinsically motivate students to further invest time and effort in the learning process (Hiebert et al., 2000). Clearly, sense-making is an integral part of the learning process to develop both understanding and positive mathematical identities. Yet, many students leave school uncertain and anxious about their mathematical capabilities, as discussed below.

1.5.1 The Rise of Mathematical Anxiety

Students often leave school anxious about their ability to use mathematics, despite ideals to develop positive dispositions. For some time now, it has been recognised that students leave school with mathematical anxiety or negative attitudes towards mathematics and mathematics learning (Battista, 1999; Ellerton & Clements, 1989; Wilson & Thornton, 2006; Skemp, 1986). Hiebert and colleagues (2000) and Skemp (1986) hold that when preliminary mathematical understandings are not developed, future learning is often challenging for students, and they become frustrated and anxious. Those who are not given the opportunity to understand mathematics experience frustration and become disillusioned with mathematics and their ability to learn it (Hiebert et al., 2000). The risk is that frustrated and anxious students may lose interest and possibly withdraw from the learning or resist mathematical situations. Considering that some of these students may go on to become teachers themselves, there is the potential for mathematical uncertainty and anxiety to be replicated, resulting in a ‘cyclic phenomenon’ (Brady & Bowd, 2005).

Indeed, some preservice teachers and teachers are anxious about teaching mathematics on account of a lack of understanding, believing that their own schooling inadequately equipped them to teach the subject of mathematics confidently or effectively (BTR, 2005; Brady & Bowd, 2005; Wilson & Thornton, 2006; Smith & Klein, 2008; Stephens, 2000). Nonetheless, teachers do not set out to intentionally cause harm to students’ mathematics learning or to their attitude towards mathematics (Skemp, 1986). Skemp, for example, blamed teachers’ lack of knowledge about how to teach or how students learn mathematics for the long-lasting effects on students. Thus, it seems timely to look at the teacher's pedagogy as well as the curriculum innovation, because the most important influence on students' performance and achievement is determined by what teachers actually do in the mathematics classroom (BTR, 2005).

However, what is significant to this research is that research-based information about what impact the new teaching practices have had on student disposition and/or engagement is limited. At present little is known about what features of teaching may help to improve student disposition and reduce mathematical anxiety, and increase students' interest in mathematics, and thereby their willingness to engage in the learning. Alarmingly, according to Jenkins (2006), researchers often attribute student interest levels in a subject to a personal quality rather than as an outcome of their education experiences. A goal of this research was to include the 'students' voice,' as Jenkins recommended, identifying effective features of the teaching and learning experience that improved, or not, the students' disposition and engagement.

1.6 Implementing Curriculum Change

Mobilising schools for innovation depends on the work of teachers (Department of Education, Science and Training [DEST], 2003). The delivery of quality mathematics teaching and learning is dependent on the teacher's access to, or uptake of, professional development ideals, such as the new curriculum approach (McPhan, Pegg, Cooksey & Lynch, 2008). In Queensland, the new approach encourages teachers to "focus on the *learning* rather than the *teaching*" (DETA, 2008, p. 1). This shift of focus is achieved when teachers adopt a process approach, where they encourage learning processes comprising thinking, reasoning and working mathematically (DETA). Therefore, curriculum change depends largely on a teacher's comprehensive development of understandings of the curriculum objectives. However, previous research reveals that even when teachers do understand curriculum innovations, implementation is often limited (e.g., Anderson & Bobis, 2005; Cavanagh, 2006; Woodbury, 2000). It is anticipated that policy initiatives may never reach fruition without research into the practical application of intended curriculum goals.

A major problem of successful curriculum implementation is the gap between ideas and aspirations, and the attempts to operationalise these ideas and aspirations (Stenhouse, 1978). Teachers need time to explore the curriculum so that they can develop curriculum ‘know-how’ (Lloyd, 2008). As Cavanagh (2006, p. 121) stated, the “path to curriculum reform is slow and replete with challenges,” and the challenges need to be understood. Even though research communities are becoming aware that a gap exists between the intended and the implemented curriculum, understanding why the gap exists still needs further research (Anderson & Bobis, 2005; Cavanagh, 2006). For innovation to be successful, teachers must be willing to adopt, appraise and implement new instructional strategies (Skemp, 1986), and research needs to focus on the teacher and the teacher’s practice (BTR, 2005; Freebody, 2005; Lloyd, 2008; Masters, 2009). Hence, this case study aimed to look closely at one teacher’s ideas about the curriculum intent, her attempt to operationalise those ideas, and the emergent barriers or circumstantial successes of implementation.

Teachers learn from experience and exploration (Lloyd, 2008). Generative change is possible when teachers inquire into their own practice, which Steinberg, Empson and Carpenter’s (2004) case study revealed. Thus, as Freebody (2005, p. 4) highlighted:

> In aiming to reform or improve education, it is important to make a strong commitment to the competence of teachers, including their competence to recognise areas needing change and their competence in learning to manage productive change.

The key point Freebody made here is that for change to occur, the teacher needs an opportunity to investigate her practice, first, to determine areas that need change, and second, to make teaching decisions based on that knowledge. Without recognition of what needs to be changed, it is possible that the teacher may withstand new curriculum incentives. The goal of this research was to support the teacher in her endeavour to implement change, and to understand the complexities of managing productive change.

1.6.1 Making a Case

A case study makes its own case. This type of study is used to investigate an individual case to “optimise understanding of the case rather than to generalise beyond it” (Shank, 2006, p. 443). As Freebody (2005, p. 3) asserted, “there should be a growing body of locally-conducted research, aimed at enhancing teaching and learning, on which to base educational decisions.” A case study, such as this research, seeks to make systematic sense of the practices undertaken at a specific site, which could then be added to other sense-making accounts of practice (Stenhouse, 1978). Hence, there is potential for this locally-conducted research to inform further decisions to advance mathematics education. However, this case study does not propose to make scientific generalisations. Rather, the interest of this study was directed towards adding insight to curricula initiatives by drawing on contextual findings and conclusions (Burns, 2000). The goal was to produce a detailed account of one teacher's experience, to assist teachers, researchers and policymakers to think methodically about where or how practices could be altered, possibly leading to improved teaching and learning outcomes (Lankshear & Knobel, 2005).

The participating teacher in this case study was 'purposively selected' (Lankshear & Knobel, 2005) because she was drawn to the new investigative thinking and reasoning pedagogical approach espoused at university. She was in her first year of teaching which attended to Lloyd's (2008) advice in that there is an increased need for detailed examples of what curriculum implementation entails during a teacher's initial experience with the mathematics curriculum. However, the main reason the teacher was selected was her openness to change and willingness to investigate the practical application of curriculum ideals. The nature of this study was not to attempt to find a best method; rather, it was to describe and share the teacher's curriculum implementation experience and the effects on the students' learning experience.

The strength of the descriptive case study is that it can provide a rich description of curriculum implementation that can eventually be compared to and contrasted with other studies to detect patterns and identify robust or quality features of teaching and learning (Ball, 2003). A corroboration of findings has the potential to provide a basis for determining what works in mathematics teaching and learning, how it works, and what could be done to improve mathematics education (Ball, 2003). As the National Council of Teachers of Mathematics (NCTM, 2007, p. 1) suggested, by detecting patterns across studies, "educators can identify robust features of teaching." In short, by sharing the experience it is hoped that other teachers may be able to compare experiences or extrapolate ideas that can be further investigated in their own practices.

1.7 Theoretical Perspective

Learning mathematics with understanding lies at the heart of curriculum documents. When mathematics curricula involve students solely in mastering facts and procedures, it is impoverished (Schoenfeld, 1992). Skemp (1986) suspected that the reason the curriculum might be considered impoverished, in the sense that Schoenfeld (1992) described, was because preliminary, mathematical understandings were insufficient to enable further success and confidence in mathematics learning. However, it is this deep level of understanding that is largely missing in mathematics teaching and learning today (Ball, 2003). For instance, much of what is being taught in Australian mathematics classrooms is being taught and learnt in the same instrumental ways as was the case in traditional methods of teaching (Hollingsworth, Lokan, & McCrae, 2003). Over twenty years ago Skemp (1986) suggested that mathematics education must depart from a presentation of unintelligible rules that were to be memorised and applied to get the 'right answer'. Even then he believed that understanding a mathematical idea involved being able to explain how and why it is true. However, "learning mathematics *without* understanding has long been a common outcome of school mathematics instruction" (NCTM, 2000, p. 19).

Skemp's (1986) theoretical viewpoint underpins this research. His psychological view of learning was similar to Piaget's (1896-1980) view that students must be active in their own knowledge construction. He theorised that when students use previous understandings to bring meaning to new situations they expand and build on their existing schema. However, he believed that the process of assimilation and accommodation of new ideas into cognitive schema must be explicitly guided to ensure students construct meaningful webs of related ideas. It is these meaningful webs of related ideas that he believed help the development of relational/conceptual understanding. It was envisaged in this research that when students learn mathematics in ways that they can detect mathematical relationships, and are able to explain how and why things work, they will develop foundational knowledge that furthers mathematical thinking.

The constructivist view of mathematics teaching and learning is also influenced by many related contextual features. For instance, knowledge is always mediated within a sociocultural domain (Cobb, 1994; Lincoln & Guba, 2000). In other words, learning does not occur in isolation; there are many contextual features that impact on the learning. This research took the stance that both the constructivist and sociocultural epistemologies also applied to the teacher's construction of knowledge about the curriculum intent and implementation. Through personal sense-making experience, there is potential for the teacher to reconceptualise her views about mathematics teaching and learning and possibly alter her practice to more closely align with the new curriculum incentives. In summary, teaching and learning cannot be separated; they both involve the construction of mathematical knowledge, practices and dispositions, which too, cannot be separated.

1.8 Significance of this Case Study

This research is significant because the emphasis on process in teaching is to improve numeracy outcomes for all students (Council of Australian Government [COAG], 2008) and to ensure students leave school as ‘successful learners’ (MCEETYA, 2008). As pointed out in the *National Numeracy Review Report* (COAG, 2008, p. xii), “the rush to apparent proficiency at the expense of sound conceptual development needed for sustained and ongoing mathematical proficiency must be rejected.” The report stressed that curriculum emphases should discourage the implementation of “low level procedural tasks,” commonly used in Australian classrooms, in exchange for a focus on the development of “understanding and thoughtful action that deep mathematical learning requires” (COAG, 2008, p. xii). Masters (2009) recommended that for numeracy outcomes to improve there is a need to focus on sharing teacher knowledge and experiences which involves investigating classroom practices ‘in situ,’ as was the goal of this research.

In addition, a person’s disposition when using mathematics is critical. However, many of the adult population fear mathematical activity (AAMT, 1997), which means that a focus on disposition is pertinent to avoid perpetuating a fear of mathematics. The Australian Education Council (AEC, 1991) asserted that, “students come to school enthusiastic and eager to learn mathematics,” and alarmingly, “a great deal leave school with quite negative attitudes” (p. 31). Therefore, when students are learning mathematics, they need experiences that help them to develop “personal confidence, comfort and willingness to ‘have-a-go’” (Australian Association of Mathematics Teachers [AAMT], 1997, p. 14). Clearly, the swing from enthusiastic attitudes to negative attitudes opposes the MCEETYA (2008) ideal that students leave school optimistic, enthusiastic and confident. Thus, a close examination of one teacher’s attempt to implement the QSA (2004) curriculum may reveal contextual features of the learning process or environment that enable, or not, changes to students' disposition and engagement (willingness to 'have-a-go').

Curriculum bodies are ambitious and aim to improve the quality of teaching and learning, yet their ambitions will only be achieved if the proposed ideals are translated into the practice of teaching. As the McKinsey report (Barber & Mourshed, 2007) noted, the only way the quality of students' learning and thereby achievement can be improved, is by improving the practice of teachers. Research has demonstrated that there is a gap between the intended curriculum and curriculum implementation in New South Wales (Anderson & Bobis, 2005; Cavanagh, 2006). However, little is known about why implementation is lacking or, in fact, effective as in some cases. Because the New South Wales studies revealed that implementation is limited, the recommendations of those projects were considered and used to guide this project. Anderson and Bobis (2005) and Cavanagh (2006), along with Masters (2009; 2009a) and the *National Numeracy Review Report* (COAG, 2008), recommended that rich, detailed accounts of a teacher's practices are needed. Significantly, they speculated that these accounts have the potential to contribute to or advance knowledge about possible solutions that will improve successful implementation and subsequent numeracy learning, as was the aim of this research.

1.9 In Conclusion: The Research Questions

In conclusion, education systems have a role to play in supporting innovative practices that will promote numeracy learning in schools. Thus, effective solutions need to be "identified, disseminated and taken up more widely" (Masters, 2009a, p. 94). However, curriculum implementation requires the work of motivated teachers (Shulman & Shulman, 2004), and fortuitously, the teacher-participant in this study had often expressed an aspiration to improve her classroom mathematics teaching practice. The significance of this research was to contribute to that body of knowledge by providing a rich, thick description of this teacher's experience. The intention was to help others construct knowledge and understandings about curriculum goals and implementation as they relate the information presented to their own experiences (Stake, 2005). The researcher's anticipation was to acquaint other teachers with curriculum ideals, to reveal effective teaching practices, to prepare teachers for possible hindrances, and to perhaps inspire the documentation and contrasting of similar cases.

Therefore, this descriptive case study aimed to understand, in depth, the teacher's attempt to implement thinking, reasoning and working mathematically as key to developing robust mathematical understandings. It also investigated the resulting effect on student disposition and engagement. The research questions were:

1. What do teachers need to know and do to incorporate thinking, reasoning and working mathematically in their practice?
2. What effect will thinking, reasoning and working mathematically have on students' engagement and disposition towards mathematics learning?

The purpose of the case study was to identify and reveal enduring demands or evidential successes of curriculum change implementation, to illuminate factors that may, or may not, inspire teachers to persevere and sustain change. The aim was to inform the 'practice' of mathematics teaching and learning, and educate policymakers about this, as recommended by the RAND Mathematics Study Panel (Ball, 2003).

1.10 Structure of the Thesis

The next two chapters review the literature related to this thesis. Chapter two examines what is involved in curriculum implementation, the curriculum, and what it means to become numerate. It then looks at the development of mathematical knowledge and conceptual understanding. Chapter three looks at another three contextual features supposed to improve mathematics teaching and learning. These include the facilitation of an investigative learning process approach, as currently recommended by DETA (2008) and QSA (2004), ways of teaching to promote investigative learning, and the development of positive mathematical identities. The chapter then reports on related research findings and subsequent research recommendations that have influenced the planning and undertaking of this study.

Chapter four outlines the qualitative interpretive research methodology and case study research method. This chapter discusses the problem that this research addressed, which in short is the gap between curriculum intention and implementation. It then describes the constructivist theoretical paradigm and the sociocultural conceptual framework underpinning this research. The research questions and objectives are also outlined in this section, which is followed by an explanation of the strengths and limitations of the single case study method. The research method and design are then discussed, including the case selection, researcher's role and data collection methods.

There were three phases of data collection and three data gathering methods (questionnaires, interviews and observations) to allow for triangulation of the data. Each is described in detail, followed by the qualitative content data analysis approach applied. The last paragraph points out the significance of this research.

Chapter five examines Reagan's beliefs. This chapter uses the data gathered through the interviews to describe Reagan's evolving interpretation of the key messages of the curriculum, as proposed by QSA (2004). It uses this data to examine her thoughts about thinking, reasoning and working mathematically, lifelong learning and making learning relevant. The data were then used to outline her beliefs about learning with understanding and how students develop conceptual understanding, and to detail and discuss her experience, as she tells it, about scaffolding investigative thinking processes. The last section draws out her thoughts about establishing a community of practice through helping her students to become mathematically smart, building mathematical identity, mathematical stamina and a working mathematical community.

Chapter six and seven then use the classroom observations, along with the students' pre- and post-questionnaires and work samples, to detail and discuss the classroom lived experience. These data were used to interpret the gradual progression of the curriculum implementation process. Chapter six describes the understandings, the 'ways of knowing,' that Reagan envisaged the students would be developing, and then describes how she incorporated the mathematics practices, the 'ways of doing' into the classroom learning environment. Chapter seven reveals how the mathematics practices' framework supported Reagan's attempt to scaffold the students' thinking and reasoning processes and then reveals the changes to the students' disposition from their point of view.

Chapter eight draws the findings together to answer the research questions. This chapter also relates the findings to the anticipated emphasis of the National Curriculum Board's (2009) paper, the *Shape of the Australian Curriculum: Mathematics*, and to other policy recommendations. It finishes with possible avenues for future research.

In summary, this chapter has introduced the research project and outlined the significance of this study. It pointed out that contemporary societal demands require students to be able to confidently choose, use and apply mathematics for successful lifelong learning (MCEETYA, 2008). Thus robust mathematical understandings need to be developed to ensure students are equipped with knowledge and understandings that will, in turn, enable and advance mathematical thinking. Such understandings are developed when students are given instructional tasks that require them to investigate the ‘how and why’ of mathematical ideas and the inherent mathematical relationships (Hiebert et al., 2000; Skemp, 1986). Key to students’ construction of knowledge, according to new curriculum documents (QSA, 2004; 2007), is to provide them with sense-making learning opportunities. However, past research (Anderson & Bobis, 2005; Cavanagh, 2006; Woodbury, 2000) indicates that there is a significant gap between curriculum intent and implementation. Consequently, it has been recommended (e.g., Freebody, 2005; Gillard, 2009; Lloyd, 2008; Masters, 2009) that research is needed that will provide detailed accounts of individual teacher’s practical experiences of curriculum implementation, which, as the chapter pointed out, was the goal of this research.

CHAPTER TWO

SHIFTING GEARS: DRIVING CHANGE

"Mathematics, the subject provides the content on which to base the development of numeracy; but mathematical knowledge is only one aspect of numeracy."

(BTR, 2005, p. 4)

2.1 Introduction

Successful navigation of the rapidly changing economy brings new societal demands. Technological innovation has lowered prices and increased the efficiency of computers, mobile telephones and the internet, which has had prominent effects on the 'new economy' (Landefeld & Fraumeni, 2001). Citizens are now required to use and understand the functionality of technologies. They must also be able to judiciously handle data because bandwidth communication networks now make it possible for the "seamless delivery of ever-higher volumes of data and services anywhere, anytime" (Commission of the European Communities, 2006, p. 5). Thus, citizens need to be able to access data and make sense of data in order to arrive at well-reasoned conclusions and informed decisions (Booker, Bond, Sparrow & Swan, 2004). The training for this competence begins at school.

In order to cope with the societal demands of contemporary times, students need a more sophisticated repertoire of abilities than was necessary in the past. In fact, the Organisation for Economic Co-operation and Development [OECD] (2006) believes that it is possible that the skills that students now learn in school will not be sufficient to serve the needs of their adult lives. The OECD has used the term 'skills,' to describe what the students learn at school; however, for the purpose of this research, the term 'mathematical competence' was preferred because it is more encompassing. These terms relate more specifically to the current emphasis placed on mathematics teaching and learning in response to the new economy. For example, government bodies and policymakers worldwide believe that more than ever students need to understand mathematics in ways that will develop confident and insightful application of mathematical ideas, and facilitate lifelong learning (AEC, 1991; AAMT, 1997; NCTM, 2000; MCEETYA, 2008; OECD, 2006). Hence, the emphasis is being placed on knowing how to work with mathematics rather than simply knowing mathematics. From the time this research began in 2007, curriculum goals in Australia have been undergoing changes to meet these new demands.

This chapter outlines new curriculum goals intended to assist students to become competent and confident users of mathematics, capable of continued learning over the span of their adult lives. Curriculum implementation is discussed, and the terms 'numeracy' and 'mathematical literacy' defined. The last section discusses the ways of knowing and understanding mathematics. The next chapter, as part of this literature review, discusses ways of learning mathematics, teaching mathematics and ways of being mathematically. The ways of being involve developing positive and productive mathematical dispositions and identities. The chapter then evaluates recent research into curriculum implementation to examine what research foci and methodologies are being recommended.

2.2 New Curriculum Goals

The emphasis on what outcomes are necessary for school leavers has changed. For instance, the second goal outlined in the *Melbourne Declaration on Educational Goals for Young Australians* (MCEETYA, 2008, p. 8) stated that: "All young Australians become successful learners, confident and creative individuals, and active and informed citizens." The declaration stressed that students must develop capabilities that enable them to become deep and logical thinkers who can confidently continue learning both in and beyond school. However, there is an uncertainty about exactly what capabilities school leavers may need, thus the emphasis is now being placed on students developing mathematical competence that enables them to continue learning throughout their lives, and on entry into the work force. To elaborate, a lifelong learner, according to the Queensland Studies Authority (2004, p. 2), is:

- a knowledgeable person with deep understanding;
- a complex thinker;
- a responsive creator;
- an active investigator;
- an effective communicator;
- a participant in an interdependent world; and

- a reflective and self-directed learner.

These attributes will be addressed throughout this literature review as relevant.

The emphasis on lifelong learning has implications for the direction of mathematics teaching and learning. For example, the *National Mathematics Curriculum: Initial Advice* (National Curriculum Board [NCB], 2008) explained that teachers now need to ensure that when students leave school they know and understand mathematics beyond simply learning procedural skills, the rules of mathematics. This change of emphasis involves learning and understanding mathematics in ways that enable students to recognise the connections and applications of mathematics knowledge, as well as its transferability (Booker et al., 2004; DETA, 2008; Hiebert et al., 2000; Kilpatrick et al., 2006; NCTM, 2000; Van de Walle & Lovin, 2006). The National Curriculum Board (NCB, 2008, p. 5) stated that a fundamental goal of the curriculum should be:

> educating students to be informed thinking citizens, interpreting the world mathematically, appreciating the elegance and power of mathematical thinking, experiencing mathematics as an enjoyable experience, and using mathematics to inform predictions and decisions about personal and financial priorities.

The national emphasis promotes two key aims for the mathematics curriculum. The first is to develop mathematical power, which involves knowing and understanding the discipline of mathematics in ways that facilitate mathematical thinking for astute decision making and problem solving. The second aim is more personal: it involves developing an appreciation of the power of mathematical thinking as a way to make sense of the world. This aim involves the development of positive mathematical dispositions through recognising that mathematics does make sense, and therefore, with perseverance and knowing how to think mathematically, problems can be solved (Kilpatrick et al., 2006). Hence, the outcome of an effective mathematics education, as suggested above, requires students to leave school equipped with mathematical knowledge, understandings and strategies that enable mathematical thinking and reasoning. Importantly, students should also leave school with a personal trust in their capabilities to use and learn mathematics.

The focus on mathematical thinking and the development of a positive disposition is not such a recent innovation. It was recognised almost twenty years ago by the Australian Education Council (1991) that it was impossible to determine exactly what mathematical competencies students would need in the 21st century. Back then, the AEC identified important components of a mathematics education necessary to prepare students for societal demands. These components, which are very similar to the current emphasis, were that students must develop confidence in their ability to apply mathematical ideas, to problem solve and to communicate with and about mathematics in systematic ways (AEC, 1991). In the U.S., a similar agenda for school mathematics was reflected in two successive documents prepared by the National Council of Teachers of Mathematics (NCTM, 1989; 2000). Hence, with the focus still being placed on the development of mathematical thinking and reasoning strategies and positive dispositions towards mathematics and mathematics learning, the question remains as to why these ideals appear to have not been translated into practice. The next section explores what is entailed in the curriculum and its implementation.

2.3 The Curriculum and Implementation: What is Involved?

First, what is the curriculum? The curriculum plan generally gives details of the what, when and how of the teaching and learning process (DETA, 2005). The curriculum is dynamic, and, according to DETA (2008), it encompasses five aspects that are all responsive to each other. These aspects can be thought of as a sequence of elements: “the intended, enacted, experienced, assessed and achieved curriculum” (DETA, 2008, p. 1). The intended curriculum is what the State or National policy documents expect students to learn. The enacted curriculum refers to how teachers deliver the intended curriculum and how students engage with the curriculum. The experienced curriculum refers to how students experience the curriculum which will differ from student to student, according to DETA. The assessed curriculum includes the teacher’s assessment practices, and the achieved curriculum, as DETA explains, refers to what students have learned.

Whilst the curriculum may appear in a written format in school or policy documents, it involves much more. The curriculum encompasses everything that occurs within a school environment that supports student learning (DETA, 2005; DETA, 2008; Stenhouse, 1978). Hence, the curriculum is more than an intention, prescription or aspiration; it includes what happens and what is achieved within real settings (Stenhouse, 1978). There will be aspects of the curriculum and learning environment that support learning and aspects that may limit learning. This research took into consideration experiences of both students and the teacher of the ‘intended’, ‘enacted’, ‘experienced’ and ‘achieved’ curriculum. To maintain focus on the research goals, the ‘assessed’ curriculum was not investigated, although the teacher’s formative assessment practices were considered. When the practical part of this research was carried out teachers were encouraged to assess students’ knowledge and understandings. Whereas, in the following year DETA (2008) introduced new assessment practices, whereby the students’ competence to think and reason, communicate and reflect on and with mathematical ideas were to be assessed alongside their knowledge and understandings. Hence, the teacher’s formative assessment processes were considered because these decisions related more specifically to thinking, reasoning and working mathematically.

2.3.1 Queensland Curriculum Goals

In 2007 when this research began, Queensland schools were using the *Years 1-10 Mathematics Syllabus* (Queensland Studies Authority [QSA], 2004) to inform their curriculum. This syllabus document was “designed to help students become lifelong learners” (QSA, 2004, p. 2). The rationale of the syllabus highlighted three essential elements aimed towards developing attributes of a lifelong learner, which were to be incorporated into mathematics teaching and learning programs. These three elements were the ability to think, reason and work mathematically, active engagement and the development of a positive disposition. The rationale of the syllabus (QSA, 2004, p. 1) stated that:

> Thinking, reasoning and working mathematically are essential elements of learning for, about and through mathematics. Positive dispositions towards mathematics learning and active engagement with mathematical tasks are integral to thinking, reasoning and working mathematically. Such dispositions are developed through student engagement in mathematical investigations relevant to a range and balance of situations from life-related to purely mathematical. When making sense of life experiences or seeking solutions to problems, individuals:
>
> - see the mathematics in situations encountered;

- plan, investigate, conjecture, justify, think critically, generalise, communicate and reflect on mathematical understandings and procedures, and
- select and use relevant mathematical knowledge, procedures, strategies and technologies to analyse and interpret information.

The key message of the 2004 syllabus was to ensure students were prepared for lifelong learning, through developing dispositions to think, reason and work mathematically in positive ways across a range of situations. Then in 2008, the 2004 syllabus document was to be considered an information resource for teachers (DETA, 2005). Schools were supposed to adapt their learning programs to align with the *Queensland Curriculum, Assessment and Reporting [QCAR] Framework* (DETA, 2005). Importantly, the key message underpinning the new documents is also to develop attributes of a lifelong learner capable of thinking, reasoning and working mathematically in productive and insightful ways.

The new framework was intended to streamline planning. For instance, the introduction of the QCAR framework was an attempt to simplify an over-crowded curriculum (Masters, 2009a) because the QSA (2004) syllabus was made up of almost 600 Core Learning Outcomes. The framework was also designed to align the curriculum, pedagogy, assessment and reporting within and across schools (DETA, 2005). In essence, the motive of the QCAR framework was to “help schools deliver more cohesive learning programs and help students achieve deeper levels of understanding” (DETA, 2005, p. 2). The new direction is an attempt to assist teachers in their curriculum planning by providing them with statements indicating what is considered essential for all students to know, understand and be able to do by the end of years three, five, seven and nine (QSA, 2007c, p.1). The *Essential Learnings* statements differ from the outcomes of the previous syllabus because the statements describe the key concepts, facts and procedures, as well as the specific ‘ways of working’ that students need to develop. These descriptive statements are intended to “free teachers up to focus more strongly on their pedagogy” (DETA, 2008, p. 1).

Importantly, and significant to this research was DETA’s (2008, p. 1) statement that the teacher is required “to focus on the *learning* rather than the *teaching.*” This change, according to DETA, is achieved when the teacher adopts a process approach where she encourages learning processes comprising thinking, reasoning and working mathematically. The learning process has been emphasised in such a way that the teachers are being asked by DETA to assess students' capabilities to think and reason, communicate and reflect with and about mathematical ideas. As mentioned earlier, assessment practices no longer involve simply assessing students' knowledge and understanding; students' thinking and reasoning, communicating and reflecting processes must also be assessed. Hence, there has been a deliberate attempt to bring the learning process to the fore, more so than in the previous documents.

The process of working mathematically has been reiterated in the new curriculum framework. Even though in 2004 the syllabus documents encouraged learning processes comprising thinking, reasoning and working mathematically, these learning processes are now captured in prescriptive assessment criteria as 'assessable elements' (DETA, 2005). Similar to the previous QSA (2004) rationale, the *Education Queensland Scoping and Sequencing Essentials: Years 1-9* (DETA) (2008, p. 45), stated that mathematical competency requires students to develop "a disposition to think and act mathematically and the confidence and intuition to apply mathematical concepts to explore and solve everyday problems that confront them." Yet, a poignant difference between the two curriculum frameworks is the inclusion of the 'working mathematically' strand. This strand must be integrated with the other five content strands of chance and data, number, algebra, measurement and space (DETA, 2005). The inclusion of the working mathematically strand is to assist teachers to ensure students develop capabilities that will enable them to contextualise and pose questions, check, verify and communicate ideas, and to develop mathematical thinking strategies (DETA, 2008). The *Scoping and Sequencing Essentials Years 1–9* (DETA, 2008) document was not available when the practical part of this research was completed. However, the key message of the *Years 1-10 Mathematics Syllabus* (QSA, 2004), which was the curriculum implemented as part of this research, reflected the working mathematically emphasis encouraged by DETA (2008).

2.3.2 Curriculum Implementation

The success of curriculum implementation depends on two factors. The first factor is internal and involves the way the teacher interprets and understands the curriculum (Cuban, 1993; Stenhouse, 1978; Shulman & Shulman, 2004; Reys, Reys, Barnes, Beem,

& Papick, 1997; Woodbury, 2000) and her beliefs about how students learn. The second factor is external to the teacher and involves the support provided by the school community or professional development programs (Masters, 2009a; MCEETYA, 2008). Both factors are integral to this research, as explained below.

First, research into curriculum implementation must begin with teachers' beliefs and practices (Board of Teacher Registration Queensland [BTR], 2005; Reys et al., 1997). The teacher's thoughts and beliefs will impact on her interpretation of the curriculum, and in turn will determine the extent to which curriculum ideals will be implemented. Handal and Herrington (2003, p. 62) emphasised that for change to be realised, researchers need to closely examine a teacher's beliefs because these beliefs will be a "significant mediator in curriculum implementation." Often though, the teacher's thinking about reform efforts has been ignored by curriculum researchers and designers (Ross, Cornett & McCutcheon, 1992; Woodbury, 2000). Ignoring teacher's beliefs is a problem because no matter what the qualities of curriculum standards, the curriculum will not teach itself (Ball, Hill & Bass, 2005). Cuban (1993) warned that unless the teacher could interpret the curriculum, she would ignore new ideals. An important intention of this research was to make transparent the teacher's beliefs and thinking about the curriculum.

Another internal factor is the teacher's sense of ownership over the curriculum approach. Teachers are often reluctant to change unless they have a sense of control over the new ideals (Farmer, Gerretson & Lassak, 2003). Just like their students, teachers are learning. Therefore, they need time to plan, teach and reflect on the experience when attempting to implement curriculum change. New knowledge cannot simply be tacked on; existing knowledge needs to be remodelled so that new knowledge can be integrated with understanding (BTR, 2005). As DETA (2005) pointed out, teachers need time and support to understand what the new curriculum approach involves throughout the implementation process. Cavanagh's (2006, p. 117) research, on some NSW teachers attempting to incorporate the *Working Mathematically* strand into their teaching practice, revealed that teachers need to "reconceptualise their views on the process of learning mathematics." Thus a gradual approach is necessary if teachers are to construct their own understandings and personal reverence for the thinking and reasoning learning process.

This research took the view that the teacher may be uncertain about what is involved in adopting proposed curriculum ideals because she had not experienced the curriculum ideals. For instance, research has found that even though some teachers appear to understand reform recommendations, translation into practice is not widespread (e.g., Anderson & Bobis, 2005; Cavanagh, 2006; Reys et al., 1997). This may be because teachers do not have a clear image of what new curriculum goals actually look like in practice, as was found by Anderson and Bobis (2005) and Bobis (2004). For example, when Bobis (2004) investigated some teachers' views as they attempted to implement the *Count Me In Too* early numeracy program, she found that because the teachers were unsure of what the program looked like in practice, they became anxious and overwhelmed. The desire to change is partly self-initiated, but change also requires knowing what is involved, and in this curriculum implementation attempt, it involved knowing what a process approach might look like in practice.

The second factor is external to the teacher and involves the culture of the workplace environment. For instance, it involves the support provided by the school community or professional development programs (Masters, 2009a; MCEETYA, 2008). However, even though DETA (2005) suggested that professional support is important, it is often limited. For example, Bobis (2004) found that a significant barrier to implementing the new program, according to the teachers, was the lack of professional support. Conversely, a beginning teacher-participant, who was willing to listen and act upon advice, found that when she did receive guidance it was invaluable. An integral part of this research was to support the teacher throughout the implementation process, as in the case of the beginning teacher in Bobis' research. Clandinin (2008) found that creating a middle space to open pathways between theory and practice increased the opportunities for teachers to make changes to their practice. It was anticipated that the collaboration between the researcher and the teacher-participant in this research may create the space that Clandinin described.

Other external factors include the contextual features within the school environment. Whilst the "teacher is the key dimension of the classroom community" (Grootenboer & Zevenbergen, 2008, p. 244), the resources, the staff, the parents and the students themselves all make up part of the classroom context. As Grootenboer and Zevenbergen pointed out, the classroom context is complex and involves a network of these interrelating aspects which also extend beyond the immediate context. Importantly, the students come to the classroom with preconceived ideas about mathematics and mathematics learning, which are often generated from peers and family (Grootenboer & Zevenbergen, 2008), or prior learning experiences (Skemp, 1986). Their existing mathematical understandings, attitudes, abilities and beliefs about their own knowledge, will impact on their learning, accessibility to the learning, as well as their willingness to participate in the learning (Hiebert et al., 2000; Kilpatrick et al., 2006; Skemp, 1986; Van de Walle & Lovin). Yet, the students' unwillingness to engage and participate in the learning, along with their development of positive mathematical identities have been ongoing concerns for mathematics education researchers for some time (Boaler, 1999; Grootenboer & Zevenbergen, 2008). The students' dispositions and engagement levels are influenced by external factors that may influence the teacher's implementation decisions, as investigated in this research.

In summing up, curriculum implementation is a complex process. It involves the teacher's beliefs, understanding and sense of ownership over the new approach. It also involves professional support and knowledge of how to manage change (Masters, 2009). Successful implementation also involves understanding and managing other contextual features, which most importantly require understanding the students and how they learn. Therefore, the aim of this research was to understand those aspects of the implementation process that might help the teacher gain a sense of control over the new curriculum ideals that have been recommended to improve students' numeracy outcomes.

2.4 Becoming Numerate

Over time, views of numeracy have broadened. The *Cockcroft Report* (Cockcroft, 1982) made the point that a numerate person would be expected to make sense of mathematics, appreciate mathematics, and understand how mathematics can be used for communication. The report defined numeracy as an "at-home-ness with numbers" which would enable a person to "cope with the practical demands of his everyday life" (Cockcroft, 1982, p. 11). However, in a changing world, when technology advances rapidly, the practical mathematical demands of everyday life also increase. Consequently, definitions of numeracy have evolved. For example, in Australia, Willis (1998) regarded numeracy as the "capacity to bridge the gap between 'mathematics' and the ‘real world’, to use in-school mathematics out-of school” (p. 37). Similarly, Coben (2000) asserted that a numerate person can think mathematically; he/she knows when to use mathematics, how to use it, and what to use. Becoming numerate, therefore, involves more than developing mathematical understandings, as will be explained.

Numeracy is a concept, a way of knowing. Willis (1990) viewed the concept of numeracy as an integration of three types of knowledge: mathematical knowledge, contextual knowledge and strategic knowledge. The three types of knowledge were later viewed by Willis (1998) as ‘knowhow,’ mathematical knowhow, contextual and situational knowhow and strategic and critical knowhow (Willis, 1998). Mathematical knowhow refers to core mathematical knowledge, understandings (concepts), skills (procedures) and processes (ways of working) that are necessary to solve quantitative problems within a real context (Thornton & Hogan, 2004). If one has mathematics knowledge, he or she will ‘know’ about and be able to ‘do’ mathematics, which involves being able to think mathematically (Ball, 1988). However, ways of thinking about mathematics in Willis’ (1998) view also involve contextual knowhow and strategic knowhow, which involve knowing mathematical strategies and applications.

Before moving on it is necessary to point out the difference between a procedure, concept and a process. Knowing how to perform the cognitive activity involved in how to add 6 + 4 is referred to as procedural knowledge (McInerney & McInerney, 2002). Thus, procedural knowledge is the understanding that there is a step-by-step ‘procedure’ to solving a mathematics problem, often also referred to as a skill. Conceptual knowledge refers to knowledge that is understood, which involves organising related ideas into a meaningful network of ideas (Van de Walle, 2004). For example, using the above number ‘fact’ 6 + 4 = 10 a student who understands the ‘concept’ of addition would also know that 4 + 6 = 10, 10 – 6 = 4, or 10 – 4 = 6 because of the relationship of addition and subtraction existing in the student’s mind. The term ‘process’ relates to the cognitive processes going on in a student’s mind, which of significance to this research are the processes comprising thinking and reasoning, communicating and reflecting, as outlined by DETA (2008) and QSA (2004). The term ‘mathematical idea,’ for the purpose of this research, refers to a fact, concept, procedure or process.

The concept of numeracy includes the capacity to think about, transfer and apply mathematical ideas across contexts. Contextual knowhow means that one has the capacity to apply particular mathematical ideas in any given situation for real purposes (Willis, 1998). Or the reverse, one knows how the context might impact on the mathematical ideas being used (Thornton & Hogan, 2004). For example, a person working in a bank knows what mathematics to use for balancing transactions and that this mathematics needs to be exact. When grocery shopping, this person knows that the same accuracy is not always required, and subsequently, may choose to use mental arithmetic estimations to approximate the cost prior to reaching the checkout. Contextual knowhow in this sense is the capability to select and apply appropriate mathematical ideas purposefully. Strategic knowhow refers to the actual processes and strategies one uses or enacts to solve everyday problems (Willis, 1998). It also involves knowing how to communicate and link mathematical knowledge. For example, the grocery shopper who likes to estimate the cost will have developed mental arithmetic strategies, such as rounding. An item that is $4.85 could be thought of as $5.00. Then the next item may cost $4.15 and be thought of as $4.00, thus 4+5=9, so the shopper estimates the cost to be $9.00.

In comparing the term ‘numeracy’ with the phrase ‘mathematical literacy’ there are many similarities. For example, the OECD Program for International Student Assessment (PISA, 2006) defined mathematical literacy as:

> an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen. (OECD, 2006, p. 72)

This definition asserts that students will need to develop mathematical knowledge and ways of working with that knowledge that will enable them to make sense of, and manage, mathematical situations. Hence, the definition aligns with Willis’ (1998) mathematical, contextual and strategic knowhow. The definition also takes into account that each individual’s life situations both in the present and future will differ, and consequently, flexible use of understandings must be developed. Underpinning both terms is recognition that learning mathematics for rapidly changing times requires not only developing knowledge and understandings of the discipline of mathematics, but also developing idiosyncratic ways of working with and thinking about those understandings.

However, becoming numerate is more complex than acquiring mathematical, contextual and strategic knowhow. As pointed out earlier, the National Curriculum Board (2008) stressed that students should also develop an appreciation of mathematics as a discipline and of mathematical thinking as a way to make sense of the world. This appreciation implies that students know how to think, reason and work with mathematics. It also implies that students develop positive beliefs about their mathematical learning abilities in order to confidently and competently draw upon and apply existing mathematical ideas and thinking processes to continue expanding their understandings. As QSA (2004, p. 1) asserted: “Positive dispositions towards mathematics learning and active engagement with mathematical tasks are integral to thinking, reasoning and working mathematically.” Curriculum documents (DETA, 2008; QSA, 2004) emphasise that a positive disposition and a willingness to engage actively in mathematical experiences are critical to becoming numerate. On the other hand, active engagement requires a willingness, desire and interest to learn and use mathematics (QSA, 2004), and therefore, depends on a positive disposition. In essence, a student’s disposition is a determining factor for successful and ongoing mathematics learning and numeracy development (Kilpatrick et al., 2006).

This research took the stance that a positive disposition is something that students construct for themselves through positive and productive learning experiences; and is not something that can be taught or given. Thus it was anticipated that students may be more inclined to cognitively engage with mathematics when they believe they have the capability to make sense of mathematics for themselves (Hiebert et al., 2000; Kilpatrick et al., 2006). In other words, when students perceive that they ‘can’ think, reason and work mathematically, that they do have the knowhow and capability to think mathematically, they may be more inclined to take up an active role in their own mathematics learning. As Kilpatrick and colleagues (2006) pointed out, students who perceive their mathematical ability as something that can be expanded rather than something that is fixed are more inclined to seek out challenging situations. They suggested that students are more willing to engage and less likely to be discouraged by failure when they view themselves as being in a process of learning mathematics.

Thus the term ‘productive disposition’ has been borrowed from the RAND study Panel (Ball, 2003) and Kilpatrick and colleagues (2006) in exchange for the term ‘positive disposition.’ The reason is that their term combines the attributes of ‘active engagement’ and ‘positive disposition.’ For instance, a productive disposition refers to:

> the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics (Kilpatrick et al., 2006, p. 131).

This definition integrates both thought and action. It suggests that students are required to develop an appreciation of mathematics as a discipline and as a way to make sense of the world, as well as an appreciation of effort (thinking and reasoning mathematically) as a way to learn and use mathematics. The definition therefore captures the essence of the terms ‘positive disposition’ and ‘active engagement,’ as outlined by DETA (2008) and QSA (2004).

To summarise, this becoming numerate is all encompassing. It involves developing mathematical, contextual and strategic knowhow (Willis, 1998) as well as a productive disposition (Ball, 2003; Kilpatrick et al., 2006). In other words, becoming numerate involves the development of mathematical understandings and thinking processes as well as idiosyncratic ways of thinking about and working with those understandings and processes. The National Curriculum Board’s (2009, p. 5) paper, *Shape of the Australian Curriculum: Mathematics*, summarised what it means to be numerate:

> Numeracy is the capacity, confidence and disposition to use mathematics to meet the demands of learning, school, home, work, community and civic life.

This definition again reiterates the importance of developing a ‘productive disposition.’ However, a productive disposition involves more than being confident; it involves a willingness to persevere in ways that students will initiate and sustain investigative thinking (Kilpatrick et al., 2006). The next section discusses some ideals that have been proposed to improve the quality of mathematics teaching and learning, and outlines the theoretical framework of this study.

2.5 Ways of Knowing and Understanding

Constructivist and social constructivist views of knowledge construction underpin this research, as will be explained in this section. The epistemological focus in curriculum documents has shifted away from simply relying on transmissive methods of teaching. For instance, the Queensland mathematics syllabus documents (DETA, 2008;

QSA, 2004), are based on the principles of constructivist and social constructivist theories of learning espoused by Piaget (1896-1980) and Vygotsky (1896-1934) respectively. Constructivist and social constructivist theories view knowledge as not being ready formed or absolutely true (Airasian & Walsh, 1997). These views of learning perceive that the brain does not passively receive information; rather, it seeks to interpret the information (Crick, 1994). However, in the past, many people viewed mathematics as a fixed body of knowledge that lacked the scope for creativity (Romberg, 1992). Romberg maintained that this narrow view resulted from the transmissive approach schools once used to teach mathematics. When teachers used transmissive methods, they positioned the students as passive recipients of information and focussed on getting them to know and memorise the rules and concise methods of working with procedures (Anghileri, 2000; Bobis, Mulligan, Lowrie & Taplin, 1999). In other words, the teacher was the 'teller' and the students were the 'receivers' of knowledge. By contrast, constructivist views of learning recognise that personal meaning-making is essential to advance knowledge construction.

Constructivism is not an instructional approach. Instead, it is a psychological view of knowledge construction, describing how learners develop and use cognitive processes (Airasian & Walsh, 1997). Constructivist theories of learning view knowledge as being constructed through the testing of hypotheses (Airasian & Walsh, 1997). Because individuals arrive at their own meaning-making in response to the assimilation and accommodation of new experiences with existing beliefs, curriculum documents (DETA, 2008; QSA, 2004) encourage teachers to adopt an investigative approach to mathematics teaching and learning. The goal is to create opportunities for students to become active in their own knowledge construction through processes of thinking and reasoning to test and prove mathematical ideas. It is in the process of interpretation, making sense of mathematical situations through testing and proving ideas, that sense is made, knowledge is constructed, and understanding is developed (Crick, 1994).

Knowledge is also constructed during social interaction. According to QSA (2004) and DETA (2008), when students communicate their thinking and reasoning processes, they make sense of mathematics for themselves. This view aligns with the work of Vygotsky (1896-1934) who perceived thinking to be an interactive dialogue that one has within the self and with others. He believed that people are not sure what they are actually thinking until they hear themselves explain their thoughts. The Queensland Studies Authority (2004, p. 3) proposed that when students can "communicate their thinking and reasoning in ways that make sense to themselves and others" they become a "knowledgeable person with deep understanding," an attribute of a lifelong learner. However, as students engage in discussion to clarify their thoughts, they are also being placed, or they are placing themselves, in potentially compromising positions exposing their mathematical ability. Thus, the learning environment needs to be supportive because communication of ideas is crucial if students are to learn with understanding, as the next section discloses.

2.5.1 Learning with Understanding

Learning mathematics *with* understanding is the vision of contemporary curriculum and standards documents (e.g., BTR, 2005; DETYA, 2000; NCTM, 2000; MCEETYA, 2008; QSA, 2004). However, according to the NCTM (2000, p. 19), "learning mathematics *without* understanding has long been a common outcome of school mathematics instruction," and subsequently, there has been much discussion and research into the problem since the 1930s. The result has been a growing body of research into the cognitive science of learning with understanding. In particular, Richard Skemp's (1986) seminal work has made a substantial contribution to mathematics education, and is frequently cited (e.g., Gray & Tall, 1992; NCTM, 2000; Van de Walle 2004; Van de Walle & Lovin, 2006; Willis, 1990). It is his theoretical viewpoint of understanding that has influenced the researcher's view of learning and knowledge construction.

Skemp (1986) referred to two concepts of understanding which he termed 'relational understanding' and 'instrumental understanding'. Relational understanding involves recognising the connections between ideas in one's mind. A real-world problem experienced by Skemp (1993) demonstrated the usefulness of developing relational understanding. Skemp asked a colleague for directions to get home from a new school he visited. The informer told Skemp (1993, online) to "go left, left, left, right, half right, and then left onto the A45." However, Skemp missed the first turn at the gate which resulted in confusion and a lengthy trip home. He had failed to accurately memorise the steps. The next time his friend sent him a map, which meant that he could then work out a plan for getting home; internally he had constructed his own cognitive map of the town. This time he understood conceptually the layout of the town. The internal representation of related ideas that Skemp (1993) had constructed is what Skemp (1986) referred to as structured knowledge, synonymous with Piaget's (1896-1980) 'cognitive schema.' Skemp (1993) explained that with this structured knowledge he could not only get home, he could vary his route to include a stop for petrol and food. The added options his cognitive map provided illustrate the 'problem-solving power' of structured knowledge.

Mental representations are central to developing conceptual understanding. As in the example above, Skemp's past experiences enabled him to relate bits of existing knowledge (knowing where north was, or knowing the scale of the map, and so on) to create an internal, mental representation of related ideas, henceforth, a conceptual whole. Thus, a 'concept' "is a way of processing data which enables the user to bring past experience usefully to bear on the situation" (Skemp, 1986, p. 27). According to cognitive scientists (Anderson, Reder & Simon, 1998; Skemp, 1986), mathematical competence largely depends on the availability of mental structures that can be drawn upon and utilised to create further mental structures in response to an experience. To explain the networks of ideas constructed in the mind, Hiebert and Carpenter (1992) used the metaphor of a spider's web. They suggested that the nodes could be thought of as mental representations of mathematical ideas, facts or procedures, and the threads that join the nodes could be thought of as the relationships or connectors that link those ideas. When facts and procedures are taught with understanding, that is when teachers help students to create mental representations by explicitly highlighting the connections between new and existing ideas, students develop deeper understandings as part of a coherent whole (Bransford, Brown & Cocking, 2000; Hiebert & Carpenter, 1992; Kilpatrick, et al., 2006).

Thus, as Siemon (2001) suggested, the primary focus of school mathematics should be to assist students to create abstract mental objects in their minds which can then be manipulated with understanding and confidence. It is important to help students to store mental objects in their minds as a part of a coherent, conceptual whole. Unrelated facts or procedures will be much harder to remember than if they were part of a conceptual whole, according to Skemp (1986). He believed that when there is not a network of mathematical ideas to draw upon, each new piece of information that students encounter will be learned and stored as an isolated fact. Even though isolated facts may be understood, they become stored in the mind as unrelated ideas, which in Skemp's opinion, is the basis of developing an 'instrumental understanding.' On the other hand, developing conceptual understanding involves drawing upon existing bits of knowledge to make sense of new ideas, which in turn expands the initial knowledge and ideas (Van de Walle & Lovin, 2006).

Continued learning will be compromised if conceptual understanding has not been developed. Skemp (1986) explained that subsequent learning that takes place for a student with insufficient schema is based on a willingness to accept. In other words, the student must be willing to receive and absorb knowledge either from the teacher or from a text book. In this situation the students are positioned as receivers of knowledge, and the learning that takes place is "rote-learning not schematic-learning" (Skemp, 1986, p. 122). The students in this position will not be well placed to develop productive dispositions. Skemp recommended that to remedy this situation, the students will need to go back to the beginning to understand the how and why of procedures. However, he suggested that often neither the teacher nor the students will realise that the students need to reorganise their understanding, and even if they did, there is often not the time to start over.

Hence, students should be given opportunities to investigative mathematical ideas in depth. They need to be able to draw on and then build upon existing cognitive schema (Anderson, Reder & Simon, 1998; Skemp, 1986). The beauty of mathematics, according to Romberg (1992), is its potential to investigate, test, prove and discuss all sorts of interesting number, geometric and algebraic patterns and relationships. Thus, as a discipline, mathematics provides scope for creative learning experiences that promote investigative thinking, and thereby, conceptual understanding. Conversely, the development of conceptual understanding also helps students to develop an insight into the discipline of mathematics itself (Romberg, 1992). For instance, when students understand how and why certain mathematical ideas work, they will understand where and why the mathematical idea is useful, and thereby know how to apply the idea, and in what contexts this idea can be used (Kilpatrick et al., 2006). This deep understanding about mathematics and its applications supports students' numeracy development.

2.5.2 Constructing Conceptual Understanding

Assisting students to construct conceptual understanding is complex, and requires 'complex thinking' (QSA, 2004). As Skemp (1986, p. 27) stated, the "actual construction of a conceptual system is something which individuals have to do for themselves." However, he affirmed that the process could be sped up with the right materials and guidance. Teachers can assist students to develop connected understandings by engaging them in reflective thinking (Battista, 1999; Kilpatrick et al., 2006; Van de Walle, 2004; Van de Walle & Lovin, 2006). Reflective thinking helps students to make explicit the links between mathematical ideas and relate their results to existing cognitive schema. As Stephens (2000) asserted, building these connections in students' mathematical thinking lies at the heart of any effective numeracy program.

Reflective thinking can be encouraged through communication. Van de Walle and Lovin (2006, p. 4) asserted that reflective thinking "is an active, not a passive, endeavour," and therefore, the challenge for teachers is to ensure students are mentally and socially engaged. They suggested that when students engage in activities that force them to use their ideas as they search for solutions, they engage in reflective thinking. It is when students communicate, explain and justify their thoughts and strategies that they reflect on their thinking; the reflective thinking process then enables them to create new mathematical ideas (Campbell & Rowan, 1997; Van de Walle & Lovin, 2006; QSA, 2004). The QCAR framework (DETA, 2008) has recognised the significance of communication and reflection, as such; teachers are now required to assess students' communication and reflection processes. Hence, to advance cognitive development and the construction of conceptual understanding, this research adopted Battista's (1999) and Hiebert and colleagues' (2000) perspectives regarding reflective thinking and communication, as explained below.

Communication and reflective thinking help the development of cognitive webs of ideas. Battista (1999) posed that when students are engaged in a learning cycle of action, reflection and abstraction (Battista, 1999), understanding is developed. The learning cycle is mobilised through communication and reflective thinking (Hiebert et al., 2000). To elaborate, in the initial stages of learning, students initiate an action. They may use concrete objects or, as understanding develops, concrete objects may be replaced with graphic or symbolic representations. The manipulation and talking about these representations helps students to create visual images in their minds to connect to in later learning. The terms ‘concrete materials’ and ‘manipulatives’, for the purpose of this research, imply objects that can be moved around by students (Hynes, 1986). This research also adopted Perry and Howard’s (1994) view that students can represent ideas pictorially, symbolically, and orally. As Hiebert and colleagues (2000) emphasised, communication is essential for cognitive development. When students communicate ideas through the use of concrete, symbolic or verbal representations, they justify and validate each other’s thoughts. In the process of communication, students also reflect on their thoughts to interpret and “incorporate new information into existing conceptual structures” (QSA, 2004, p. 3). Reflection is the individual aspect of the learning process that happens in students’ minds when they replay the experience (Hiebert et al., 2000; Battista, 1999). This stage of the learning cycle helps students to create abstract images in their minds (Battista, 1999) and to connect those ideas in a meaningful network (Hiebert & Carpenter, 1992).

In summary, for students to think and reason mathematically, mathematical understandings need to be developed as a conceptual whole through building cognitive webs of related mathematical ideas (Skemp, 1986, Hiebert & Carpenter, 1992). Below is a synopsis of the psychological cognitive view of learning theoretically conceived to develop Skemp’s (1986) notion of relational understanding:

1. The learning process begins with an action (Battista, 1999) where students test ideas (Airasian & Walsh, 1997); the experience involves using past experience and a variety of representations (concrete, pictorial or symbolic) to make sense of the situation.
2. Abstract images are created in the student’s mind, as a result of manipulating representations, which students can connect to in later learning (Skemp, 1986). This abstraction process occurs when students reflect on the experience (Battista, 1999). Through continued experiences in varying contexts, the abstract images become cognitive schema (Romberg, 1992; Skemp, 1986).
3. As students test new ideas, the activity will force changes to thinking in ways that modify existing ideas to accommodate new schema (Romberg, 1992, Skemp, 1986). In the process of reflective thinking, connections between ideas will be highlighted and ideas will be stored as part of a related conceptual whole (Battista, 1999; Hiebert & Carpenter 1992; Skemp, 1986).

4. Schemas are idiosyncratic to the individual student and are not fixed, thus learning involves more than the aggregation of ideas; learning also involves modifying and remodeling existing schema (Romberg, 1992).
5. Communication of ideas supports cognitive development. As individuals communicate ideas they reflect on their thoughts and in the process interpret new ideas and remodel existing schema. As Vygotsky (1934) proposed, thinking is an interactive dialogue that occurs as inner and outer speech.

Curriculum documents (e.g., DETA, 2008; QSA, 2004), as well as this research, are underpinned by the above constructivist and social constructivist cognitive views of learning. The next section outlines the importance of developing relational understanding.

2.5.3 Highlighting Mathematical Relationships for Deep Learning

A surface approach towards mathematics learning will not highlight the connections between ideas that can be achieved when a deeper approach is adopted (QSA, 2007). Yet, often teachers of mathematics in Australia focus on asking students to follow repetitive procedures without reasons rather than encourage them to explain their thinking or discuss alternate solutions (Hollingsworth, Lokan & McCrae, 2003). The paradox of this situation is that the contemporary curriculum documents are informed by constructivist theories where students are active in their own knowledge construction. Bruner (1985, p. 8), in his discussion on learners, made the point that "if you see children learning mathematics by rote, you can also say … that somebody got confused about models." However, Bruner did go on to say that education is not a scripted exercise and that the best choice is not always the best choice. Rather, he suggested that an appreciation of the variety of models and methods that are possible to use is more productive. What is important is that teachers help students attend explicitly to important mathematical relationships which can be done via any method (NCTM, 2007). Masters (2009a) asserted that if just one teaching method is used there is the risk of shallow teaching, and the consequence of shallow teaching is limited cognitive growth and depth for the students. Whilst developing understanding through recognising mathematical relationships spans all content strands of the mathematics syllabus, for the purpose of brevity, the number strand will be discussed here briefly.

2.5.4 Recognising Number Relationships

Students need to learn mathematics in ways that assist them to understand the logic and structure of mathematics through recognising the inherent mathematical patterns and relationships in mathematics (Mulligan 2006). To elaborate, Anghileri (2000) wrote about two purposes of calculation, one for exploring structure and one for problem solving. She suggested that there are two types of problems used in classrooms, routine problems and nonroutine problems. Routine problems are often referred to as word problems (Bobis, Mulligan, Lowrie & Taplin, 1999), where there is one correct solution to a straightforward procedure (Baroody, 1993). Nonroutine problems are problems that require thinking and reasoning (Anghileri, 2000). For example, shoppers face nonroutine problems when given the choice between purchasing a four kilogram bag of potatoes at a cost of $3.99 or a 10 kilogram bag at a cost of $8.99. Shoppers need to also consider how many potatoes are consumed each week to reason which option is the best option. If they purchase more than what is needed, there will be wastage, and subsequently, what may appear to be a more economical purchase may be the expensive alternative. Anghileri (2000) explained that if students learn to calculate simply for solving a problem they generally learn that there is one correct way to solve the problem, which could inhibit divergent thinking.

Even though problem solving is important, using repetitive problems with low complexity runs the risk of becoming a shallow teaching approach. The *National Numeracy Review Report* (Council of Australian Government [COAG], 2008) expressed concern that often people still hold the belief that the standard algorithmic procedures that they were taught at school are all that is necessary. In fact, Hollingsworth, Lokan and McCrae's (2003) report revealed that the percentage of repetitive problems and problems with low complexity used in Australian mathematics classrooms was alarmingly high. Whilst these procedures are still important, the fear is that when students' attention is focused only on following steps, they may perceive mathematics as a collection of memorised rules and procedures (Burns, 1994). The outcome of such an approach, according to von Glaserfeld (1991, p. xiii), is that students learn mathematics in ways that cause them to develop an aversion to numbers rather than understand the "enchanting things we can do with them [numbers]."

On the other hand, calculating for exploring structure involves students in investigating and exploring all sorts of number relationships (Anghileri, 2000). For example, the teacher may pose an investigation something along the lines of ‘how many goal/point combinations can be found if the final score for a football team was 181 points?’ Knowing that there are six points to each goal, the students may work in various ways to investigate all possible number combinations. They may use multiplication, addition, division or repeated subtraction. They could use a variety of concrete, diagrammatic or symbolic representations to communicate their ideas. The investigation has the potential to help students recognise the connections between numbers and operations. The difference when students calculate to explore structure is that the investigation will be open ended, inviting many levels of investigations, and through investigation the students recognise that mathematics is structured, that it is logical and that it does make sense.

An awareness of ‘number’ and number relationships is important if students are to develop mental thinking strategies (Anghileri, 2001; Brown & Liebling, 2005; Van de Walle, 2007). As the *Education Queensland Scoping and Sequencing Essentials Years 1-9* (DETA, 2008, p. 45) document stated, students must develop strategies for “mental computation and deep understandings of how numbers work,” which aligns with Willis’ (1998) strategic knowhow. Brown and Liebling (2005) suggested that teaching traditional written algorithmic procedures has largely failed because students could not envisage the big picture of how certain procedures were related. They explained that students were so focussed on making sums work when learning traditional written procedures that they did not even realise if the resulting solution was reasonable, or in fact a mistake. For example, a student may competently perform a written procedure such as 22 x 0.5; they may line up the digits, multiply 22 x 5 to get 110. Some may be satisfied that the answer is 110, while others may take the next procedural step and ‘count back one decimal place’ to get the answer of 11. By contrast, a student with number sense may recognise the relationship between the symbolic representations, that 0.5 represents ½, and they could then respond that ½ of 22 is 11. Thus they would be recognising the relationship between division and multiplication and between decimals and common fractions, and would realise that 110 is more than ½ of 22, and that therefore, it is a mistake.

Often new methods of teaching to develop flexible computational strategies include the use of concrete materials. However, Owens and Perry (2001) cautioned that concrete materials themselves will not help students to construct visual imagery, although they believe that there is potential in the use of concrete materials if the teacher can use the materials wisely. For instance, concrete objects, such as empty number lines and bead strings with bead sections of ten marked in different colours can assist the development of place value knowledge (Menne, 2001). These materials help the students to think about multiples of ten. The multiples are marked as landmarks along the empty number line or in the bead sections of ten, and students make jumps either forwards or backwards to these landmarks. The aim is for students to develop a sense of 'ten-ness' (Menne, 2001), and thereby help students to create visual images of the concept of groups of 'ten' in their minds to connect to in later learning. It is envisaged that as students develop a sense of ten-ness, they understand the concept of place value knowledge which then supports mental computation strategies.

Almost twenty years ago the AEC (1991) were suggesting that students should develop mental computational strategies and that teachers should recognise that mental strategies may be idiosyncratic to the particular task and to the individual. One wonders why shallow teaching, rather than flexible thinking about numbers and number relationships, appears to still be the modus operandi in some classrooms across Australia. Possibly teachers are concerned about students mastering the basics, like single digit number facts. Mastery of the basics implies that a student can respond to a number fact in a three second time period, without resorting to counting or other inefficient methods (Van de Walle, 2007). However, it is when students recognise mathematical relationships that they develop thinking strategies that assist in number fact recall, as elaborated on below.

2.5.5 Thinking Strategies to Master the Basics

Students develop idiosyncratic strategies to recall number facts through recognising and making sense of number relationships. Therefore, the initial stages of learning should focus on securing a firm foundation of whole number numeration, strategies and relationships (Booker, Bond, Briggs, Davey & Lovitt, 1998; DETA, 2008; Hunting, Davis & Pearn, 1997; Siemon, 2001; Van de Walle, 2007). The development of conceptual understanding of number facts from the very onset of learning will help students to master the basics. Once conceptual understanding has been developed, the next stage of the learning process requires "well-timed practice" (Kilpatrick et al., 2006, p. 122). At this stage the teacher's role is to guide students to develop and practise efficient, fact retrieval 'thinking strategies' (Booker et al., 1998; Van de Walle, 2004). For example, a fact retrieval strategy may be recognising 'rainbow facts', the addends that make 10. A student may recognise automatically that 6+4 is 10, and thus know that 60+40=100 or that 24+26=50. If students' learning is directed towards recognising the relationships between number facts, they will invariably develop idiosyncratic strategies that will assist recall and flexible computational methods (Anghileri, 2000; Booker et al., 2004; Kilpatrick et al., 2006; Skemp, 1986; Van de Walle, 2007). As students develop efficient retrieval strategies they will not have to resort to guessing and it is expected that their dependence on aids such as tables or charts will diminish (Booker et al., 1998; Van de Walle, 2004).

The early stages of learning are critical because these stages set the foundation for the development of further mathematical ideas. The capability to reason about and with basic number facts will be pivotal towards developing the competence and confidence to think and reason mathematically (Booker et al., 2004). Thus each child should be able to master the basics. By the end of year three, at age eight, students should have mastered addition and subtraction facts (Van de Walle, 2007). Then by the end of year six, at age 11, students should have mastered multiplication and division number facts (DETA, 2008; Van de Walle, 2007). On completing year seven, at age 12, students should be able to mentally multiply double-digit by single-digit numbers (DETA, 2008). Yet alarmingly, in year eight and above many students do not have complete command over basic number facts (Van de Walle, 2007). When students cannot recall facts, they focus their attention on working out the fact which in turn prevents them from recognising key mathematical relationships (Kilpatrick et al., 2006). For students to move on to solving more complex mathematics they must have developed sound knowledge of basic number facts (Anthony & Knight, 1999; Battista, 1999; Booker et al., 1998; Madell, 1985; Sun & Zhang, 2001; Van de Walle, 2004; Van de Walle & Lovin 2006; Westwood, 2003; Willingham, 2003). Mathematicians (e.g., Howe, 1997; Klein, Braams, Parker, Quirk, Schmid & Wilson, 2005; Mathematics Standards Study Group [MSSG], 2004; Quirk, 2005; Wu, 1999) believe that it is the automaticity of recalling a basic number fact that frees up mental energy to be used for solving more complex mathematical problems. This research took the view that the ability to recognise and recall number fact relationships has the potential to support further thinking, reasoning and working mathematically.

2.6 In Summary

In summary, this chapter has revealed that students are now required to leave school equipped to handle rapidly changing societal demands. Policy makers and government bodies have made it clear that students need to understand mathematics in ways that will develop confident and insightful application of mathematical ideas, and facilitate lifelong learning (AEC, 1991; AAMT, 1997; NCTM, 2000; MCEETYA, 2008; OECD, 2006). Teachers are being encouraged by curriculum documents (DETA, 2008; QSA, 2004) to help students leave school meeting MCEETYA's (2008) ideal that all Australian students become successful, responsive and creative learners. Hence, the focus is now on sense-making to learn mathematics with understanding. This focus is underpinned by an emphasis on learning processes comprising thinking and reasoning, communicating and reflecting (DETA, 2008). This curriculum emphasis calls for a move toward comprehensive teaching practices. According to DETA, teachers must ensure that the desired 'understanding' and 'learning' has occurred, rather than whether it simply has been 'taught.' It is envisaged that through the aforementioned learning processes, students will construct connected webs of mathematical ideas and develop robust knowledge based on understanding and being able to communicate the 'how' and 'why' of mathematical ideas, as suggested by Skemp (1986).

However, curriculum intentions need to be translated into the classroom teaching practice if change is to be made. Whilst the curriculum documents are laid out for teachers in written format, the curriculum and its implementation involve much more (Cuban, 1993; Stenhouse, 1978). There are internal factors that will influence the implementation, such as the teacher's beliefs, including her beliefs about the curriculum, her students and how they learn, and her pedagogical and mathematical understandings (Ball, Hill & Bass, 2005; BTR, 2005; Farmer, Gerretson & Lassak, 2003; Stenhouse, 1978; Shulman & Shulman, 2004; Reys, Reys, Barnes, Beem, & Papick, 1997; Woodbury, 2000). External factors, such as the school environment, staff, parents and the students will also affect the implementation process, along with professional development opportunities (Clandinin, 2008; Masters, 2009a; MCEETYA, 2008). At present, little is understood about how the curriculum intentions have been implemented into mathematics classrooms (Anderson & Bobis, 2005; Cavanagh, 2006; Reys et al., 1997) or about what factors enable or suppress the process of implementation, as was the goal of this research. The emphasis on process in learning has brought new implications for mathematics teaching and learning, as will be discussed in the following chapter.

CHAPTER THREE

TEACHING AND LEARNING AS PROCESS

"When a class knows that it will progress, everyone–the teacher included–comes to class with more intention and wit, more sense of being in this together."

(Palmer, 1999a, p. 6)

3.1 Introduction

The previous chapter discussed the new curriculum emphases of developing mathematical power and an appreciation of the power of mathematical thinking to advance learning throughout one's life. The chapter explained what is involved in becoming numerate and what knowledge and understandings are proposed to support the development of numeracy. However, becoming numerate involves more than developing mathematical content knowledge and understandings. As Sfard (1998, p. 507) pointed out, "mathematics must be seen not only as a ready-made product, but also as a process." Therefore, mathematics learning means developing an understanding of both the subject-matter of mathematics, such as mathematical concepts, facts and procedures, as well as how to work with that knowledge, such as knowing how to think, reason and work mathematically.

Thus, this chapter looks at the ways of learning, teaching and interacting that researchers and curriculum documents have proposed to support numeracy development. The first section focusses on learning; it outlines what it means to think and reason mathematically, how to scaffold investigative thinking and the types of tasks some researchers suggest. The next section focusses on teaching; it discusses possible features of the environment to help establish a mathematical 'community of practice.' It looks at the teacher's pedagogical content knowledge and some features of effective teaching other researchers have pointed out. This is followed by a discussion about the students' construction of positive and productive mathematical identities. The chapter finishes off with previous research recommendations that have informed the decision to collect qualitative data through a descriptive case study methodology (Merriam, 1998).

3.2 Ways of Learning: The Process Approach

Policymakers encourage teachers to facilitate an inquiry-based learning process where students are engaged in mathematical investigations to promote thinking, reasoning and working mathematically (AEC, 1991; NCTM, 2000; QSA, 2004). For students to develop attributes for lifelong learning they need to become “active investigators,” which means students need to “value a spirit of inquiry and a questioning habit of mind” (QSA, 2004, p. 3). The learning process needs to be organised through an approach that stimulates intellectual engagement and encourages students to ‘struggle’ with important mathematical ideas (DETA, 2008; Hiebert et al., 2000; NCTM, 2007; QSA, 2004; Savery & Duffy, 2001). Instructional tasks, thereby, should involve students in struggling over and with mathematical ideas. The NCTM (2007, p. 2) explained the term ‘struggling’ to be a process where “students expend effort to make sense of mathematics, to figure something out that is not immediately apparent.” The teacher’s role is to create an environment that engages students mentally and socially in ways that they will ponder and discuss new ideas (Van de Walle & Lovin, 2006). In fact, the QSA (2004, p. 3) believe that students “operate as active investigators when they contribute to, and share ideas about, mathematical knowledge, understandings, procedures and strategies.” Hence, the students’ role implies a willingness to actively engage in thinking and reasoning learning processes that enable them to figure out the task (DETA, 2008; QSA, 2004). This role thereby requires a disposition to persevere.

Research has consistently pointed out that it is the thinking processes which students engage in that results in learning (e.g., Burns, 1994; DETA, 2008; Hiebert et al, 2000; Kamii & Joseph, 1988; Madell, 1985; QSA, 2004; Skemp, 1986; Van de Walle & Lovin, 2006). However, as Van de Walle and Lovin (2006, p. 5) warned: “We can’t just hold up a big THINK sign and expect children to ponder new thought.” Students need opportunities to learn how to think mathematically and at the same time they need to be able to think mathematically to learn (Kilpatrick et al., 2006). Hence, the learning experience needs to facilitate the development of mathematical thinking and reasoning processes, reflecting the practices of successful mathematics users. Successful mathematics users work through a reasoning process to discover, justify and prove the logic of mathematics (Romberg, 1992). Romberg and the RAND Panel (Ball, 2003) proposed that if teachers understood the specific ‘mathematics practices’ that successful learners of mathematics *do*, then there is a greater likelihood of improving the quality of mathematics teaching and learning. This research envisaged that the quality of mathematics teaching and learning will improve if the quality of the learning process is 'enabling,' that is, it supports the students to learn mathematics with understanding.

The problem is that teachers are often uncertain about what an inquiry based process involves and what it might look like in practice (Ball, 2003; Hartland, 2006; Hunter, 2008). Many teachers have not experienced either teaching or learning as a process of inquiry, where investigative thinking underscores the learning process (Klein, 2000). Consequently, some mathematicians, educators and members of the public argue that the new curriculum ideals have 'watered down' mathematics instruction (Ball, 2003). The RAND Study Panel (Ball, 2003, p. 33) blamed "weak implementation of instruction" as the cause of debates and not the instructional approach itself. As Klein (2000) argued, unless teaching approaches shift from direct teaching to establishing a culture of inquiry that promotes cognitive challenge, students will not be equipped with mathematical understandings or dispositions that facilitate continued learning. However, teachers need to understand what is involved in implementing a process approach and what this process might look like in practice. The next section describes a proposed framework to assist teachers in their endeavour to establish a culture of inquiry.

3.2.1 Learning How to Think and Reason Mathematically

In an attempt to clarify a process approach, the RAND Study Panel (Ball, 2003) described a set of three core *mathematics practices.* They suggested that these practices can be thought of as "the way in which skilled mathematics users are able to model a situation to make it easier to understand and solve problems related to it" (p. xviii). The three practices involve the use of *representations* to illustrate and communicate mathematical ideas; making reasoned *justifications* about the truth of mathematical ideas; and seeking and recognising relationships and patterns to make mathematical *generalisations.* The RAND Study Panel (Ball, 2003) proposed that the teacher's effectiveness would be improved if she understood the practices, and students' learning would benefit if they were enculturated into these practices.

This research speculated that a framework, such as the RAND Study Panel's (Ball, 2003) 'mathematics practices,' has the potential to bring curriculum goals into fruition. For instance, as mentioned previously, many teachers are unsure what a process approach looks like in practice. Even though researchers such as Savery and Duffy (2001) warned that the challenge for teachers is to be aware not to 'proceduralise' the thinking, scaffolding thinking processes could benefit through a framework from which the teacher and students could work. It was envisaged in this research that the 'mathematics practices' could provide such a framework. As the RAND Study Panel (Ball, 2003, p. 34) suggested, the "mathematics practices play an important role in a teacher's capacity to effectively teach." The framework may benefit both teaching and learning. For instance, the teacher may use the framework as a model to make an idea easier for the students to understand, or the students may use the framework to scaffold investigative thinking processes. The RAND Study Panel (Ball, 2003) hypothesised that the quality of mathematics teaching and learning could be improved by investing time into understanding these 'process' dimensions of mathematics, which was a goal of this research.

To elaborate, when an investigative task is given to the students they need to know how to approach the learning from the outset and then how to validate the truth of the mathematical ideas involved. The first step involves thinking about how to 'represent' the problem. As the QSA (2004, p. 3) suggested, students "need to use a range of representations to communicate mathematical understandings." A range of representations could include manipulating concrete materials, or using pictures, symbols or words to assist the students' thinking and reasoning, that is to 'justify' and explain 'how' and 'why' their ideas, or the mathematical ideas, work. Justification and representational practices are integral features of the reasoning process towards developing understanding because to understand an idea "means both knowing it and knowing why it is true" (Ball, 2003, p. 37). As Wu (1996) suggested, when students are not encouraged to investigate why something is true, they miss the opportunity to make sense of and validate mathematical ideas for themselves. The justification process may also enable students to clarify their own idiosyncratic problem solving and thinking strategies, and therefore, potentially help to improve their disposition as mathematics learners and users. The third core practice is 'generalisation' which involves searching for and recognising mathematical patterns, relationships and structure (Ball, 2003). This practice is essential for helping students to see some regularity or similarity in mathematical representations, as was also pointed out by Mulligan (2006).

The development of the three practices involves a focus on the process of learning and doing mathematics, as well as a focus on the cognitive processes in which students engage (Ball, 2003). For example, the practice of representing and justifying ideas may assist communication and reflective thinking, which enables students to construct abstract mental representations (Battista, 1999), and detect mathematical relationships (Kilpatrick et al., 2006; Van de Walle & Lovin, 2006). Thus it was anticipated that the mathematics practices may assist the students, as well as the teacher, to communicate and think about mathematical ideas in ways that make sense to themselves and others. Importantly, the mathematics practices may help students to develop robust mathematical understandings through being able to represent and justify mathematics in ways that enable them to recognise mathematical relationships, as emphasised by Skemp (1986). To clarify, when students have the opportunity to investigate mathematical ideas in ways that assist them to discover mathematical patterns, relationships and structure, they develop both cognitive webs of ideas and mathematical thinking strategies. It was anticipated that as students used the practices they would learn how to think, reason and work mathematically and thereby possibly regard themselves as being mathematically capable.

3.2.2 Taking up an Investigative Approach

As teachers become aware of students' thinking processes, they can plan tasks to ensure that all students at all levels sense that their existing knowledge and understandings will aid their learning (Hiebert et al., 2000). There is often a great disparity of learner's levels of understandings within school classrooms, which means that the teacher's awareness of students' thinking is important. For example, Masters (2009a, p. vi) explained that research into Australian students' literacy and numeracy levels reveals significant gaps; he stated: "By Year 5, the gap between the top and bottom 20 per cent of students is equivalent of about 2.5 years of school, and between the top and bottom 5 per cent of students, about five years of school." However, research has also found that when teachers adopted a hands-on investigative approach, they became more aware of children's thinking, and consequently, they developed a greater capacity to cater for varying levels of understanding (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge & Gould, 2005). Hence, a challenge for teachers is to understand where the students are currently at and where they are headed (von Glaserfeld, 1991).

To promote an investigative task the teacher needs to be aware of how the framing question and resources will target the mathematical ideas the investigation proposes to develop. For instance, the question should be posed in such a way that the activity will encourage students to investigate mathematical ideas, not to search for one right answer (Hiebert et al., 2000). In this sense, the task needs to be the "*tip of the iceberg*" (Lovitt & Williams, 2004, p. 7), in other words, it needs to be the start of what has the potential to become a deeper investigation. Lovitt and Williams have prepared materials in affiliation with the Curriculum Corporation to help teachers promote 'working mathematically.' Their *Maths300* project is an accumulation of investigate activities that encourage investigative thinking. They believe that in order to help all students access an activity, tasks should provide scope for students to work with either abstract or concrete representations depending on their levels of understanding. As the RAND Study Panel (Ball, 2003) along with Kilpatrick and colleagues (2006) proposed, students need opportunities to represent their ideas and then justify those representations to promote investigative thinking. Hence, the teacher needs to be clear about what mathematical ideas the activity will target and be aware of the students' prior understandings, and then create and pose the investigation in a way that students can choose and use a variety of representations applicable to their level of understanding.

However, concrete materials themselves do not translate into the development of abstract mathematical ideas (Perry & Howard, 1994). Rather, it is how the materials are used that promotes cognitive development and meaningful understandings (Lovitt & Williams, 2004; Van de Walle & Lovin, 2006). The focus should be on the students' development as mathematical thinkers, and therefore, they must learn *with* resources not *about* resources (Mannigel, 1998; Van de Walle & Lovin, 2006). In other words, the materials need to support the learning of mathematical ideas instead of how to master the manipulatives. The point is that visual and concrete representations support cognitive development when used to promote mathematical thinking and reasoning about mathematical ideas.

3.2.3 Concerns about the Types of Mathematical Problems

Care also needs to be taken when using real world problems to promote mathematical investigations. Wu (1996) and Raimi (2000), both mathematicians, were concerned about teachers using real world problems to facilitate learning. They asserted that when problem solving was interpreted as solving 'real world' problems, and used as a way to 'learn' mathematics, the mathematics content became watered down. They argued that real world problems can help to illustrate how mathematics can be applied, but should not be used to develop understanding. Wu's (1996) opinion was that real world problems can obscure the basic mathematics and, as a result, obstruct the very skills that students should be learning. It appears that they were both particularly concerned about the demise of mathematical thinking and the lack of intellectual rigour. Wu (1996) and Raimi (2000) emphasised that the solutions to problems used in classrooms need to be discussed explicitly to assist students' understanding about how and why certain mathematical ideas may or may not have worked.

In developing numeracy, which involves developing contextual and strategic knowhow, Wu (1996) and Raimi's (2000) argument about using real world problems seems counter-intuitive. Yet, on deeper reading, it appears that they were concerned that the 'mathematics' sometimes gets lost in the context. For instance, teachers may set up shop in the classroom as one way of using real world problems. The activity involves students in purchasing items with play money, involving routine problems of addition and subtraction. This activity could simply be a procedural activity. On the other hand, the activity has the potential to facilitate investigative thinking and mathematical reasoning. For example, if the students were encouraged to represent and justify the strategies they used to perform mental computation, thinking strategies would be clarified. In the process, there is the potential for students to recognise many relationships, such as the link between addition and subtraction or between addition and multiplication, which could scaffold the development of number sense. Through discussion, the teacher can help to point out the underlying mathematics, and distil the key mathematical idea of a solution from its original (real world) context. Furthermore, these mathematicians proposed that students will recognise subsequent applications of the mathematical idea in entirely different situations. Hence, as students recognise subsequent applications they develop mathematical thinking strategies that will assist them beyond the immediate context to solve ‘real world’ problems (Wu, 1996; Raimi, 2000).

Another concern when using problem solving to promote investigative thinking relates to the teachers’ interpretation of creating ‘authentic’ learning experiences. Often teachers make the mistake of thinking that authentic learning opportunities mean that the students need to generate the problem, which does not necessarily result in authenticity (Savery & Duffy, 2001). Authenticity pertains to matching the task to the cognitive challenges reflected in the discipline (Ball, 2003; Romberg, 1992; Savery & Duffy, 2001), which also aligns with Wu’s (1996) and Raimi’s (2000) argument. A task is authentic if it is designed to target and develop specific mathematical understandings and practices, because, as Boaler (1999) asserted, knowledge and practice are intricately related. Boaler found that the students in her study learnt more when they engaged in mathematical practices in the classroom, and then used these practices in different contexts. In short, when the focus is on developing mathematical understandings and ways of working (practices), and then students are given opportunities to practise and apply this knowledge to other contexts, the learning is authentic. Authenticity in this sense means that when students are presented with problems to solve in school that target the development of specific mathematical ideas, they may find problems encountered out of school, that are rarely so specific, easier to manage (Kilpatrick et al., 2006).

In summary, the process approach is complex because it involves scaffolding cognitive development, developing mathematical practices and encouraging productive dispositions. Some key features have been pointed out. The learning process must promote mathematical thinking. The inquiry-based process approach is achieved through investigative tasks that are “purposeful and relevant, and stimulate inquiry, action, reflection and enjoyment” (DETA, 2008, p. 2). To be purposeful and relevant, however, the task needs to invite cognitive challenge (Klein, 2000; NCTM, 2007), and reflect the practice and discipline of mathematics (Ball, 2003; Boaler, 2000; Romberg, 1992; Savery & Duffy, 2001). As emphasised by Romberg (1992) and the RAND Study Panel (Ball, 2003), the culture of the classroom needs to reflect the social practices associated with working like a mathematician. Therefore, inquiry-based learning involves a whole new way of thinking about the classroom culture. Characteristics of this culture are outlined in the following section.

3.3 Ways of Teaching: Promoting Inquiry

The classrooms of the past where students were often passive recipients of information will no longer suffice, to promote inquiry thinking mathematically must underpin the learning (Bobis et al., 1999; DETA, 2008; Hiebert et al, 2000; QSA, 2004). The classroom community needs to become investigative, and students need to be enculturated into practices reflecting those of successful mathematics users (Ball, 2003; Cobb, 1994; Romberg, 1992). As found by Goos, Galbraith and Renshaw (2004, p. 112), when students participate in "the intellectual and social practices that characterise the wider mathematical communities outside the classroom" they learn to think mathematically. For the purpose of this research, those practices involve representing, justifying and forming generalisations as a way to promote learning processes comprising thinking, reasoning, communication and reflection. The notion of "communities of practice'' (Lave & Wenger, 1991, p. 41) has become influential in mathematics teaching and learning. In a community of practice, the classroom community assumes the practices of the relevant discipline in similar ways that an apprentice assumes the practices of the trade (Lave & Wenger). Other researchers (e.g., Hiebert et al., 2000; Van de Walle & Lovin, 2006) recommend creating a 'community of learners.' However, Lave and Wenger's (1991) phrase will be used in this research because it characterises the development of mathematical practices.

3.3.1 Establishing a Community of Practice

Within a community of practice, students are expected to challenge each other's ideas. Therefore, communal conflict must be valued as a learning tool, which means students need to develop a sense that conflict within the community is healthy (Palmer, 1999). Thus communal conflict needs to be encouraged in supportive ways where everyone can benefit from the conflict. Palmer asserted that when communal conflict is mistaken for competition, the competitive atmosphere can be detrimental to learning. In competitive environments, winning is for personal, rather than communal gain, and the act of competing is often silent and passive, rather than collaborative and celebratory. Palmer emphasised that learning cannot happen without a supportive environment being established, simply because students fear being ridiculed, not knowing, or being exposed. Such fears, according to Palmer, will diminish the students' willingness to actively participate in the classroom learning experience.

In order to reduce fear a hospitable environment must be established (Palmer, 1999). Palmer believes this type of environment can be created by a teacher who can embrace every idea, no matter how off the mark it may seem, as a contribution to the group's and to the individual's search for truth. Other researchers (e.g., Anthony & Walshaw, 2007; Askew et al., 1997; Groundwater-Smith, Ewing & Le Cornu, 2003; Hiebert et al., 2000; McInerney & McInerney, 2002; Savery & Duffy, 2001; Smith, 1997) concur that students need to sense that their contributions are worthwhile and will be valued if they are expected to actively engage in the learning process. As Groundwater-Smith and colleagues (2003, p. 214) pointed out, "in truly collaborative cultures, there is support, trust and openness as well as tolerance of disagreement." When students feel safe to share ideas, they are more willing to participate actively in the learning process and to be part of the learning community (McInerney & McInerney, 2002). The learning setting, therefore, needs to alter in ways that the students feel a sense of belonging (Klein, 2000; Osterman, 2000). In fact, one of the reasons numeracy teachers were more effective than others in Askew and colleagues' (1997) research, was because these teachers valued each student's contribution and used each misconception to guide further learning.

Hence, the mathematics learning community needs to be one in which all responses are accepted positively to increase opportunities for facilitating growth for each learner (Groundwater-Smith, et al., 2003; Palmer, 1999). When students learn that their ideas will contribute to the learning, they start to feel confident to express their ideas, and they develop positive self-perceptions of themselves as capable 'learners' (Palmer, 1999). This aligns with Kilpatrick and colleagues' (2006) view that when students sense they are in a process of learning, and that their mathematical abilities and ideas are not fixed, they are less likely to fear failure and more likely to initiate their own mathematical inquiry. It was envisaged in this research that when a supportive environment is established, students may become more confident to participate actively, socially and mentally in the learning process because they sense that they have the capacity, power and ability to learn. Thereby, the environment has the potential to support students to become successful learners, as MCEETYA (2008) idealised.

3.3.2 Building Bridges: Pedagogical Content Knowledge

The teacher's role is complex. For instance, to be able to use each contribution to build understanding and build, or sometimes restore, students' confidence in themselves as successful learners is not an easy task. In Grootenboer and Zevenbergen's (2008) view, the teacher's role is to create a bridge between the student and the mathematics, which requires a well developed affinity with mathematics and ways of working with mathematics. This bundling together of knowing the mathematics, which is disciplinary content knowledge and knowing how the learner learns was described by Shulman (1986) as 'pedagogical content knowledge'. Shulman believes that research needs to focus on the teacher's pedagogical content knowledge.

Effective teachers understand the content of what is to be taught (Ball, Hill & Bass, 2005). This view was also reflected in the *National Numeracy Review Report's* (COAG, 2008, p. 65) statement: "Teachers need robust content knowledge to enable them to support, direct and guide their students." Content knowledge is especially important when using students' misconceptions to facilitate discussion and further learning. As pointed out previously, it is also important that teachers can explicitly highlight the meanings or reasons for mathematical procedures based on mathematical relationships (Ball, 1988; Ball, 2003; Hiebert et al., 2000; Kilpatrick et al., 2006; MSSG, 2004; Skemp, 1986; Van de Walle & Lovin, 2006; Wu, 1999). Ball's (1988) research on preservice teachers' mathematical content knowledge indicated a concern when teachers themselves hold limited understanding of 'how and why' procedures work. Some participants Ball interviewed had limited conceptual understanding of place value which made it difficult to resolve a given misconception about solving a written algorithm of 123 x 645. Ball (1988, p. 9) described the misconception as a place value mistake which occurred when students were "forgetting to 'move the numbers' (i.e., the partial products) over on each line." The teachers all knew the steps of the written algorithmic procedure, yet few could explain why the numbers 'move over'. Clearly, the teacher's disciplinary content knowledge is important.

However, it is also important to understand how the teacher's content and pedagogical knowledge may influence her thoughts about what and how to teach (Ball et al., 2005; Brown & Borko, 1992; Manouchehri & Goodman, 1998; Shulman, 1986; Woodbury, 2000). Pedagogical content knowledge involves understanding the misconceptions or preconceptions that students bring with them to the learning situation, and what strategies to employ to reorganise the students' understanding (Shulman, 1986). Ball's (1988) example highlights the significance of understanding content and pedagogical knowledge. Some of the participants in Ball's study did not recognise the place value relationship, that the '4' in 123 x 645 represents '40'; 123 is being multiplied by forty, or four groups of ten 'ones', and not four 'ones'. Without knowing what the '4' represents in relation to 'ones' or 'tens' or 'hundreds,' it was difficult for the participating teachers to insightfully address the misconception. The preservice teachers were not in a position to effectively assist students to make sense of the mathematical ideas for themselves through investigative experiences. Ball's (1988) research indicated that the preservice teachers were accustomed to following rules without understanding. These teachers may benefit from a framework, such as the RAND Study Panel's practices, to make it easier for themselves and students to understand multidigit multiplication conceptually.

3.3.3 Effective Teachers

Effective teaching is complex as is the goal to improve the quality of mathematics teaching and learning. Yet, in Askew and colleagues' (1997) quest to understand what effective teaching involves, they have pointed out some attributes pertaining to effective teachers. First of all these teachers understood mathematics in connected ways and understood how to help students recognise those connections. In this study eighteen teachers were interviewed three times each so that the researchers could understand the approaches teachers took towards teaching. The teachers were classified as discovery, transmission or connectionist teachers. Discovery teachers perceived mathematics to be a body of knowledge to be discovered by students and, hence, were oriented towards methods of teaching that were student-directed. Transmission teachers viewed mathematics as a collection of rules and procedures and were oriented towards teacher-directed approaches. The connectionist teachers differed because they took a more holistic approach. These teachers were more oriented towards developing connections within and between mathematical ideas. These teachers, according to Askew and colleagues, appeared to adopt a balanced view. They valued both teacher-directed and student-directed teaching strategies, and valued the students' idiosyncratic methods as important components of the teaching-learning process.

Valuing students' thinking and holding high expectations of students were key features of effective teaching. One of the most effective schools in Askew and colleagues' (1997) study held high expectations of the students. They were expected to explain their thinking processes from the initial stages of their schooling. Importantly, because they were accustomed to justifying their thoughts, rather than simply giving an answer to a problem that they perceived the teacher to already know, they were confident to challenge and share ideas (Askew et al., 1997). Hence, the students learned the art of validating the truth of mathematical ideas for themselves from the beginning of their schooling, which has the potential to set them in good stead for later learning.

Challenging students' thinking and encouraging them to communicate their thoughts assists the sense-making process. For example, the teachers who were interviewed, as part of the *Early Numeracy Research Project* (Clarke, Cheeseman, Gervasoni, Gronn, Horne, & McDonough, 2002), mentioned that they used various questioning techniques to challenge students' thinking and reasoning. Clarke and colleagues explained that the teacher-participants in this study encouraged their students to justify their ideas, and to listen and evaluate others' explanations. They believe that because the teachers resisted the urge to tell students everything, their teaching practice was more effective towards promoting quality learning. Like the teachers in Askew and colleagues' (1997) study, these teachers were effective because they invited mathematical discussion whereby students justified their ideas, listened and evaluated each others' thoughts, and debated alternative strategies. Thus as the students communicated their ideas, they were in a process of integrating knowledge and practice.

However, this research took the view that to connect the student with mathematics knowledge and practice also requires a well developed affinity with the students both as individuals and as a group, and experiential knowledge about how they learn. Knowing how the learner learns is an integral part of the teaching process because it is the student who constructs her/his own knowledge, understandings and idiosyncratic ways of working (Skemp, 1986; Thomas, 2005). Yet it is equally important to know what parts of the learning context facilitate cognitive challenge and investigative thinking within the classroom community. As pointed out by Putman, Lampert and Peterson (1990, p. 90), “knowledge and thinking are inextricably intertwined with the physical and social situations in which they occur.” Hence, part of this research was to understand what features of the learning environment shaped the students’ experiences towards establishing themselves as competent and confident mathematics users. The next section points out some features that may support the development of mathematical identities.

3.4 Ways of Being: Developing Mathematical Identity

It is when students actively engage with mathematical ideas that researchers and reformers believe students will develop mathematical competence and mathematical identities (Walshaw & Anthony, 2008). In an attempt to understand the relationship between the students’ construction of knowledge and identity, Boaler (2002) researched different pedagogical practices that engaged the students. Boaler found that the students who participated in reform-oriented classrooms were given more opportunity to use and apply strategies than those of traditional approaches. Even though some students can successfully apply procedures learned in more traditional ways, they can also experience an inner conflict (Boaler, 2002). The students Boaler interviewed expressed the view that they wanted the freedom to express their own interpretations and ideas. When they were not recognised as capable learners, they became disinterested in learning mathematics. If the goal for teaching is for students to become successful learners who can think about mathematics in logical ways, then the learning experience needs to promote such characteristics.

Discussion-oriented classrooms have been found to help students identify themselves as capable, confident and competent learners. Boaler’s (2002) and Hunter’s (2008) research revealed that in discussion-oriented classrooms, the teachers assumed more facilitative roles and helped the students to alter their perceptions about their roles. These students recognised that they were developing abilities that enabled continued learning and did not perceive mathematics learning as a ritual of reproducing procedures (Boaler, 2002; Hunter, 2008). The students understood that their role was to learn and understand mathematical relationships. Boaler (2002) and Hunter (2008) found that how the students were positioned, or how they felt they were being positioned, impacted on the ways they engaged with mathematics and mathematics learning. In other words, when the students perceived that their role was important in the communal construction of mathematical knowledge and understandings, they identified themselves as capable, confident and competent mathematics learners.

How students perceive their abilities will affect their attitudes towards mathematics and their capacity to continue learning. For instance, successful learners identify themselves as having the capacity and capability to solve a broad spectrum of challenging mathematical problems (Kilpatrick et al., 2006). They also develop a "productive relationship with the discipline of mathematics" (Boaler, 2002, p. 47). Conversely, often students are anxious about their mathematical capabilities (Battista, 1999; Brady & Bowd, 2005; Ellerton & Clements, 1989; Skemp, 1986; Wilson & Thornton, 2006). Anxiousness can be a result of a child being fearful of an authoritarian teacher; however, this study is more concerned with the fact that anxiety can be a result of not knowing, or not believing in one's own mathematical ability. Students experience feelings of anxiousness when the learning, or mathematics, becomes more complex (Skemp, 1986). If students have not developed the cognitive schemas necessary to understand the task at hand, or have not established mathematical practices that enable them to think and reason mathematically, it is possible that they will become anxious. The anxiety can occur because students may be simply trying to remember more rules and methods (Skemp, 1986), rather than believing that they have the knowhow to figure out the mathematical problem.

In summary, to develop a productive disposition depends largely on the quality of the learning experience. Being productive demands active engagement in the learning process, in the sense that students will explore and test ideas to develop mathematical competency and confidence. In Klein's (2000) view, the learning experience should be 'enabling;' the students should be viewed as capable and respected learners who can, and should be able to, speak their constructed ideas. This view summarises what Boaler (2002) and Hunter (2008) both found in their research. Klein (2000) added that what is to be learned should also 'interlock' with the students' past experiences in meaningful ways. Therefore, the entirety of the students' mathematical experiences constitutes the ways in which students construct their mathematical identities. Thus mathematical competence and identity involve a culmination of the students' past experiences, the mathematical knowledge and understandings they have constructed, the idiosyncratic ways of working they have developed and their appreciation of mathematical thinking and of mathematics as a way to make sense of the world. The integration of these ways of knowing, ways of working and ways of being may impact on the students' motivation and interest for further learning, as investigated in this research.

The current curriculum aim is focussed on assisting students to develop positive dispositions. Both QSA (2004) and DETA (2008) proposed that students develop positive dispositions through active engagement with mathematical ideas. They believe that when students are socially and mentally engaged in learning processes comprising thinking, reasoning and working mathematically, they will learn mathematics with understanding. However, unless teachers understand what is involved in implementing curriculum change, these goals will never be supported with circumstantial evidence. This research argues that the classroom environment must be conducive to, and support an investigative process approach to mathematics teaching and learning. The following section discusses some research studies that have investigated various attempts to implement reform oriented approaches.

3.5 Related Research Findings and Recommendations

An assumption informing this research was that if the nature of mathematical thinking were better understood by teachers, then students' numeracy development would be strengthened (Department of Education, Training and Youth Affairs [DETYA], 2000). The findings of another three studies highlighted the value of mathematical thinking, inquiry-based teaching and teachers as inquirers. The first study, by Hartland (2006), revealed the value of teachers as inquirers. Hartland used reflective journals to document and analyse her own attempt to implement inquiry-based teaching in her science classes. The second study, by Hunter (2008), revealed the value of using the mathematics practices as described by the RAND mathematics Study Panel (Ball, 2003) to frame inquiry-based mathematics teaching and learning. The third, a single case study by Steinberg, Empson and Carpenter (2004), revealed the value of understanding students' mathematical thinking processes to scaffold learning and numeracy development.

To take up an approach to teaching that encourages inquiry and promotes thinking also requires the teacher to adopt an inquiring habit of mind, as Hartland found (2006) in her teacher-as-researcher experience. Hartland (2006) explained that it was particularly challenging to relinquish her role as a teacher who was accustomed to giving the students the facts after she had spent many years planning her curriculum to impart knowledge to her students. She reported that the transition from teaching using traditional methods to adopting an inquiry-based approach required a “massive paradigm shift” (p. 6). Hartland believes that her past desire to ‘cover’ the curriculum meant she resisted any attempt to adopt a more facilitative role. When teaching is reduced to “coverage”, according to Wiggins (1998, p. 298), the “corresponding actions are *teach, test and hope for the best.*” This phenomenon often leads to surface teaching and superficial understandings (Wiggins, 1998). However, as Hartland (2006) altered her practice and resisted the urge to simply cover the curriculum, she observed changes; her students become engrossed in the inquiry process, they discovered facts for themselves, and subsequently, gained much deeper understandings. Hartland believes that it was continued reflective inquiry into her teaching practice that enabled her to persist with the take up of a more facilitative role.

The second study by Hunter (2008) investigated how the mathematics practices, those described by the RAND Panel (Ball, 2003), can support and organise mathematical thinking and inquiry. Hunter (2008, p. 31) designed a “communication and participation framework” to scaffold thinking and discussion. The framework was centred on developing a community of inquiry and proficient use of mathematical practices. Hunter found that the “practices provided the students with a predictable framework for strategy/solution reporting, inquiry and argument, and resulted in extended reasoned dialogue” (p. 36). She concluded that the students’ cognitive development was effectively supported. Hunter (2008) believes that a framework, such as the one she devised, can help implementation of new ideals. However, she suggested that for change to be sustained, teachers require ongoing support through collegial trialling, examination and discussion of implementation experiences.

The third study, by Steinberg, Empson and Carpenter (2004), revealed that paying attention to students’ mathematical thinking and communication can be especially productive in terms of student learning. They examined, through a single-case study methodology, one teacher’s implementation of reform-oriented ideals that promote learning with understanding through processes of thinking and reasoning. The teacher used Cognitively Guided Instruction (CGI), a framework devised by Carpenter, Fennema, Franke, Levi and Empson (1999), to plan the learning experiences. The framework requires teachers to plan for instruction based on problem-solving and their knowledge about children's mathematical thinking, rather than using prescribed lessons. Steinberg, Empson and Carpenter's (2004) analysis focussed on the mathematical discussion this teacher had with her students. They concluded that discussion proved to be an effective instructional strategy to support the students' cognitive development. Yet, important to this research they reported that whilst the teacher made remarkable changes over the period of the research, the process was difficult. However, they believe that the participant-researcher relationship provided a means for the teacher to ask and answer questions about the specifics of her teaching.

Some key features of the learning process have been outlined by these three studies (Hartland, 2006; Hunter 2008; Steinberg, Empson & Carpenter, 2004). In particular, it appears that as teachers experience the benefits of new curriculum ideals, there appears to be an increased desire to implement change. However, sustaining change is complex, and the question remains as to what features of the learning environment support teachers in their endeavour to maintain new teaching approaches. Nevertheless, reflective inquiry has helped the process of implementation of new ideals. For instance, Hartland (2006) gained a deeper understanding about the learning process and effective planning decisions in her teacher-as-researcher experience. With this new insight, Hartland believes she is now more equipped, and willing, to make and sustain change. She recommended that teachers need to recognise, for themselves, the importance of understanding how students learn and the importance of engaging in a process of professional renewal. In the other two studies (Hunter 2008; Steinberg, Empson & Carpenter, 2004), collaborative effort and inquiry helped to implement change. In both projects, a culture of inquiry was not only established in the classroom, but also about the classroom, through developing productive teacher-researcher relationships. The teachers and the researchers assumed roles of co-inquirers and engaged in reflective dialogue, as in this research.

Various studies have examined teacher interpretations of reform ideals, two of which have been the impetus for this research. Both studies were conducted in New South Wales, Australia. The studies examined teachers' interpretations of the 'Working Mathematically' strand of the Board of Studies of New South Wales (BOSNSW) (2002) syllabus documents. Anderson and Bobis (2005) looked at how teachers interpreted Working Mathematically from the *Mathematics K-6 Syllabus* (BOSNSW, 2002) and the teachers' willingness to implement working mathematically into their practice. They surveyed teachers to determine how their practices reflected the reform-oriented syllabus, and then conducted follow up interviews with those teachers who incorporated working mathematically into their teaching. Cavanagh (2006) focussed on surveying secondary school teachers' interpretations of the aims of the 'Working Mathematically' strand of the *Years 7-10 Mathematics Syllabus* (BOSNSW, 2002a). He held 39 interviews to examine the extent to which these teachers implemented working mathematically in their classrooms. Both studies revealed that teachers were unsure what 'Working Mathematically' actually looked like in classroom practice (Anderson & Bobis, 2005; Cavanagh, 2006).

The studies found that teachers drew upon diverse experiences and knowledge to interpret curriculum recommendations and therefore engaged with practices differently. For example, depending on teachers' personal knowledge and beliefs about the actual role of working mathematically in teaching and learning mathematics, engagement with syllabus recommendations varied (Anderson & Bobis, 2005). Despite teachers reporting that they used practices and planned learning experiences to incorporate processes such as reasoning and communicating, some teachers were more informed than others (Anderson & Bobis, 2005). Cavanagh (2006) found that the majority of the teachers he surveyed had an incomplete understanding of working mathematically and, therefore, had not made substantial changes to their practice.

Both of the NSW studies (Anderson & Bobis, 2005; Cavanagh, 2006) have recommended future research orientations, which have informed this research. Anderson and Bobis (2005, p. 71) advised that further research is needed to "explore particular teacher's practices in detail to form a picture of the successful implementation of working mathematically." Providing rich descriptions of successful teachers' efforts may support implementation for other teachers (Anderson & Bobis, 2005). This information could also clarify what working mathematically actually looks like (Anderson & Bobis) or provide examples of best practice to be distributed among the wider mathematics community (Cavanagh, 2006). Both emphasised that further interviews and lesson observations are needed to document successful implementation of reform-oriented curriculum materials, as is the goal of this research.

3.6 In Conclusion

In conclusion, to be numerate involves a capacity to use and enact mathematical practices, knowledge and skills, *as well as* the "disposition to use and appreciate all that is powerful and beautiful in mathematics" (Siemon, 2001, p. 46). However, there is no one way to approach mathematics teaching and learning because all learners are different (Graham & Fennel, 2001). Thus, as Klein (2002) suggested, it is time to abandon the quest for a one-size-fits-all teaching method and instead aim to empower students with robust mathematical understandings, effective practices and productive dispositions that might ensue from a variety of instructional techniques. Thus Skemp's (1986) theoretical viewpoint has informed this research. In short, he believes that if students understand how and why mathematical ideas work and how they are related they will have developed robust mathematical understandings. Second, knowledge and practice are inextricably intertwined (Boaler, 2002), and therefore, to develop robust understanding the students must also develop robust practices. Hence, the RAND study Panel's (Ball, 2003) notion that investing the time to understand the process dimensions of the three mathematics practices has influenced the researcher. Hence, it was envisaged that the practices framework may assist thinking, reasoning and working mathematically. Last, this research speculated that once a hospitable environment was established (Palmer, 1999), and once the students recognised that learning is a process (Kilpatrick et al., 2006), they may be better placed to develop productive mathematical dispositions.

From this perspective, encapsulating the empowerment of students with robust mathematical understandings, effective practices and productive dispositions, the teacher's role is complex. She is required to teach based on her theoretical and experiential knowledge drawn from her own questioning, observations and reflections about what curriculum documents are suggesting and what works in practice. However, as Walshaw and Anthony (2008, p. 517) suggested, "an understanding of what quality mathematics pedagogy looks like, specifically in relation to the vision of communal production and validation of mathematical ideas, is still in its formative stages." Researchers have pointed out that even when teachers do understand curriculum ideals implementation is not widespread because teachers are uncertain what a process approach looks like in practice (Anderson & Bobis, 2005; Cavanagh, 2006; Reys et al., 1997). At present there is a gap between curriculum intention and implementation.

However, change is possible. Other researchers have found that supportive studies (Steinberg, Empson & Carpenter, 2004) and reflective practices (Hartland, 2006; Hunter 2008) have helped teachers to make changes to their practices. Whilst the goal of this research was to investigate and understand the participating teacher's existing and changing beliefs about the curriculum, it was also to support the teacher throughout the implementation process. It was envisaged that the second pair of eyes in the classroom may support the teacher's practical inquiry and thereby encourage deeper reflection about her teaching practice. Through co-reflection it was hoped that a link between theory and practice may be established, as recommended by Clandinin (2008). Then as the data are interpreted, analysed and reported in rich detail, it was anticipated that the findings may highlight challenges and successes of curriculum change and the circumstantial effects on the students' engagement and dispositions.

The aim is to inform and inspire teachers and researchers, to explore new approaches to mathematics teaching and learning. As Cavanagh (2006) noted, the teacher participants in his research acknowledged the importance of sharing their success stories. These teachers believed that shared stories helped them to learn and benefit from each other's knowledge and experiences. The review of the current situation in literacy, numeracy and science learning indicated that at present there has been "insufficient focus on professional sharing of knowledge and experience within and between schools" (Masters, 2009a, p. 49). Masters stressed that a "greater focus is required on learning *in situ* (in the classroom)" (p. 49), as in this research. The next chapter discusses the research method and design, the researcher's role and the basic interpretive and constructivist lens through which the qualitative data will be gathered, interpreted, analysed and reported.

CHAPTER FOUR:
METHODOLOGY AND METHOD

"Technique is what teachers use until the real teacher arrives, and we need to find as many ways as possible to help that teacher show up."
(Palmer, 1997, p. 11)

4.1 Introduction

The previous two chapters pointed out that empirical research is needed in order to understand and share the practicalities of the process of curriculum implementation (Ball, 2003; Masters, 2009a). As Stenhouse (1978) asserted, emergent curriculum must be studied and grounded in classroom practice. He advocated case studies because they seek to make systematic sense of the practice undertaken at a specific site, which could then be added to other sense-making accounts of practice. Hence, because implementation is dependent on the actions that take place within the individual classroom, a descriptive case study (Merriam, 1998, p. 38) was the most applicable methodology for this research. As Merriam suggested, the detailed information presented in a descriptive case study provides scope to attend to practical and contextual details about how and why the curriculum implementation did or did not work.

The case study was conducted in a Year Six classroom of a small private school in North Queensland. The participating mature age teacher, who will be referred to as Reagan, was in her first year of teaching and was drawn to the new investigative thinking and reasoning pedagogical approach espoused at university. Thus Reagan was purposively selected (Burns, 2000; Lankshear & Knobel, 2005; Stake, 2005) because she had often expressed an interest in learning about, and working with, investigative mathematical thinking processes to enhance students' learning. The intention was to provide a rich detailed description of Reagan's experience and the circumstantial effects of the new curriculum approach; the curriculum change being implemented and investigated was the 'process approach' to mathematics teaching and learning. This change required adopting an inquiry-based investigative approach, comprising learning processes of thinking, reasoning and working mathematically (DETA, 2008; QSA, 2004). The process approach is intended to improve numeracy outcomes for students by focusing on the development of mathematical understandings, practices (ways of working mathematically) and positive dispositions.

As a researcher, my goal was to investigate, interpret and analyse the contextual features of the learning environment, which first, facilitated or constrained new curriculum implementation, and second, facilitated changes to students' engagement levels and dispositions towards mathematics and mathematics learning. The two focus questions were:

1. What do teachers need to know and do to incorporate thinking, reasoning and working mathematically in their practice?
2. What effect will thinking, reasoning and working mathematically have on students' engagement and disposition towards mathematics learning?

This chapter outlines the methodological approach and method I used to conduct this study and to help understand and identify the limiting and facilitating contextual features on teaching practice and student learning. It also outlines my role as a researcher.

4.2 Research Purpose

This research addresses a practical problem; that problem being the gap that exists between the curriculum ideals as outlined by curriculum documents and what happens in the classroom. It also addresses a second concern which involves the development of a positive disposition towards mathematics and mathematics learning. Aware that Reagan's setting goals would not automatically improve her students' outcomes, I set about researching external and internal factors that affect curriculum implementation (Ball et al., 2005; Ball, 2003; BTR, 2005; Cuban, 1993; Stenhouse, 1978; Shulman & Shulman, 2004; Reys et al., 1997; Ross et al., 1992; Woodbury, 2000). First, and external to Reagan's practice was the support she received, or did not receive, from the school community and professional development programs. Even though data from questionnaires, interviews and observations were gathered to determine the level of support Reagan received, an important goal of this inquiry was to provide an opportunity for her to investigate her own practice through reflective dialogue with me, the researcher. Hence, in order to produce a detailed account of curriculum implementation and its effects a qualitative case study methodology was most suitable, as explained below.

4.3 Qualitative Research Methodology: A Case Study

A qualitative methodology was chosen to carry out this research. As Merriam (1998) described, qualitative research seeks to understand in depth the nature of the setting to reveal how all parts of the lived experience work together to form a whole, which was the goal of this research. A quantitative study would not have been suitable. The focus of quantitative research examines each component (variable) as a separate entity with the intention to form generalisations (Lankshear & Knobel, 2005; Merriam, 1998). By contrast, qualitative interpretive approaches, like this study, are more concerned with the process than the product (Burns, 2000; Merriam, 1998). Hence, quantitative study was not applicable, nor any form of approach focussed on an outcome from which generalisations could be made. The focus of this study was not to examine the macro political nature of the institution; rather, it was concerned with understanding what classroom conduct facilitated or constrained collaborative inquiry.

One of the distinct advantages of using a qualitative case study is to understand "how" and "why" questions through inductive reasoning (Yin, 1994, p. 9). Because the curriculum innovation is new in Queensland, little is understood about the actual implementation process or consequences. Hence, a similar approach was taken to that of learning mathematics with understanding, where students investigate mathematical ideas until they understand how and why things work. To understand the challenges or success of curriculum implementation, it was my intention to understand 'how' and 'why' certain ways of knowing, doing , thinking, being and interacting enabled or constrained learning, as well as to draw out relationships between these features. Thus I applied inductive logic to identify cause and effect relationships between contextual features, such as the curriculum ideals, mathematical process and practices, classroom environment and teacher beliefs and student dispositions, to build understandings about what works, how and why. In this sense, what the research induces is drawn from information that is not already commonly available (Lankshear & Knobel, 2005). As Merriam (1998) pointed out, inductive logic builds understanding as it seeks to portray the interactions and relationships between many possible contextual features. In short, the nature of this study was to understand how components interact to produce a rich holistic account of a bounded system in order to offer insights that may expand the reader's experience (Merriam, 1998).

It was important to develop an understanding of the implementation process starting from Reagan's initial beliefs. As Anderson and Bobis (2005) and Cavanagh (2006) found, even when teachers do understand new curriculum ideals, implementation is often limited. Hence, questionnaires and interviews were used to gain insight into Reagan's understanding of the curriculum and her beliefs about how students learn. Following Lankshear and Knobel's (2005, p. 184) advice, observations were used to "check for consistency" in Reagan's responses, and to examine whether her understandings informed her practice. However, I have also taken into consideration researchers' (Anderson & Bobis, 2005; Cavanagh, 2006; Hartland, 2006; Hunter 2008; Klein, 2000) views that unless the teacher has experienced what a process approach looks like or feels like in practice, implementation may be difficult. Thus throughout the period of the practical research, my intention was to open up opportunities for reflective dialogue and to support Reagan when she sought guidance based on my theoretical understandings.

Evaluative judgements and decisions founded on reflective inquiry are important when implementing new approaches. As Lankshear and Knobel (2005, p. 10) pointed out, there is "often a temptation in education to hope for 'magic bullets' or 'quick fixes', at the level of theory and practice alike." By contrast, serious researchers, as in this research, wish to understand how and why something works or to understand features that may make it work in other cases (Lankshear & Knobel, 2005). Stenhouse (1978) pointed out that it is important for teachers to have the opportunity to investigate their practice to accomplish any sort of change. He stressed that too often academic researchers examine a teacher's practice, and, whilst the practice may be central to the research, the teacher remains at the periphery of the investigation. Even though his advice was made over thirty years ago, there is still need to understand the implementation process and this requires a collaborative teacher and researcher effort, as was my intention in this research.

The value of reflective discussion between the teacher and the researcher has been recognised in other studies (e.g., Hunter, 2008; Steinberg, Empson & Carpenter, 2004). Reflective dialogue occurred in these studies as the teachers and researchers assumed roles of co-inquirers. The researchers believe that the co-inquiry helped the teachers to reconceptualise their views about student learning and to remodel their practices. Reflective dialogue, as in this case study, involved a cycle of action and reflection which was based on the work of Paulo Freire (1921-1997). Freire (1998, p. 43) believed that the process of "doing" and "reflecting on doing" could be used to gain an understanding of the way things are (the classroom experience), to then use 'informed' action to transform the practice for further action and reflection. Masters (2009) believes it is possible to identify teaching strategies that improve student learning through active reflection. Hence, I anticipated that the interviews would create a middle, reflective space to open pathways between theory and practice, and thereby increase Reagan's' opportunity to change her practice and my understanding of the implementation process, as was suggested by Clandinin (2008).

In addition, recognising the importance of developing productive dispositions, I also endeavoured to investigate what a productive disposition involved and what features of the learning context supported this development. At present curriculum implementation is limited (Anderson & Bobis, 2005; Cavanagh, 2006; Reys et al., 1997), and therefore, research about how the ideals support students' development of productive dispositions is also limited. To revise briefly, according to QSA (2004), for students to think, reason and work mathematically they need to be 'actively engaged' and have a 'positive disposition' towards mathematics and learning. However, I have used the term 'productive disposition' (Ball, 2003; Kilpatrick et al, 2006) because it incorporates the terms 'active engagement' and 'positive disposition.'

Through my constructivist orientation, and after a review of the literature, I surmised that constructing a productive disposition involves the development of three foci. The first is for students to construct deep mathematical understanding which includes knowing how and why ideas work and how the ideas are related (Hiebert et al., 2000; Skemp, 1986). The second is for students to be able to confidently and competently develop idiosyncratic ways of thinking about and working with mathematical ideas (Ball, 2003; DETA, 2008; Kilpatrick et al., 2006; QSA, 2004). The third involves establishing a positive and supportive classroom community of practice that encourages cognitive challenge and values communal conflict (Boaler, 2002; Klein, 2000; Lave & Wenger, 1991; Palmer 1999; Savery & Duffy, 2001). In short, students need supportive experiences that enable them to become actively involved in the construction of their own mathematics knowledge, mathematics practices and mathematical dispositions.

Hence, classroom observations were conducted to gain insight into what active involvement in the learning process looked like in practice, as well as what contextual features influenced the construction of understandings, practices and dispositions. The observed data and work samples were used to help determine whether an investigative learning process helped students to understand the 'how' and 'why' of mathematical ideas. Then, the classroom observations along with data collected from pre- and post-student questionnaires were used to investigate whether thinking, reasoning and working mathematically promoted productive dispositions towards mathematics and mathematics learning. The written responses and observations were also used to determine possible reasons why (or why not) the students changed their perceptions. In short, this case study involved gathering data from interviews with Reagan, classroom observations, work samples and pre- and post-questionnaires, from both Reagan and the students to help understand two problems. The first problem was the persistent gap that exists between curriculum intention and implementation and the second was the need to improve the students' dispositions towards mathematics and mathematics learning.

4.3.1 Research Aim

As a researcher my aim was to contribute to new understandings in the area of teacher practice and student learning at a micro level. The aim was to interpret relationships between features of the learning context that support or constrain curriculum implementation and/or student learning, which can then be added to other sense making accounts of practice, as suggested by Merriam (1998) and Stenhouse (1978). The end goal was to provide a rich, thick description of this single case (Glesne, 2006; Merriam, 1998; Stake, 2005) about the effects the implementation had on Reagan's teaching practice and the students' learning. It is anticipated that eventually, through an accumulation of sense making accounts of this nature, teachers can extrapolate patterns and trends between the cases and their own experiences to evaluate or modify ideas about their own teaching practices. It is from this basis that informed curriculum implementation can proceed (Lankshear & Knobel, 2005). For instance, the findings of this research might reveal successes which could inspire other teachers to take up a process approach. The findings could also disclose challenges which would help other teachers or schools anticipate the types of things that may happen and to learn how to handle potential problems when implementing a process approach.

However, to ensure situational verifiability, the findings needed to be presented in a way that invites other teachers to test and evaluate the findings within their own situation, rather than expecting them to accept the findings as a generalisation about curriculum implementation (Stenhouse, 1978). Hence, an important aim of this research was to ensure that the data are respected for their integrity. Therefore, a process of "participant checking" was carried out, as emphasised by Lankshear and Knobel (2005, p. 183). The many opportunities for one-to-one correspondence between the teacher and researcher gave Reagan the opportunity to clarify meaning and thus make the findings more precise. The next section discusses the case study methodology.

4.3.2 Case Study Limitations

The major criticism of qualitative research is reliability and validity because of the subjective nature of the data (Burns, 2000). The researcher's presence in the classroom can have an effect on the participants known as the "Hawthorne effect" (Kumar, 2005). This effect occurs when participants alter their behaviour simply because of the presence of the researcher. The ideal situation is to ensure that the observer has a neutral effect; however, this is unlikely to be met (Burns, 2000). In an attempt to position myself as part of the regular classroom discourse, I often visited the classroom prior to starting the data collection phase. As the research began, hourly visits were conducted twice weekly over two school terms. In order to examine holistically the specific activities and interactions that take place, it was necessary to spend a considerable amount of time in the setting, as suggested by Burns (2000) and Stake (2005). Within the classroom I was neither a total participant nor a total observer; where possible, I interacted with the students to help reduce any effects my presence might have on their learning.

Another limitation of this study was the sample size of one case. However, the intention was to produce one account, one piece of a bigger more complex puzzle, as suggested by the RAND Study Panel (Ball, 2003). Burns (2000) asserted that the problem of inference from a sample of one is the uncertainty about what could be expected from the same teacher given a different class. Hence, replication of this study would not be possible because both the participants' and the researcher's understandings would have evolved making replicability impossible. As Lankshear and Knobel (2005) pointed out, a limitation of inductive logic is that it is based on probable conclusions that could possibly be disproved later in other situations. To replicate an identical social context can never be guaranteed (Rist, cited in Burns, 2000), and thus, rather than focus on replicability, this case study was more concerned with reliability.

Reliability in case studies is focussed on the dependability of the results, as opposed to replicability, such that “the results make sense and are agreed on by all concerned” (Burns, 2000, p. 475). A characteristic of qualitative case study research is that the researcher spends considerable time on site, and continuously interprets data (Stake, 2005). The time spent on site and in weekly interviews enabled many opportunities for meanings to be clarified and checked. As Stake suggested, the researcher is placed in the thick of the situation. Through reflective thinking, the researcher deliberates over meaning and works to relate interpretations from interviews and observations to context and experience, as in this research. Reliability and credibility are therefore gained by “thoroughly triangulating the descriptions and interpretations, not just in a single step, but continuously throughout the period of the study” (Stake, 2005, p. 443). The aim was to understand as much about the complex meanings inherent in the case, as Stake emphasised, that would enable a rich, descriptive account of the experience for readers to arrive at their own conclusions.

The case study method has also been criticised for its lack of ability to afford generalisation (Burns, 2000). Guba and Lincoln (1981, p. 377) noted that in descriptive case studies there can be a tendency to exaggerate a situation or perhaps generalise the study to a whole rather than present it as a “slice of life,” leading the reader to make inaccurate interpretations. However, this case study was used to investigate an individual case to “optimise understanding of the case rather than to generalise beyond it” (Shank, 2006, p. 443). In this sense, the case study was focussed on “circumstantial uniqueness and not on obscurities of mass representation” (Burns, 2000, p. 474). The aim was for readers to draw their own conclusion from the data presented and from the suggestions made, rather than to present a proof. As Merriam (1998) explained, the case study itself is unique, and it is this uniqueness that has the potential to produce information that is otherwise not available. Hence, because case studies generate rich data, even though the data are subjective, they bring light to phenomena and processes that may lead to further intensive investigations (Burns, 2000).

4.3.3 Case Study Strengths

The strength of a case study is its potential to help structure future research; therefore, the case studied "plays an important role in advancing a field's knowledge base" (Merriam, 1998, p. 41). Merriam asserted that a descriptive case study can be particularly useful in an area of research where research has been minimal, such as curriculum innovation and implementation. The advantage of a qualitative case study is the potential to "illuminate reasons for action and provide in-depth information on teacher interpretations and teaching style" (Burns, 2000, p. 13). In this research a close proximity with the participants and the activities was maintained which helped to develop an insider's view, as Burns (2000) suggested, of the ways things work. The insider's view enabled the researcher to examine closely the subtleties and complexities of the educational and social interactions which can otherwise be overlooked through using standardised measures (Burns, 2000).

The products of organised cases, whereby data are analysed carefully, will eventually provide the basis for illuminating descriptions (Lankshear & Knobel, 2005; Stenhouse, 1978). Case studies are preliminaries to larger scale studies (Burns, 2000); they begin to form a data base for future comparisons to be made between other sense making accounts for theory building (Lankshear & Knobel, 2005; Merriam, 1998; Stenhouse, 1978). Such descriptions enable the researcher and teacher themselves, as well as other readers, "to understand and evaluate what has been observed, and to think methodically about how and where changes could be made that might lead to improved teaching and learning" (Lankshear & Knobel, 2005, p. 12). The RAND Study Panel (Ball, 2003) also advocated descriptive studies. RAND believes that whilst there has been active debate in the U.S. about what curriculum incentives and teacher knowledge are important, little knowledge exists about the actual implementation. Even though each classroom is unique and bound by school and policy expectations, and curriculum study is empirical and thereby not completely replicable, contextual data about the problems or effects of curriculum implementation can be useful (Stenhouse (1978). Hence, the collaborative teacher-researcher effort was mutually supportive in this endeavour to provide an illuminating description of the implementation experience, process and circumstantial effects.

Significantly, the key strength of the case study is its potential to inform based on the richness of data collected from documents, participant observations, interviews, and questionnaires (Merriam, 19998; Stake, 2005). Case studies have proven to be useful in the area of researching education innovations and for informing policy (Merriam, 1998), as is the goal of this research. Furthermore, the RAND Study Panel stated that over time correlations between descriptive case studies can "provide a basis for generating hypotheses, models, and theories about how mathematics learning and instruction work and about what might be done to improve it" (Ball, 2003, p. 60). The following section outlines the theoretical and conceptual framework underpinning this research.

4.4 Theoretical Conceptual Framework

The constructivist and sociocultural stance of this research implies that learning is constructed through personal experiences (Skemp, 1986) and knowledge is mediated through social interactions (Bruner, 1985; Cobb, 1994; Lincoln & Guba, 2000; Vygotsky, 1978). Hence, coming from this stance, this research is framed around valuing personal experiences as a means of sense making and knowledge construction, for the learners and the teacher. For instance, as the students actively investigate mathematical ideas and communicate their thoughts, they develop mathematical understanding (DETA, 2008; QSA, 2004). Similarly, as the teacher investigates the curriculum ideals, actions the ideals, and then engages in reflective dialogue, she links theory to practice (Clandinin, 2008), develops new understanding and alters her teaching practice. This section outlines a conceptual framework firstly for the mathematics teaching and learning process and secondly for the curriculum implementation process.

4.4.1 Becoming Numerate: Enabling Continued Successful Learning

Mathematics learning occurs in a socially bound context. Learning mathematics involves both a process of individually constructing knowledge and a process of enculturation into mathematical ways of being; one provides the background for the other (Cobb, 1994). To become enculturated into mathematical ways of being, the classroom needs to reflect the practices of mathematics users and the students need to take on attributes of successful mathematics users. In this sense the classroom becomes a "community of practice" (Lave & Wenger, 1991, p. 41). However, as mentioned earlier, when teachers are unsure what a process approach looks like in practice, it can be difficult for them to facilitate knowledge construction and the development of mathematical ways of working and being. It has been proposed (e.g., Ball, 2003; Kilpatrick et al., 2006; Romberg, 1992) that if teachers understood the specific ‘mathematics practices’ that successful learners of mathematics *do*, their capacity to teach effectively may improve, and if the students were enculturated into these practices, their learning would benefit (Cobb, 1994). The specific mathematics practices include an ability to *represent* mathematical ideas, to *justify* the ‘how’ and ‘why’ of mathematical ideas and to form *generalisations.* This research proposes that the practices may scaffold the teacher’s and students’ thinking and reasoning processes.

Fundamental to the cognitive and sociocultural domain is the notion of identity, which is how learners perceive themselves in relation to others in a particular context. Engagement in a quality learning process does not necessarily facilitate changes to students’ dispositions, that is, changes to their interest levels, willingness to engage actively, or their appreciation of mathematics as a discipline and way of thinking. It was anticipated that when students sense they are in a process of learning, and that their mathematical abilities and ideas are not fixed, they are less likely to fear failure and more likely to initiate their own mathematical inquiry (Kilpatrick et al., 2006; Palmer, 1999). Thus the learning process requires a supportive environment where communal conflict is valued as a learning tool (Palmer, 1999) and where students can establish themselves as productive and positive learners of mathematics.

Hence, classroom observations and pre- and post-questionnaires were used to investigate the sociocultural context of identity, focussing on how the students’ dispositions are constructed or transformed in relation to other students, and in relation to the mathematics and the learning of mathematics. Similar to Kilpatrick and colleagues (2006), I envisaged that as students learn ‘how’ to think and reason, through using the mathematics practices, they will learn ways to work mathematically which will support cognitive development and the construction of productive dispositions. Below is an illustration that conceptualises features of learning that contribute to numeracy development and successful learning (see Figure 4.1). The table draws on the cognitive (Skemp, 1986) and socio cultural (Cobb, 1994) domains and is more fully explained below.

At the time when this research began, the QSA (2004) was emphasising learning experiences that would encourage students to think, reason and work mathematically. Now this emphasis has been incorporated into the 'working mathematically' strand which is to be integrated with all other content strands of the mathematics curriculum (DETA, 2008). The inclusion of working mathematically has meant that the students' processes of thinking and reasoning, communicating and reflecting are now 'assessable elements' (DETA, 2008). This research took the view that a process approach, to encourage *working mathematically,* involves an interdependent relationship of ways of knowing, thinking, doing and being. In other words, mathematical knowledge, processes, practices and dispositions are all related, as are mathematical ideas. Hence, becoming numerate occurs within an environment where students' internal attributes and external interactions and actions impinge on each other. The internal attributes are the first four frames presented along the bottom row of the table, and the frame to the right is the external factor. Each frame has the capacity to support the other in some way. For instance, the ways of interacting have the potential to contribute to the development of the students' ways of knowing, doing, thinking, or being. For the purpose of clarity each frame will be explained below, as interpreted by the researcher.

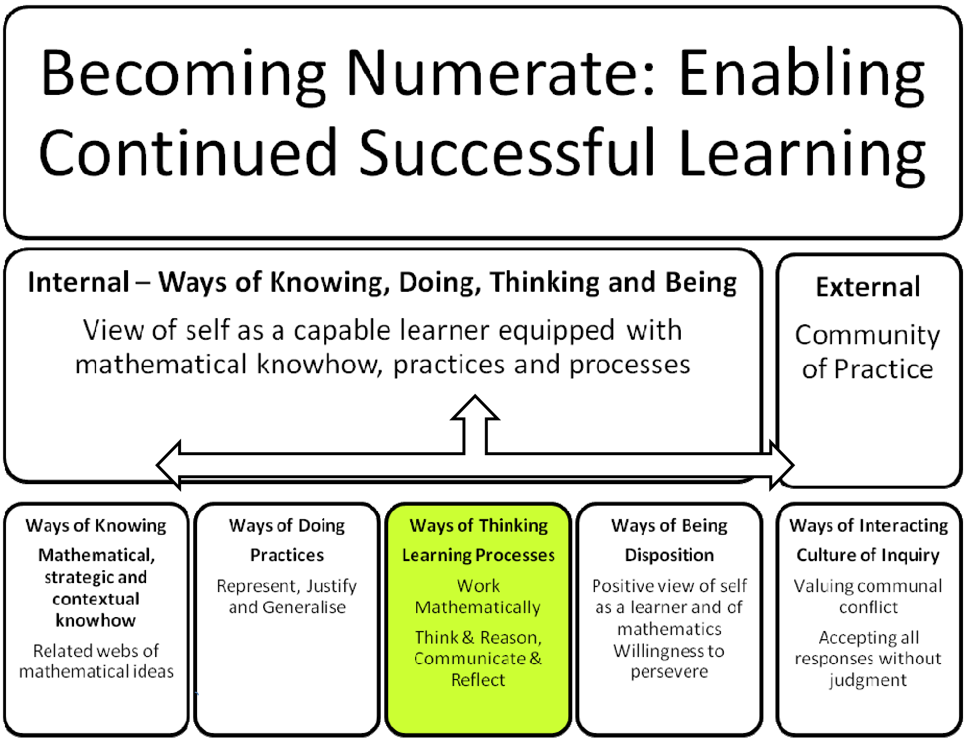


Figure 4.1. Conceptual framework to illustrate becoming numerate

Working from left to right of Figure 4.1 the first frame refers to the development of *ways of knowing.* This frame refers to the students' construction of internal cognitive structures in their minds, through a process of action, reflection and abstraction. It was envisaged that as students understand the how and why of mathematical ideas and recognise the relationships between these ideas they will develop mathematical, strategic and contextual knowhow, as described by COAG (2008) and Willis (1998). The second frame refers to the *ways of doing*, which for the purpose of this research includes an ability to use the three mathematical practices described by the RAND Study Panel (Ball, 2003). The third frame relates the cognitive learning processes of thinking and reasoning, communicating and reflecting recommended by DETA (2008) as *ways of thinking.* The fourth frame is to do with developing mathematical *ways of being* which refers to the development of positive (QSA, 2004; DETA, 2008) and productive dispositions. The term productive has been included because it relates more to the students' attitudes towards mathematics and mathematics learning. A positive disposition implies that students sense they are capable users of mathematics and that they appreciate mathematics as a way to make sense of the world. However, a productive disposition is more, it also refers to students' involvement in learning and willingness to persevere; it implies a belief that "steady effort in learning mathematics pays off" (Kilpatrick et al., 2006, p. 131). The last frame, *ways of interacting*, relates to the external features of the learning environment; the ways in which the student-student and teacher-students' interactions are conducted.

4.4.2 Successful Implementation: A Process Approach

A similar framework has been devised to conceptualise what successful curriculum implementation involves (see Figure 4.2). As Farmer, Gerretson and Lassak (2003) suggested, it is important for teachers to develop a sense of control over new ideals. Reagan, just like her students, is learning through a process of inquiry. Thus her existing knowledge about how students learn, or about new curriculum goals, may need to be remodelled in response to the new experience. There are also internal attributes and external contextual features that impinge on each other during Reagan's process of curriculum implementation. The external features involved the classroom community of practice that Reagan and her students established, as well as the community of practice beyond the classroom. This community included the support provided by the school community or professional development programs (Masters, 2009a; MCEETYA, 2008). Reflective dialogue between the teacher and the researcher was considered an integral contextual feature to support implementation and subsequent change to teaching practice (Clandinin, 2008). The internal attributes included the ways in which Reagan interpreted and understood the curriculum (Cuban, 1993; Stenhouse, 1978; Shulman & Shulman, 2004; Reys, et al., 1997; Woodbury, 2000), her pedagogical-content knowledge (Shulman, 1986) and her belief in herself as a capable and effective teacher. An important intention of this research was to make transparent Reagan's evolving beliefs and thinking about the curriculum and its implementation.

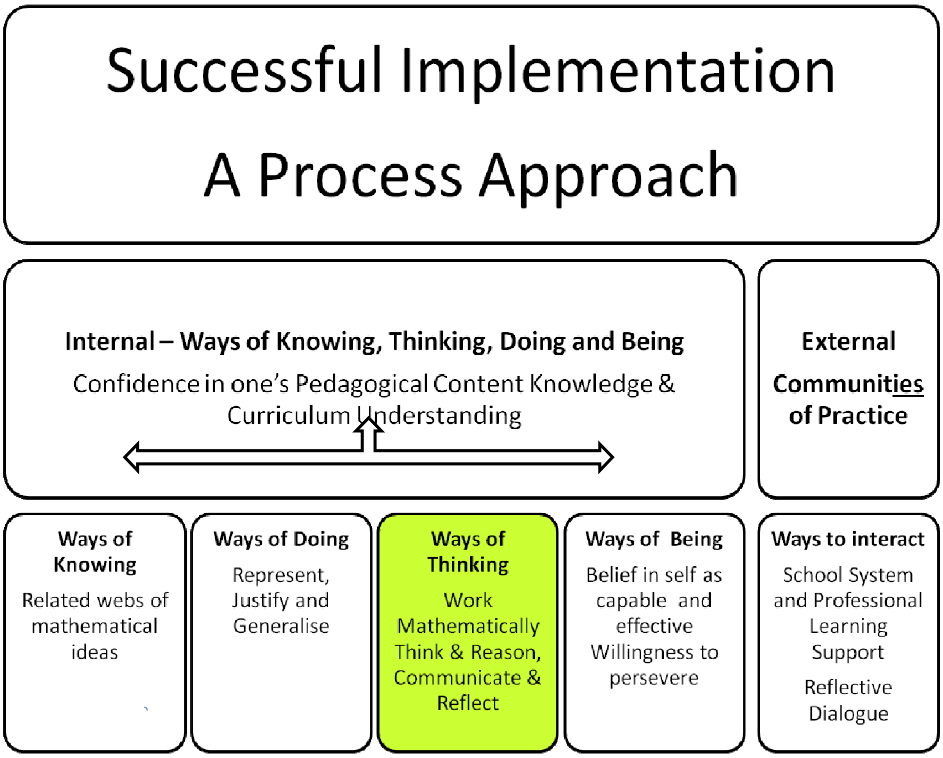


Figure 4.2. Conceptual framework to illustrate the successful implementation of a process approach

To explain the illustration, moving from left to right along the bottom, the first frame refers to Reagan's curriculum, pedagogical and mathematics content knowledge. The next frame refers to Reagan's understanding of the three mathematics practices as a way to enhance her pedagogy. These practices, according to the RAND Study Panel (Ball, 2003), would assist Reagan to effectively scaffold the students' thinking processes and communication of ideas. The third frame relates specifically to the curriculum goals; it refers to Reagan's ability to establish an environment whereby investigative thinking and reasoning processes would remain at the forefront of students' learning, as encouraged in the literature (Battista, 1999; DETA, 2008; Hiebert et al., 2000; Kilpatrick et al., 2006; Van de Walle, 2004; 2007; Van de Walle & Lovin, 2006; QSA, 2004). The fourth frame relates to Reagan's existing and new beliefs about herself as a mathematics teacher and her willingness to persevere despite pending challenges. The last frame refers to the established and founding communities of practice. These communities are important because they may facilitate, constrain or sustain the uptake of a process approach towards mathematics teaching and learning.

4.5 Research Objectives

To fulfill the research aim, that is to contribute to new understandings in the area of teacher practice and student learning at a micro level, this research needed to focus on specific components. Hence, the research objectives were:

1. to investigate and understand the teacher's interpretation of the curriculum and her beliefs about how students learn, as recommended by researchers (Anderson & Bobis, 2005; Brown & Borko, 1992; Cavanagh, 2006; Cuban, 1993; Hartland, 2006; Short & Burke, 1996) and policy documents (BTR, 2005; DETYA, 2000);
2. to investigate and understand how mathematics practices can support thinking and reasoning, communicating and reflecting and how these practices and thinking processes fit into the teacher's daily classroom practice, as recommended by the RAND Study Panel (Ball, 2003);
3. to investigate the "features of the classroom or classroom activities that make it possible for students to develop and engage in mathematical practices," again as recommended by the RAND Study Panel (Ball, 2003, p. 40) and thinking processes, including the way classroom interactions are conducted;
4. to investigate "what features, specifically, shape the learning of different students" (Ball, 2003, p. 40), such as what characteristics of instruction help students to build effectiveness as they use thinking processes and practices to work mathematically;
5. to reveal the possible barriers and/or enabling factors that contribute to successful implementation of thinking, reasoning and working mathematically;

6. to investigate the students' experience to understand whether an investigative process approach supports cognitive development, intellectual and social engagement, and the development of a productive disposition; and
7. to prepare the findings to inform teachers, educators and policymakers of effective teaching practices, or implementation challenges, that contributed to successful (or not) curriculum implementation and the development of successful, numerate learners.

4.6 Research Method and Design

Data collection is not a neutral process. As Lankshear and Knobel (2005) pointed out, if two people were to observe the same situation, it is unlikely that they would both record identical field notes. They assert that the same occurs for collecting spoken and written data, because each researcher will tend to focus on what most interests them. The goal of this research was to portray the occurrences as they happened, with honesty and integrity. Hence, to improve trustworthiness of the data, where possible I, as the researcher, collaborated with Reagan to cross check that the message was interpreted as she had intended (Burns, 2000; Cresswell, 2003; Lankshear & Knobel, 2005). In addition, the process of data collection was 'selective' in the sense that it was not possible to collect all data relating to the research focus (Lankshear & Knobel, 2005; Stake 2005). Thus I collected multiple sources of data, maintaining a tight focus on the research objectives, to allow for triangulation and to improve reliability and validity of the data and findings (Burns, 2000; Stake, 2005). The data sources included questionnaires, observations and interviews, which were collected over three phases of research, within a period of two school terms, as described in the table and paragraphs below. First, I explain the case selection and my role as the researcher.

4.6.1 Case Selection

This case study was a *purposive sample* (Burns, 2000; Lankshear & Knobel, 2005; Stake, 2005), aimed to serve a real purpose, which was to understand the gap between curriculum intention and implementation and to bring light to curriculum change implications. Lankshear and Knobel (2005, p. 149) explained that purposive sampling can "provide data that are more specific and directly relevant to a research concern or interest." To add strength to this case study it was essential to select a motivated participant and setting that would best help me to understand the research problem, question and objectives, as suggested by Creswell (2003) and Shulman and Shulman (2004). Reagan had often expressed a desire to improve the effectiveness of her mathematics teaching practice, and furthermore, the school principal was interested in helping her to advance her professional knowledge. Subsequently, Reagan and the school were selected. The decision was also influenced by her willingness to learn, because this meant that the research could be done *with* Reagan and not *on* Reagan. Glesne (2006) found in her research that by researching *with*, the process helps contribute productively to the lives of the participants, as was the intention of this research. The case thereby offered an opportunity for the researcher and participants themselves to learn (Stake, 2005).

Reagan was in her first year of teaching and had been drawn to the new curriculum ideals whilst at university and was expected to be well-placed to bring contemporary understandings to numeracy development, as recommended by the AAMT (1997). According to the AAMT, entrants to the profession are in a better position, than maybe a teacher accustomed to more traditional methods of teaching, to operationalise contemporary ideals. I also acknowledge that a teacher's inexperience could be a limitation of this research. However, all teachers are agents of change whether they are entrants to the profession or experienced teachers. For instance, Lloyd (2008) found that novice teachers were distinctly different from experienced teachers in their use of the curriculum, and thus, were classified as 'piloters' of curriculum implementation.

There were 450 students attending this small school in North Queensland. The 29 teachers were all committed to the philosophy and ideals of Catholic Education. The school promoted a collaborative effort between the school and parents to develop their children spiritually, academically, emotionally, socially, physically and culturally. In particular, the school was committed to challenging all students to develop and achieve their full potential, and to ensure that the students were prepared for lifelong learning. A core value of the school was to develop in their students a belief that they had the capacity to positively influence their achievements in all areas of life. This value was something that I as a researcher shared. There were 33 students in the Year Six (age 11) classroom, two of whom were classified as having intellectual learning disabilities. These two students were often removed from the classroom for learning support. According to Reagan, the students varied in their mathematical understandings, yet were all working towards the same learning outcomes, except for the two aforementioned. Reagan was motivated to explore the new curriculum ideals to improve the students' learning and her practice, and thus, our interests in this research were mutual. This meant that research could be conducted in ways that informed the teaching and the teaching informed the research, as explained below.

4.6.2 Researcher's Role

In essence, this research was a "reflexive" practice where the primary focus was on classroom practice contributing to research aimed towards improving the quality of mathematics teaching and learning (Ernest, 1998, p. 73). My role as the researcher was to be a sensitive observer, effective communicator and honest reporter. Merriam (1998) used the analogy of the investigator being a detective, searching for clues and leads to fit the missing pieces of the larger contextualised puzzle together. In qualitative research the investigator is primarily the research instrument (Guba & Lincoln, 1981; Lankshear & Knobel, 2005), and thus all observations and analyses are filtered through the researcher's worldview and values (Merriam, 1998). Hence, to be a sensitive and reflective researcher, it was necessary to maintain a focus on the research objectives and theoretical/conceptual framework as a way to view, make sense of and interpret meaning from the data observed and collected. Paying heed to Merriam's advice, I was sensitive to the contextual features and interactions that took place and remained alert to potential biased views throughout the investigation. In this sense, the research process was a reflexive process, whereby I was "reflecting critically on the self as researcher" (Guba & Lincoln, 2000. p. 183) to ensure that any interpretations were based on the theoretical framework.

However, I am also aware that there are many ways of knowing (Lankshear & Knobel, 2005; Lincoln & Guba, 2000). Hence, it is important to disclose my bias and understandings. First, my bias was that all learners must believe that they are capable and that with effort they have the capacity to achieve success, whether they are adult or child learners. Second, as a researcher I had accessed, interpreted and analysed many documents to determine what was meant by thinking, reasoning and working mathematically (e.g., DETA, 2005; 2008; QSA, 2004; 2007; 2007a; 2007c). Then upon further reading, such as the RAND Study Panel (Ball, 2003) and Kilpatrick and colleagues' (2006) text, which have underpinned the new *proficiency strands* of the

Australian Mathematics Curriculum (Commonwealth of Australia, 2009), I surmised that the mathematics practices may assist thinking and reasoning processes. Therefore, as a researcher, I had preconceived ideas that a belief in oneself as a capable learner and/or teacher and an understanding of the mathematics practices may assist both curriculum implementation and successful learning. Aware that new understandings about curriculum ideals and implementation were, and will continue to be, constructed by the participating teacher, the researcher and the readers themselves, I maintained a sensitive focus on the research objectives and theoretical framework to report the information honestly.

To communicate effectively was important in this investigation, just as effective communication is important to support mathematical inquiry (Hunter, 2008). For the purpose of this research, Groundwater-Smith and colleagues' (2003, p. 216) definition of 'effective communication' was used. They defined effective communication as "that which produces the intended results, that is, the receiver receives the message the way the sender intended." To assist effective communication and to ensure credibility of the study it was important to build an empathetic rapport with the teacher participant (Merriam, 1998). Empathy in this sense, involved accepting responses without judgement, whereby the interviewee felt that she was being understood and not evaluated (Groundwater-Smith et al., 2003). Empathetic communication required listening to the participating teacher, which, as Guba and Lincoln (1981, p. 142) explained, involves more than "hearing." Thus, it was necessary for me to listen attentively by speaking *with* and not *to* Reagan, as suggested by Freire (1998). To help Reagan feel that her thoughts were valued, as a researcher I resisted the urge to speak, as Freire advised. Therefore, it was my role to pay attention to the nonverbal cues and allowed silences to occur during the interview process, as suggested by Burns (2000) and Freire (1998). Silences provided opportunities for Reagan to hear the question and also for me to clarify interpretations. Also, jottings about nonverbal cues were noted immediately after the interview, to minimise distractions, and to ensure a sensitive focus was maintained during the interview process. These jottings helped to clarify interpretations, and to enhance the richness of the data and reporting.

My role was to serve an existing practical need and provide a rich description that may offer possible insight into the curriculum implementation process (Ernest, 1998). However, to achieve this aim also meant that the nature of my role was to participate in the implementation process in helpful and supportive ways. Throughout her university degree, Reagan had explored *Years 1 to 10 Mathematics Syllabus* (QSA, 2004) describing thinking, reasoning and working mathematically; however, she had not experienced practical implementation of these processes before, during or beyond university. Thus, as a researcher, it was important to value Reagan's preconceived ideas, but it was also necessary to sensitively offer suggestions where applicable. Hence, again, it was imperative that I filtered my views through the lens of the theoretical and conceptual framework. In recognising that the process of change is ongoing, situations needed to be interpreted through the theoretical lens to ensure reliability and validity (Burns, 2000; Merriam, 1998). In addition, it was important to be aware that my presence in the classroom possibly had the potential to provide direction to Reagan's reflection, both in action and on action. In other words, whilst all observations and analyses are filtered through the researcher's worldview and values (Merriam, 1998), it was possible that my presence, as a researcher, may impact on Reagan's view about the way things are. Thus honesty, sensitivity and sincerity were paramount, as were regular "member checks" (Merriam, 2000, p. 26). As a researcher I frequently asked whether my interpretation of the observations and Reagan's comments 'rang true' with her perceptions of the experiences, as recommended by Merriam.

To be an honest reporter was important because in qualitative research the data are referred to as 'constructed' data (Lankshear & Knobel, 2005). The qualitative epistemology holds the view that "reality is socially constructed, complex, and ever changing" (Glesne, 2006, p. 6). In this sense, Stake (2005) believes that the researcher assists the reader's construction of knowledge. The reader will bring their own experience to the reading to make meaning of the observations and interview data described. Hence, because case study data can be subjective, whereby the participants testify their reality or lived experience, the researcher needs to differentiate between experiential knowledge, and that of opinion or preference (Stake, 2005). This differentiation was essential to enhance internal validity, and thus report honestly. Hence, methods to improve internal validity were applied, first, through being in the classroom for extended periods of time; second, through triangulation, using observations to cross check the participants' view points; and third, through participant checking, as suggested by Merriam (1998; 2000). The results then needed to be presented in a way that was consistent with the data collected (Lincoln & Guba, 1985).

The results were interpreted, analysed and presented according to the research objectives, because the data were collected with these objectives in mind. However, it is important to note that interpreting and analysing the data was an evolving process, and new meanings were constructed upon successive readings of the data. The next section maps the data gathering process and method.

4.6.3 Data Collection Phases

There were three distinct data collection phases as displayed in Table 4.1 below.

Table 4.1 **Data Collection Phases** (*Table adapted from Lankshear and Knobel, 2005, p. 31)*

Research purpose and objectives	**Data to collect**	**Data analysis approach**	**Guiding and informing sources**
Phase one: To determine student attitudes toward mathematics learning (Q206) To determine teacher interpretation of curriculum ideals Q101)	Self-report, short answer questionnaire (anonymous, coded) Self-report, short answer questionnaire	Qualitative content analysis (Lankshear and Knobel, 2005; Kumar, 2005)	*Years 1-10 Mathematics Syllabus* (QSA, 2004) *Education Queensland scoping and sequencing essentials: Years 1-9*. (DETA, 2008)
Phase two: To observe students engaged in mathematics tasks that develop thinking processes, mathematical practices and relational understandings. (Q102, Q103, Q104) To clarify teacher's intent and perception of success, or not, of implementing an investigative process approach. (Q102)	Participant observation (Burns, 2000) in natural setting (Kumar, 2005) Un-structured interviews (Burns, 2005)	Qualitative content analysis to determine themes, using index cards	Burns (2000); Lankshear and Knobel (2005); Kumar (2005) Skemp (1986) *Years 1-10 Mathematics Syllabus* (QSA, 2004) *Education Queensland scoping and sequencing essentials: Years 1-9*. (DETA, 2008) The RAND Study Panel's description of mathematics practices (Ball, 2003)
Phase three: To determine students' change of dispositions toward mathematics learning. (Q206) To determine changes of teacher beliefs, understandings and practices about curriculum goals and students' learning. (Q105)	Self-report, short answer questionnaire (anonymous, coded) Self-report, short answer questionnaire Un-structured interview	Qualitative content analysis to map changes according to determined themes on index cards	Burns (2000); Lankshear and Knobel (2005); Kumar (2005) Skemp (1986) *Years 1-10 Mathematics Syllabus* (QSA, 2004) *Scoping and sequencing essentials: Years 1-9*. (DETA, 2008) The RAND Study Panel (Ball, 2003)

The purpose of the table was to summarise the purpose and objectives of each data collection phase, the types of data that were collected, the analytical approach and the guiding and informing literature. The coding of objectives on the left will be explained in the data analysis section of this chapter as will the coding of the observation notes and interview transcripts.

4.6.4 Phase One: Questionnaires

Self-report, short answer questionnaires were used, first, to understand Reagan's interpretation of the curriculum and her perspective about how students learn, and second, to gain an insight into the students' dispositions towards mathematics and mathematics learning. The first questionnaire was designed to determine Reagan's perceptions of the key messages intended by the *Years 1-10 Mathematics Syllabus* (QSA, 2004). At this stage of the research, the 2004 syllabus was still being used. This was an open-form questionnaire, as Van Dalen (1979) advised, to permit Reagan to answer freely in her own words to draw out her motives, beliefs and attitudes, and to reveal the background experiences upon which her responses were based. The words described in the QSA (2004) curriculum documents were not used to avoid leading questions, and thus achieve a true indication of Reagan's beliefs, attitudes and understanding of the curriculum ideals. However, as Van Dalen (1979) warned, because Reagan was not given any clues to guide her thinking, I was aware that she may unintentionally omit important information. Thus parallel questions were used to check for consistency of answers and to provide successive opportunities for Reagan's beliefs and thoughts to become evident. It was my intention to ensure Reagan was comfortable and did not feel like she was not being 'tested,' so we chose a relaxed setting to conduct this first meeting. To promote critical thought, I also asked the questions and gave her sufficient time to respond orally. This process was recorded and transcribed.

Pre- and post-questionnaires were also used to draw out and record the students' perceptions of mathematics and learning mathematics, and to note changes. It was my intention to find out about students' attitudes and dispositions, which was not always evident in the natural sequence of events, apart perhaps from fidgeting or yawning. The mode of delivery was in class time to ensure full participation. Reagan gave out the questionnaires, explaining to the students that this was their turn to express their feelings towards mathematics, learning mathematics or what they believed would make them more willing to learn mathematics. She read out each statement/question and gave the students one to five minutes to answer, depending on whether the question required a justifiable comment, before moving on to the next statement/question. Reading out the statements/questions ensured that the students understood and responded to each one. This process enabled the students to clarify meaning, and also placed them in a less confronting situation than if I as the researcher, with whom they were unfamiliar at this stage, had given the questionnaire. The questionnaires were then coded to enable comparisons with the data collected in phase three.

However, the students’ anonymity was paramount to assist the collection of honest data. For instance, Kumar (2005) pointed out that often respondents feel reluctant to disclose their personal perspectives with an investigator, which in this case included both Reagan as the teacher and me as the researcher. This reluctance was evident when a student raised his hand and said, “What if we offend you miss?” Hence, Reagan assured the students that the questionnaires were coded to maintain anonymity, as the consent forms had explained. It was my intention to draw out the students’ open and honest responses, which Kumar suggested is more likely to occur when they believe their opinions are not traceable to their identities. The questionnaire was also developed in an interactive style, using language suitable for year six students and to help the students to feel as though someone was talking to them (Kumar, 2005).

The questions were ‘attitudinal questions’ (Lankshear & Knobel, 2005), and a rating scale was developed with a smiley face at one end and a frowning face at the other. Words instead of numbers were used for the students to rate their attitudes, these were, strongly agree, agree, undecided, disagree, and strongly disagree. The questions were posed as statements and the students marked on a scale whether they agreed or not with the statement. For example, two statements were: “I enjoy giving things a go in maths, even if I don’t know if those things will work,” and: “I enjoy puzzling things out for myself in maths lessons.” Below several of the statements, the students were asked to justify their response, for example: “Give an example of a time when you enjoyed puzzling something out.” The last section of the questionnaire was designed to encourage the students to think reflectively about mathematics as a discipline and their disposition towards mathematics learning. These questions were planned to check for consistency of the students’ responses, as Van Dalen (1979) suggested. The questions were:

- What is Maths?
- Name three important pieces of maths you have learned at school, and give a reason why you believe these are important:
- What is interesting in your maths classes? Why?
- When you hear the word *maths***,** what are the first five words that you think of:

4.6.5 Phase Two: The Interview Process

Each week, I observed two one-hour lessons over two school terms, ten weeks in total. At the end of the week I conducted a thirty to sixty minute interview with Reagan. It was anticipated that the repeat contact between the teacher and researcher would build rapport and enhance mutual understanding and confidence, and develop a thick description of in-depth, accurate information (Glesne, 2006; Kumar, 2005). I took notes during and after the observations and interviews. These recounts were added to interview transcriptions to determine recurring themes. The interview process will be explained first, followed by an explanation about how the observations were conducted.

Critical analysis of the teaching and learning process was required to examine what worked and how and why. As Shulman and Shulman (2004, p. 264) put it: “At the heart of that learning is the process of critical *reflection.*” Hence, it was necessary to open up the space and provide Reagan with opportunities to reflect critically on her practice and on how she believed the students responded to the new approach to teaching (Clandinin, 2008; Shulman & Shulman, 2004). Throughout the first stage of the research Reagan planned mathematical tasks that she believed would promote investigative thinking (DETA, 2008; QSA, 2004), and thereby conceptual development of mathematical ideas (Skemp, 1986). The interviews then provided an opportunity for Reagan to reflect on the implementation and subsequent learning of those tasks. To ensure that the results were going to be dependable, I wanted to comprehend fully Reagan’s experience and perceptions, thus I took an empathetic stance to become a partner in the study (Fontana & Frey, 2005; Groundwater-Smith et al., 2003). In my view, Reagan was more an informant than a respondent, and hence, the interview questions were open-ended and conducted in a conversational tone (Burns, 2000). My goal was to accept responses without judgement in order to create an atmosphere whereby Reagan understood that her thoughts and experiences were valued (Groundwater-Smith et al., 2003) and to construct an equitable relationship. The interviews were conducted in a relaxed setting at my home, which was a joint decision. To maintain a focus on collecting credible and trustworthy data, I frequently checked with Reagan to allow her the opportunity to reflect, respond and clarify meanings when needed.

A sensitive focus was adopted throughout the interview process to generate non-leading and open-ended questions (Burns, 2000; Lankshear & Knobel, 2005; Merriam, 1998). Even though as a researcher my role was to draw out what was on the interviewee's mind (Patton, 1990), it was also important to reduce the risk of leading responses (Burns, 2000). Rather than use lead in questions, I used interpretive and gentle probing questions (Merriam, 1998), as well as "minimal encouragers" (Burns, 2000, p. 424). These questions and encouragers provided an opportunity to confirm my tentative interpretation and to prompt Reagan to reveal more information. For instance, the interpretive questions were framed around words like: 'Would you say that…', and when it was necessary to elicit more information, probing questions such as, 'How did you feel about…?' or, "How did you feel when…?" or "Do you need time to think?' were used. At times, I used minimal encouragers such as, 'Yes,' 'I see,' or 'Can you tell me more?' to evoke more details. In this respect, the interview process enabled me to respond to the immediacy of the situation, to Reagan's perspectives and to the new ideas being expressed, and thereby enrich the data collected (Merriam, 1998). My aim throughout the interview process was to generate an insider's perspective about the process of curriculum implementation and to access Reagan's evolving understandings of the inherent learning processes, as suggested by Lankshear and Knobel (2005).

The successive interviews were semi-structured, with pre-prepared questions to guide the discussion (Merriam, 1998). An interview will not capture everything a person thinks, values, or believes; thus, the successive interviews were designed to build upon data generated at previous interviews and observations, as encouraged by Lankshear and Knobel (2005). However, following Merriam's (1998) advice, the order in which the questions were asked and the exact wording were not determined ahead of time to ensure I could respond to the immediate situation. By not relying on a prepared script I was free to listen attentively and respond to the situation at hand (Merriam, 1998). As Van Dalen (1979) suggested, attentive and analytical listening helped me as the interviewer to discern when more information was required or if a question needed to be repeated or reworded. This listening opened opportunities to ask Reagan to recall classroom instances that would provide concrete evidence to help illustrate her viewpoint, and thereby ensure reliability of the data.

However, it was also my role to support Reagan in this complex and demanding learning experience. The semi-structured interview was an opportunity to co-construct data between the researcher and the participating teacher (Lankshear & Knobel, 2005). For instance, the interview process prompted Reagan's reflective thinking about her teaching and the students' learning; this reflective thinking then informed subsequent planning. After two weeks into the research Reagan sought support occasionally. For example, she was unsure how to elicit the students' thoughts, thus, as politely as I could, I suggested she invite the students to use the whiteboard to communicate their ideas. Then at midpoint, Reagan believed her students were prepared to investigate mathematical ideas more deeply and asked for help to plan an investigative algebraic sequence of activities. She was open to change and willing to investigate the practical application of new teaching practices.

Hence, over the school holidays we met and co-planned activities indicating opportunities for the students to represent and justify their thoughts, or to make generalisations (see appendix). The co-planning enabled reciprocity in communication whereby we shared and valued each other's ideas has having equal importance in the investigation of teaching ideas. The activities were sequenced and based on ideas adapted from Van de Walle and Lovin (2006) and Reys, Lindquist, Lambdin, Smith and Suydam (2004). Reagan took these activities home and then over the last part of the research she selected activities based on her reflective thoughts about what the students needed in order to help them to develop algebraic reasoning. The reflective nature of the interview process helped her to clarify her thoughts about teaching and learning and inform further teaching decisions. Her reflections also helped me to gather data based on her personal experience of the process of implementation. Because Reagan was in control of the classroom, it was her subjective perceptions about student learning that were critical for understanding the conscious habits of teaching, as pointed out by Stenhouse (1978) and Clandinin (2008). Observations then helped to clarify and cross check responses to maintain data reliability, to make the empirical data less subjective and to reduce the risk of misinterpretation (Stake, 2005), as explained below.

4.6.6 Phase Two: Conducting Observations

The observations were both unstructured descriptive observations (Lankshear & Knobel, 2005) and participant-observations (Burns, 2000). I discreetly observed from the back of the classroom during the introduction phase of the lesson and wrote descriptive notes. The second stage of the lesson involved the students' active engagement in the mathematical investigation. In this stage, I became more of a participating observer. For example, Reagan engaged my services to hand out materials, supervise group activities, or inform her if a student needed an inspirational prompt. It was anticipated that the peripheral interaction would help the students to perceive me as a classroom member rather than the classroom critic. As Burns (2000) believes, an advantage of the participant observation is that the setting is kept as natural as possible. Work samples were collected by Reagan and myself after the lesson and photocopied. These samples were coded to maintain anonymity and filed with the questionnaires. I used the preliminary observation notes and the interview records to prepare guiding questions to structure follow up observations, as encouraged by Lankshear and Knobel (2005). The purpose of preparing guiding questions was to maintain a sensitive focus on the research objectives (Merriam, 1998).

Importantly, I felt that it would be the interdependency between Reagan as the practising teacher and me as an informed researcher that could bring about changes to the perceptions and actions of both parties, as pointed out in other research (Clandinin, 2008; Hunter, 2008; Steinberg et al., 2004). Thus, the reason I chose to conduct participant observations was because they are believed to position the teacher as a co-worker rather than as the object of the research (Stenhouse, 1978). In other words, the participant observations assisted Reagan in her goal to improve her practice and me in my goal to better understand the implementation process from a teacher's point of view. At all times I valued the collaborative effort of the researcher and the teacher. To clarify, Stenhouse (1978, p. 156) cited Hamilton's perspective that participant observations are effective when the observer can find "some way to 'give' as well as to 'take'." Thus, it was my intention to uphold a co-working relationship with Reagan to ensure she felt that the investigation was not just to benefit the research, but to also assist her teaching decisions.

4.6.7 Phase Three: Questionnaires

Identical questionnaires from phase one were used in this third stage. The objective was to generate comparative data through pre- and post-questionnaires to identify changes to beliefs and attitudes. The student post-questionnaire was conducted in the same way as in the first phase of the investigation. The same anonymous coding was used to ensure a match between each student's pre- and post-response. The purpose was to examine the data to identify changes, or not, to the students' dispositions and attitudes toward mathematics and mathematics learning. There was also a post-questionnaire phase for Reagan with the same questions as the pre-questionnaire, although this was conducted as an interview. I used an interview to draw out more of Reagan's perceptions and understandings about what she or her students achieved throughout the implementation process. The goal was not so much to draw out her opinions and feelings, but rather her sensory experience (Stake, 2005) about what facilitated or constrained curriculum implementation, student learning, active engagement or the development of a productive disposition and how and why this occurred. The aim of this final interview was to generate data that would enable me to develop an understanding of changes to Reagan's beliefs about how students learn, about the process approach towards mathematics teaching and learning and about her willingness to sustain change.

4.7 Data Analysis

The data were analysed using a qualitative content analysis, described by Lankshear and Knobel (2005, pp. 332-341). They explained that content analysis "rejects a 'one word/one meaning' relationship" (p. 333) to make valid inferences from the text. A valid inference implies that the analysis is specific to the text being analysed and that there is an awareness of biased perspectives (Burns, 2000; Weber, 1985). Hence, at all times, especially with the interview data, my focus was on interpreting the truth in the message as portrayed by Reagan or the students. Analysis of the observations had two foci, as guided by Lankshear and Knobel (2005). The first was to ensure that the process was done with honesty and integrity. The second was to ensure that the content to be analysed, was specific to what occurred in the mathematics classroom and how these occurrences related to the theoretical/conceptual framework and the research objectives. The goal of the analysis was to look for patterns across the data, not norms (Glesne, 2006), which involved content category construction (Lankshear & Knobel, 2005; Merriam, 1998), as will be explained below. Duplicate copies of all data were made to maintain the original format for crosschecking once the data had been mechanically manipulated, that is, cut up and placed on charts.

The first step of the content category construction related to the research objectives. Each objective was coded according to the research question ‘one’ or ‘two’: research objective number one was coded Q101, referring to question one and objective one, objective two was Q102, objective three was Q103, objective four was Q104, objective five was Q105, and objective six was Q206. Colours were then assigned to each category. The interview transcripts were also coded, for example, ‘IV1’ for the first transcript, the numbers correlate to two interviews in the sequence. Hence, the transcript ‘IV1’ involved the first two interviews. The observation notes were also coded, for example, ‘T3WK1D2’ was a code for term three (T3), week one (WK1), day two (D2).

The next step involved reading the first interview, observation field notes, curriculum documents and the conceptual framework tables: *Successful implementation: A process approach* and *Becoming numerate: Enabling continued successful learning*. The tables were used to ensure that further conceptual categories reflected the research purpose, as suggested by Lankshear and Knobel (2005) and Merriam (1998). Any pieces of data that I considered important, interesting or requiring further investigation were noted by jotting down comments on the margins of the page and on index cards, as Burns (2000) recommended. The jottings helped me to group pieces of information into concepts for further analysis. The next interview transcript, observation field notes and relevant informing documents (see table in section 3.6.3) were categorised in the same way, building on the concepts disclosed in the previous stage. This process continued until all interviews and transcripts were coded into research objective categories and abstract concepts.

To maintain conceptual congruence (Merriam, 1998), it was important to code the data in a way that would ensure they answered the research questions in a sensible way. To maintain congruency the data were tabulated. I used the research objectives as headings. There were seven headings altogether, one for each objective and one for an area of inquiry that had not already been recognised as an objective in this research. I then used the bottom frames on the theoretical framework tables as side headings, ten altogether. Each cell on the table was transferred to index file cards to guide further analysis and to record and thereby link the data back to the original format/context. The data were also classified into two further categories within each cell; these were ‘how’ and ‘why’ categories. For example, the data for objective five (Q105), which was related to barriers or enabling factors of curriculum implementation, were further analysed to infer ‘how’ an instance supported, or not, curriculum implementation and/or ‘why’. This step was particularly important to provide a rich, thick description that gave enough detail for the reader to indirectly experience the happenings and thereby draw their own conclusions, as Stake (2005) recommended. However, to ensure harm was not caused to the participants as a consequence of this case study, ethical considerations were taken into account.

4.8 Ethics

Formal ethical codes and procedures were adhered to, as approved by university *Human Research Ethics Committee,* ethics clearance number H2727. To enable use of the given data, the participating teacher, principal and participating students’ parents were asked to sign an informed written consent form. These forms explained in detail the research purpose and procedure, and informed the participants that they could withdraw from the research at any given moment. Participants were assured that their identity would be masked to assure confidentiality and anonymity. The covering letter also explained that if a parent chose to withhold consent that their child would not be disadvantaged in any way; however, all parents consented. Lankshear and Knobel (2005, p. 110) pointed out that a researcher needs to “err on the side of ‘under-intruding.’” They explain that intrusion is not only spatial, as in the occupying of space in the classroom community of practice, it can also occur when the data collection vacillates from the research purpose. Hence, the empirical data collection (interviews and observations) were checked consistently to ensure that the questions asked and observations made were focussed on the research objectives and conceptual framework. Punctuality was also important to avoid intruding on the participants’ time. Lankshear and Knobel also suggested that research is best practice when favours are returned to honour the goodwill and generosity of the participants. In this case, the researcher was mindful of the extra time spent each week during the interview process and thereby prepared a meal for the participating teacher to take home to her family after each interview.

4.9 In Summary

This chapter has outlined the rationale, method and methodology chosen to conduct this research. To summarise, because there have been limited understandings about why a gap exists between curriculum intention and implementation, I followed recommendations of previous research (Anderson & Bobis, 2005; Cavanagh, 2006) attempts into understanding the problem. After researching the literature (Glesne, 2006; Lankshear & Knobel, 2005; Merriam, 1998; Shank, 2006; Stake, 2005; Yin, 1994) I concluded that the richness of data that a single case study could produce would best serve my research purpose to understand the 'how' and 'why' of the process of curriculum implementation. I envisaged that the various data sources used would enable perceptions to be cross checked and thereby become more reliable and valid. The data sources consisted of documents, questionnaires, interviews and observations. I also anticipated that the process of implementation would be challenging and thus relaxed interviews were conducted to open up a space for co-reflection. This reflection was intended to link theory with practice and vice versa through a process whereby Reagan reflected on her teaching practice and I reflected on the theory that framed this study, as suggested by Clandinin (2008). In addition, observations, work samples and pre- and post-questionnaires were used to determine change in students' dispositions towards mathematics and mathematics learning.

The aim now is to present the data that has been analysed through a basic interpretive content analysis in a way that invites other teachers to test and evaluate the findings within their own situation, rather than expecting them to accept the findings as a generalisation about curriculum implementation, as suggested by Stenhouse (1978). The next chapter uses the data gathered through the interviews to describe Reagan's evolving interpretation of thinking, reasoning and working mathematically. The chapter then describes her beliefs about learning with understanding and how students develop conceptual understanding. It then details and discuss her experience, as she tells it, about scaffolding investigative thinking processes. The last section draws out her thoughts, from the data, about establishing a mathematical community of practice. The aim of this chapter is to report on the findings of the interview and to set the scene for what is to come, and hence, is an important chapter. In the following chapter Reagan's beliefs can be seen as underpinning the work she does in the classroom.

CHAPTER FIVE:

INTERPRETING CURRICULUM CHANGE

"Beliefs about teaching mathematics underpin teachers' effectiveness at teaching mathematics."

(BTR, 2005, p. 22)

5.1 Introduction

This chapter describes and analyses Regan's beliefs about the teaching and learning of mathematics, and incorporates brief glimpses of her own experiences in learning mathematics and how to teach it. As McPhan and colleagues (2008) pointed out, the delivery of quality mathematics teaching and learning is dependent on the teacher's uptake of the new curriculum approach. Hence, it was important to investigate Reagan's preliminary beliefs to determine if these thoughts reflected the mathematics curriculum precepts outlined by QSA (2004) and DETA (2008). First, the chapter outlines her initial interpretation of the curriculum and her beliefs about the importance of learning with understanding. It then looks at her thoughts about thinking, reasoning and working mathematically, and her beliefs about how students learn. The chapter then describes her thoughts about how she anticipated incorporating working mathematically into her daily classroom practice, through encouraging thinking processes (DETA, 2008; QSA, 2004) and mathematics practices (Ball, 2003). Features of the classroom environment, instructional strategies and types of tasks that Reagan believed would help students to effectively work mathematically are described, as well as her thoughts about foreseeable challenges towards implementing curriculum change. The last section reveals the barriers and successes of curriculum implementation as Reagan tells them. Chapter six and seven will then focus on how her instructional strategies impact on the students' learning experience and their dispositions towards mathematics and mathematics learning.

5.2 The Queensland Mathematics Curriculum: Reagan's Beliefs

As revealed in the literature review, the *Years 1-10 Mathematics Syllabus* (QSA, 2004) has been undergoing review and change for some time. The 2004 document, which Reagan referred to as being 'new' in this interview, was a revised draft of the previous 2002 draft syllabus document. However, in the year following this research, the 2004 draft syllabus was to be renounced and the new *Education Queensland Scoping and Sequencing Essentials: Years 1-9* (DETA, 2008) was to be implemented. This section describes Reagan's beliefs about the key messages of the 2004 syllabus. It starts with her preliminary beliefs and then finishes with a brief description of her transformation.

In this first interview Reagan appeared disconcerted when she spoke about curriculum implementation. She explained that often she felt isolated because she did not have a role model to follow or a colleague with whom she could share and discuss teaching, learning and curriculum ideas (IV1). In her opinion, it was difficult to find colleagues who were competently and confidently implementing the mathematics curriculum, regardless of their years of teaching experience. She would have preferred to have participated in more professional learning programs (IV1), or to have been able to observe what implementation looked like in practice in order to familiarise herself with the syllabus goals, as she explained:

> I would like to see how other people implement the syllabus … getting people who are actually confidently working with the syllabus is probably the hardest thing, because they can be an experienced teacher but not have necessarily taken on the syllabus(IV1, p. 1).

Reagan then explained that at her school the teachers were encouraged to implement a mathematics program that was affiliated with the QSA (2004) curriculum documents. The parents purchased a corresponding journal for the students to complete in class. For ethical reasons the mathematics program will be referred to by the pseudonym, *Maths at School.* She had attended a conference about the goals of this program and found the experience inspiring. She reflected:

> Going to something like the *Maths at School* conference was really exciting and stimulating because they were people who were really using the syllabus. The conference just made me think yes, this is what I was taught, this is the focus that I have brought from university, and this is where maths is going. It's not going into worksheets. (IV1, p. 3)

In this statement, when Reagan mentioned 'this is what I was taught,' she was referring to her four years at university. She believed that the program was more aligned with the syllabus than what the school had encouraged. For instance, she makes a reference to the use of worksheets. She viewed the journal activities that the students were supposed to complete as worksheets (T3WK1D2). There appeared to be a contradiction in the way the school staff and Reagan had interpreted the program and its implementation.

Reagan believed that the QSA (2004) had attempted to make the process of curriculum implementation more manageable for teachers because the syllabus now presented fewer core learning outcomes:

> The new QSA syllabus looks a lot simpler, like it kind of cuts down your outcomes to about forty percent. But the actual reality for children to be able to truly meet an outcome means they need multiple opportunities, within that 18 month period, to address one outcome. There are so many that you need to cover that they [the students] don't get to revisit; so, that is where I think the new QSA syllabus, if it is true to what they are saying, will be much easier to implement. It seems to still allow for what I call the teacher to be a thinking teacher (IV1, p.3-4).

Reagan was hopeful that the curriculum would open opportunities for students to revisit learning outcomes, but cautions that students need many learning opportunities over an eighteen month period for the desired learning outcome to be achieved. Here she hinted at her dedication to students' construction of meaning over an extended period of time.

Reagan explained that the syllabus document was "not so prescriptive" (IV1, p. 1) and positioned her as a 'thinking teacher.' Being positioned in this way was an important ideal to her, as was revealed when she described what she meant by the term 'thinking teacher':

> I mean a thinking teacher in that I believe that outcomes education acknowledges the fact that I am an intelligent enough being who can actually create learning experiences from the guidelines, that I can unpack that outcome and then brainstorm and come up with activities to meet that. (IV1, p. 4)

Reagan's thoughts revealed that it was important to her to be positioned as a teacher who had the capacity to plan learning activities to target specific learning outcomes. Clearly, she valued the freedom to plan according to what she perceived would enable her students to meet the aims of the core learning outcomes.

Reagan believed that the structure of the core learning outcomes framework of the *Years 1-10 Mathematics Syllabus* (QSA, 2004) was an attempt by QSA to encourage teachers to plan learning experiences that were relevant to students' lives, as follows:

> While it [the syllabus document] is very structured, in that it gives you your guidelines, the main basis is to ensure there is a linking back, making the connection between the class and outside-of-school life. (IV1, p. 1)

Reagan believed that it was her role to explicitly highlight the connections between mathematics learned in school and its application beyond school. Her thoughts reflected those of the 'connectionist teachers' of Askew and colleagues' (1997) study, who were focussed on highlighting connections between school mathematics and real world mathematical applications. Although these teachers also focussed on highlighting the connections between mathematical ideas, Reagan had not spoken of this sort of connection in this initial stage of the research.

5.2.1 Interpreting Relevance

In this first interview Reagan described what she meant by making learning relevant. In her view she was focussed on connecting mathematics learning with the students' real world. However, she believed that the real world was not something that existed beyond the school grounds, as she explained:

> The real world is what the children are doing at the time; that is their real world. (IV1, p. 2)

She believed that to make the learning relevant meant integrating mathematical learning opportunities both in and beyond the mathematics classroom. She clarified her thoughts with the following example:

> Even when we did HPE, I put the maths into that, where they had to convert a 15-second pulse into one minute, then taking their pulse over a series of five minutes, and being active in taking their pulse and then graphing it. So that they do maths without realising they are doing it. (IV1, p. 5)

In Reagan's view, by extending the mathematics learning experience beyond the classroom she could help the students to recognise the connection between in-class mathematics and mathematics used beyond the classroom. In other words, by incorporating mathematics into her planning for Health and Physical Education (HPE), she believed that the learning was relevant simply because it related to what the children were doing at the time.

One of the key messages of the syllabus is for students to become “active investigators” whilst being engaged in learning experiences “related to a range and balance of situations from life-related to purely mathematical” (QSA, 2004, p. 3). Reagan’s use of the term ‘active’ in the statement above reflected this key message of the syllabus. She also indicated that whilst the students were active in this HPE lesson, they were ‘doing maths without realising they were doing it.’ Thus it appeared that Reagan wanted to promote mathematical learning experiences that were fun, and, in this sense, I wondered whether perhaps she had attempted to conceal the mathematics within the HPE lesson. Even though she was focussed on the mathematics, she was also concerned about the students’ developing dispositions towards mathematics and mathematics learning.

In another comment, Reagan alluded to a desire to make mathematical learning enjoyable. She was inspired by a game demonstrated at the *Maths at School* conference. The game involved rolling numbered cubes to help students grasp a double-halving, multiplication-thinking strategy. She introduced the game to the students when she returned to the classroom. She believed that game helped the students to grasp the strategy more easily than when she had introduced the double-halving strategy previously, as she described:

> When I went to the *Maths at School* conference, they did the doubling and halving, and, while I had tried to do that, the kids just hadn’t grasped it. At the conference, they had little foam cubes, so you played a rolling the cube game. One of the cubes had all two-digit, even numbers on it, the other one had two-digit, odd numbers, so they rolled it, and then they had to halve one and double the other. Then they had a sheet with numbers on; so they put their counter on the sheet, and they had to mentally work it out. I took it back [to the classroom] and used that straight away. I made up little mind maps, and straight away they got it. (IV1, p. 4)

Reagan perceived the game as a way of developing a thinking strategy, although the game as she has described it here seemed to be focussed on reinforcement of ideas as opposed to developing understanding. For instance, the students would roll the dice, and then either halve or double the numbers rolled. Thus, to work this way with numbers meant that the students must have already developed a ‘feel’ for numbers, which was a notion Anghileri (2000) wrote about. At this stage, I should have clarified with Reagan what prior learning the students had experienced because it may have been this learning that made it easier for the students to grasp, rather than the game itself. Nevertheless, the point being made here was that the enthusiasm that Reagan expressed towards the game may have been because it had potential for the students to perceive mathematics learning as fun. However, she also said that she had made ‘little mind maps,’ which may have been an attempt to help the students recognise connections between number facts, as stressed by various educators (Anghileri, 2000; Brown & Liebling, 2005; Van de Walle, 2007).

In the final interview, Reagan altered her perception of ‘relevance’ in mathematics education. In this interview she explained that a lesson was a “living thing” and that it needed to “be able to flow in the direction that it wants” (IV5, p. 12). She now viewed mathematics learning as a ‘lived’ experience; it was not an experience to be added to the students’ lives; rather, there was potential to create relevance in mathematics learning experiences that interlocked with the students’ regular day to day activities. For example, at the beginning of the year, she purchased an expensive mathematical resources kit, yet she now believed it was necessary to use everyday items that the students were familiar with, as she explained:

> What I have found is you are better off just getting what you have got, like when we were looking at shapes, picking up a tissue box and saying, ‘Well what kind of 3D shape is this?’ and, ‘How many faces?’ So it isn’t separating it [the learning] from every day. (IV5, p. 4)

Reagan’s intention was to enable the students to see that mathematics learning was relevant because mathematical ideas existed all around them. Hence, as she described, when learning about three dimensional shapes, she believed it was important to discuss the properties of shapes that existed in the regular classroom environment.

In summary, Reagan perceived that the *Years 1-10 Mathematics Syllabus* (QSA, 2004) had attempted to make curriculum implementation easier for teachers by reducing the number of core learning outcomes. She also believed that the syllabus had positioned her as a thinking teacher; however, she did not believe that the school shared this vision. Rather, in her view, the school encouraged the teachers to use the *Maths at School* program in a way that did not align with the curriculum ideals to promote learning that was relevant to the students’ lives. Initially, she believed that this limited view of the program inhibited her opportunities to plan learning that was relevant to the students’ lives. Making learning relevant was the main idea Reagan had gleaned from the syllabus, although as time progressed, so did her view of relevance. While, initially, she was focussed on making the learning relevant to the students’ lives, towards the end of this research she was more intent on the students being able to recognise the relevance of mathematical practices in the world in which they lived. How she altered her view will be revealed later in this chapter. In the next section, Reagan’s beliefs about what it means to learn with understanding are discussed.

5.3 The Importance of Understanding

Learning with understanding was something that Reagan had come to value, which is stressed by mathematics researchers (Hiebert & Carpenter, 1992; Hiebert et al., 2000; Kilpatrick et al., 2006; NCB, 2008; NCTM, 2000; Skemp, 1986; Van de Walle, 2004; 2007). She explained that her schooling was based on traditional teaching approaches, where she believed she was required to rely on memory rather than learn mathematics with understanding. She reflected in the final interview:

> It wasn’t until university that I actually discovered why I did a lot of the maths that I did at school. At school, I remember you just did the sums to know how to do the sums. We didn’t learn why; you did the multiplication tables, without really understanding what the concept of multiplication meant or what the concept of division actually meant (IV5, p. 2).

Reagan reiterated in the last two interviews that, as a result of her schooling, she has had to relearn many mathematical concepts. Consequently, she did not want her students to be faced with the same challenge; she wanted her students to leave school understanding mathematical knowledge.

In her experience, the lack of conceptual understanding had impacted on her confidence as a mathematics user, and subsequently, she lacked confidence in her own mathematical content and pedagogical knowledge. Reagan explained that when standing at the whiteboard, momentarily she had thought to herself, “I actually don't know whether I am teaching this right” (IV1, p. 6). To compensate, she admitted that she spends a lot of time researching mathematical ideas on the internet both to clarify her understandings and to search for teaching ideas:

> I spend a lot of time web surfing to bring in other ideas and to help to understand that concept as well. (IV1, p. 6)

Thus she was motivated to teach mathematics effectively, even if this required spending her free time clarifying her own mathematical understandings. Her willingness to investigate both mathematical and pedagogical ideas was a significant factor in why I chose to conduct this research with Reagan, in her classroom.

At the beginning of this research, developing understanding through use of concrete materials was important to Reagan. She was concerned that many of her students, even though they were in year six, did not understand the concept of multiplication or division (IV1). She explained that whilst a teacher-directed approach may suit some students, she believed most of her students needed to be able to manipulate objects to develop understanding (IV1). To clarify, a teacher-directed approach relates to a transmissive approach where the students are passive recipients of information, and the focus is on memorising rules and procedures (Anghileri, 2000; Bobis et al,. 1999). Reagan believed that her students were accustomed to a more transmissive mode of teaching; whereas, in her view, the students needed learning experiences that required the manipulation of concrete objects. For instance, she reiterated:

> What I have seen of my students so far is that there is only probably a third really, a third of the class, that can grasp concepts purely by going through it with text and chalk and talk, but the rest really need a hands-on approach to it [learning], like actually seeing [using representations]. There are some who are actually struggling with their multiplication, and just showing them the concrete examples of how many groups there are and four times three equals four groups of three or three groups of four, which is the type of manipulation I am referring to. (IV1, p. 2)

Reagan's thoughts about developing understanding through manipulating concrete materials to develop mental representations reflected many researchers' views of learning (e.g., Anderson, Reder & Simon, 1998; Battista, 1999; Hiebert & Carpenter, 1992; Piaget, 1896-1980; Skemp, 1986). Battista (1999) suggested that for students to develop understanding they need to follow a cycle of action, reflection and abstraction. Battista believed students create abstract visual representations of mathematical ideas when they replay the action in their minds. In other words, when students have actively manipulated concrete materials they reflect on the learning, which assists the development of cognitive schema or abstract mental representations. It is these abstract representations that students use to make sense of other mathematical situations (Battista, 1999; Hiebert & Carpenter, 1992; Skemp, 1986). It is when students can connect these abstract representations in their minds that conceptual/relational understanding is developed (Skemp, 1986). Reagan was particularly drawn to this learning cycle because she found it beneficial in her own redevelopment of conceptual understanding about multiplication:

> I grew up not understanding multiplication; I had never used counters for grouping (IV3, p. 2).

Reagan hoped that through manipulating and grouping concrete materials, the students would create visual images in their minds to be used when thinking about multiplication. Her interpretation of understanding related specifically to developing mathematics knowledge and understandings. At this point in the research, she had not mentioned that students also needed to recognise the connections between concepts and ideas or, as the following section reveals, ways of working mathematically.

5.4 Interpreting Thinking, Reasoning and Working Mathematically

In the initial interview, Reagan's thoughts about the key messages of the *Years 1 to 10 Mathematics Syllabus* (QSA, 2004) were equivocal; she had not mentioned that the underpinning emphasis was to encourage a learning process of thinking, reasoning and working mathematically. This was an important point because the focus of this research was to investigate her attempt to implement activities that promoted thinking, reasoning and working mathematically. In contrast to Reagan, I had researched the topic in depth and was very aware that the QSA (2004) believed thinking, reasoning and working mathematically was necessary to develop understanding and enable lifelong learning. Queensland Studies Authority (2004) and DETA (2008) stressed that students need opportunities to think and reason mathematically to ensure that they leave school equipped with the capabilities and confidence to continue learning throughout their lives. The emphases on lifelong and successful learning are important features for curriculum development, as asserted by MCEETYA (2008) and NCB (2008). Hence, a teacher of mathematics must help students to develop mathematical understandings, inclusive of ways of working mathematically (DETA, 2008; QSA, 2004). Thus the underpinning key message of the *Years 1-10 Mathematics Syllabus* (QSA, 2004), as I had understood from the literature reviewed, was to implement an inquiry-based approach. This approach was intended to facilitate learning with understanding through developing students' abilities to think and reason with and about mathematical ideas.

However, as Klein's (2000) research pointed out, many teachers have not experienced a process of inquiry, where investigative mathematical thinking underpins the learning process. This was Reagan's situation, as was revealed earlier; she was taught through transmission based methods (IV1), and consequently, did not understand what an investigative approach looked like in practice. To recapitulate, an investigative approach requires the teacher to engage the students in mathematical investigations that promote mathematical thought (AEC, 1991; NCTM, 2000; QSA, 2004). This approach implies that students need to struggle or wrestle with mathematical ideas in ways such that they initiate their own thinking processes to prove to themselves how and why mathematical ideas work (DETA, 2008; Hiebert et al., 2000; QSA, 2004; NCTM, 2007; Savery & Duffy, 2001; Skemp, 1986). Even though this research was focussed on adopting an investigative process approach (DETA, 2008), the first interview revealed that Reagan was uncertain what was involved in encouraging thinking and reasoning, as she explained:

> I am sure there are things about the actual concepts of reasoning that go into the syllabus outcomes, um so it's, while it's very structured, it gives you your guidelines that main basis of linking it back, making the connection between the class and outside of school life. (IV1, p. 2)

Hence, she was aware that the syllabus encouraged mathematical reasoning, yet she was unsure what this meant. She interpreted her role as helping the students recognise where mathematics was used beyond school.

In addition, from the literature reviewed, I had taken the view that if students were expected to think mathematically to learn, they first needed to learn how to think mathematically, as pointed out by Kilpatrick and colleagues (2006). I had envisaged that the three mathematics practices, as suggested by the RAND Study Panel (Ball, 2003), would assist Reagan to scaffold investigative thinking processes, as was outlined in the theoretical framework of this study. Thus, I needed to understand from the outset of this research what she understood about these practices. Hence, the last question I asked her in this interview was what she understood about the mathematics practices of representation, justification and generalisation. She responded:

> Well, to be perfectly honest, I have very little understanding of the three mathematical practices that you just said. If I think, just looking at, thinking of what maths is and what the words mean, to me 'representation' would be where you actually are looking at a particular mathematical concept and representation involves you constructing, using manipulating materials, to represent what you think that concept is; so a showing things visually or physically. 'Justification' is your rationale of thinking, why you have come to that conclusion, being able to explain your representation. And 'generalisation,' I am just going off the top of my head here, would be how that then applies to the mathematical knowledge you have or everyday situations that you use maths. It is probably completely wrong but that is my interpretation. (IV1, p. 7)

Even though Reagan had not consciously used these practices prior to this research, her interpretation of two of the mathematics practices aligned with the meanings attributed to each practice by the RAND Study Panel (Ball, 2003). *Representation,* according to RAND, involves representing ideas with a variety of concrete, symbolic or diagrammatic representations, and *justification* involves being able to explain the representation mathematically. However, *generalisation*, as RAND described, involves being able to detect patterns to formulate a mathematical rule.

Toward the end of this research, Reagan had changed her views of the syllabus emphases considerably. In the first interview she had not mentioned either of two important key messages of the *Years 1 to 10 Mathematics Syllabus* (QSA, 2004), which were to promote 'thinking, reasoning and working mathematically' for 'lifelong learning.' Interestingly in the last interview, she concluded that the key message of the syllabus was to promote life-long learning, as she explained:

> The key teaching and learning ideas from where I am now, I mean in regard to the syllabus, is lifelong learning and applying that to their mathematics by having them thinking that they are mathematicians and actually having functional mathematics. Within the classroom, that is what I see my role is to implement that, so they can actually see the linkage between what they are doing in the classroom to real life. So if you look at all the key areas, all of the strands, you can really help children link that to lifelong learning and the mathematical abilities that they will need throughout their life, and also, to help them to feel competent with their maths. (IV5, p. 1)

Her thoughts reflected the new curriculum incentives to ensure that students would leave school as successful learners, equipped with knowledge and understandings and ways of working to enable effective application of mathematical ideas across contexts (DETA, 2008; MCEETYA, 2008; NCB, 2008; QSA, 2004). What had become noticeable between Reagan's before and after thoughts, was a more explicit focus on 'doing' and 'applying,' as opposed to simply 'knowing.'

In summary, Reagan's thoughts about the syllabus intentions had altered. She started out believing the syllabus document encouraged teachers to plan learning that was relevant to students. By the final interview, she still interpreted her role as making explicit links between mathematics and daily living, although there was a more explicit focus on students' 'doing', which implies the process of working mathematically. She clarified her point when she explained that her role was to ensure that the students had developed 'functional' mathematical understandings to sustain lifelong learning. Her reference to the term 'functional' suggested that she believed her role now was to help the students develop ways of working with mathematical knowledge and understandings. In other words, she appeared to be more focussed on ensuring that the students recognised that they had the knowhow to work like little mathematicians both in and beyond school. Her intention aligned with Boaler's (1999; 2002) view that knowledge and practice are intricately intertwined. In short, Reagan perceived that her responsibility was to ensure that her students could work competently with mathematical ideas in any given situation. Her aspiration to develop mathematical competency was a derivative of her schooling experience. Reagan was not satisfied with the way she was taught mathematics, and therefore, she was adamant that her students would learn mathematics with understanding, as described below.

5.5 Encouraging Learning with Understanding

The *Years 1-10 Mathematics Syllabus* (QSA, 2004) emphasised that quality teaching and learning involves a move from transmission modes of teaching towards constructivist modes. This shift is achieved when a teacher encourages the learning processes of thinking, reasoning and working mathematically to enable students to construct mathematical understandings in ways that make sense to themselves (DETA, 2008; QSA, 2004). Reagan was drawn to the constructivist epistemologies; she explained that at the beginning of the year she welcomed her students and their parents to the classroom, wearing a construction hard hat. The message she hoped to portray was that the students were going to be actively constructing their own knowledge. She had placed a sign on the door that read, "The 6R Construction Crew." Her intention was to reflect the nature of the classroom community that she wanted to establish. Inside the classroom, she had painted a banner that was placed across the top of the whiteboard, it stated, "Learning Community under Construction." Hence, her metaphor symbolised that she believed knowledge was constructed within a community of learners similar to buildings being constructed by a team on a worksite.

5.5.1 Reagan's Orientation Towards Constructivism

Reagan had developed an orientation towards some constructivist principles of learning. She was particularly drawn to hands-on learning. Her motivation was a consequence of her own learning experiences. She believed that the way she was taught caused her to develop mathematical knowledge through memorisation of many mathematical ideas she did not understand (IV1). In other words, she believed that she did not understand at a conceptual level, how or why some mathematical ideas worked. She explained that she has had to relearn many mathematical concepts and she believed that by representing mathematical ideas with concrete materials her understanding had evolved (IV1). It seemed that her intention was to encourage manipulation of concrete objects to assist the students to construct abstract images in their minds which they could connect to in later learning, thereby reflecting Battista's (1999) and Skemp's (1986) views of learning. Thus, hands-on learning was an important aspect of constructivist learning that had captured Reagan's attention, as this section reveals.

5.5.1.1 Using Hands-on Learning to Construct Knowledge

Often Reagan spoke about the students needing 'visual, hands-on learning experiences' to construct understanding (IV1; IV3; IV5). For example, in the first interview she described an experience that had occurred prior to this research, where the students were exploring the concept of area. She recounted a discussion that had occurred during the lesson:

> 'But why are they doing that?' One of the students asked. I said, 'Well, see, if you cut that bit off here and put it over here, it is actually a rectangle.' The student said, 'Oh right.' Then we did a rhombus and tried to work it out. I just let them guess how they could work out that, and then they came up with that,

> 'if you folded it in half it is actually two triangles,' and even the kids who struggled with some of the mathematical concepts were able to see that the shapes were just made up of other shapes. (IV1, p. 1)

In this lesson, Reagan started the learning with a physical action in the hope that the manipulation of shapes would assist cognitive development. For instance, her intention appeared to be assisting the students to visualise the relationship between the shapes.

Reagan reflected on the lesson, and explained why she believed this lesson was important to help students learn with understanding.

> In the lesson today, because it was the manipulation of the paper to create the polygons, it made it [the learning] so much more real than if I was just standing there doing, 'this is area here, this is a triangle, and this is the area of it.' And then we linked it with what they were doing in their books. So it's a visual hands-on experience, as well as the actual textual writing. (IV1, p. 3)

Her intention was to clarify understanding for the students as they worked through a page of their *Maths at School* journal. She was focussed on helping the students to understand the diagrams in their journal through manipulating concrete materials. The journal activities were structured like worksheets. The students were given a question and they had to apply their mathematics knowledge to solve it; thus, it was her intention to link the images in the journal to physical representations. Hence, when Reagan mentioned that 'it was a visual hands-on experience, as well as textual writing,' she alluded to Battista's (1999) cycle of action, reflection and abstraction.

Early in this research, Reagan discovered and was alarmed that many of her learners could not recall basic number facts, such as 3x4 (IV3). She had cause to be alarmed because, by the end of year six, at age 11, students should have mastered multiplication and division number facts (DETA, 2008; Van de Walle, 2007). Hence, she decided to revisit the multiplication concept. Reagan explained that it was not until she was at university, investigating mathematical ideas, when she realised that multiplication can be represented with "groups of objects" (IV3, p. 2). Therefore, she believed it was necessary to use counters as a way to help the students develop foundational knowledge about the grouping concept of multiplication (IV2). She reflected on the lesson and stated that it was a "disaster" (IV2, p. 1). When I asked her why, she responded:

> I just thought that I should have got out the resources, the counters, sooner. Some kids were struggling, and the counters may have helped. (IV2, p. 1)

However, when I later asked her whether she thought the counters had helped the students develop understanding, she was uncertain, as this statement reveals:

> Yes and no, I need to have some structure and rules about the counters. … [a facilitator at the *Maths at School* conference] said that when she uses counters she has a sheet of paper that is a grid, and the students all have their own canister of counters. They are not allowed to move their counters off the grid; if they do they will be removed for the lesson. (IV2, p. 1)

Reagan was concerned about the students who played with the resources rather than use them for learning. Her concern reflected Van de Walle and Lovin's (2006) warning that care must be taken to ensure that the hands-on materials support the learning of mathematical ideas rather than learning how to master the manipulatives. She explained to me during the lesson that this was the first time that the students had used manipulatives this year (T3WK2D2). Thus, the use of hands-on materials was a learning experience for her as well as for the students.

One week later, Reagan reflected on the multiplication learning experience. Despite believing there were problems with using resources, she believed the counters had helped the students to construct meaningful understandings about multiplication, as she stated:

> They used counters to represent the concept of grouping, what it looks like, 3 x 4, what actually 3 x 4 looks like. Transferring learning into more hands-on experiences helped them to actually grasp the concept of multiplication. So doing a bit more hands-on has really helped them. (IV3, p. 1).

She concluded that through active manipulation and grouping of the counters she believed the students were able to create visual representations in their minds about groups of items.

5.5.1.2 Using Investigative Thinking to Construct Knowledge

Reagan persisted with hands-on learning, and, towards the end of this research, it appeared that she was using the hands-on learning in more investigative ways. For instance, in one particular lesson, she gave the students a variety of objects that they were to use to construct a pattern. She had not given any instructions other than that they were to make a pattern, and record the pattern in their workbooks (T4WK1D1).

She explained to me that the most important part of this lesson was for the students to grasp the idea that patterns consisted of repeating parts. It was not until the next day that Reagan's intentions were revealed. The students were required to pair up, sit back to back, and give directions to each other about how to construct the pattern (T4WK1D2). After the practical part of the lesson, she facilitated a whole class discussion asking the students what they had discovered. Her goal was for the students to recognise that if their pattern did not consist of repeating parts, it was too difficult to describe.

Hence, Reagan's enactment of students' 'active construction' differed from the previous lessons. Even though she was still focussed on the students' own construction of knowledge, in the first lesson she showed them how to use the concrete materials to assist individual knowledge construction of multiplication. In the second lesson, she let the students investigate and communicate their thinking and reasoning to promote knowledge construction about repeating parts of patterns. The second lesson included a focus on the students' communication and reflective thinking. The students were required to justify their thoughts about why the patterns could, or could not, be reconstructed.

5.5.1.3 Using Communication to Construct Knowledge

Communicating one's thinking and reasoning processes is an integral part of knowledge construction (DETA, 2008). Reagan also came to believe that as the students communicated their reasoning processes, they clarified their own and others' understandings (IV2). She had come to value the process of communication to clarify understanding because of a personal experience she had midway through this research. As she communicated her thinking about place value, she clarified and enhanced her own understanding of the place value concept, as she explained:

> While I was teaching place value the other day, and because I am the type of person who thinks things out as I talk, I realised something. It wasn't until I was up at the board talking about naming the number 0.362, that I thought "You just use whatever the last digit is, that is how you say it. You just say 362 thousandths." I thought to myself, "If this has just dawned on me, what about the students." This is where I think talking about thinking is important. (IV3, p.6)

In summary, Reagan was drawn to hands-on learning as a means of constructing knowledge. She was particularly drawn to this method because she found that manipulation of materials had helped her to clarify her own conceptual understanding of multiplication. She also valued communication of one's thinking processes, again because as she spoke about her thinking, her existing knowledge and understandings of place value were clarified and extended. Her reflection about the importance of communicating one's thinking processes aligned with the recent emphasis on working mathematically. For instance, DETA (2008) and Van de Walle (2007) believe that as students communicate their reflective thinking, they draw upon existing ideas to help make sense of new ideas, and, in the process, clarify existing or develop new understandings. In the later part of this research, Reagan was more focussed on the students' communication of their thinking and reasoning processes to construct meaningful understandings than in the beginning. However, in the fourth week of this research, she taught a lesson which seemed to contradict her developing beliefs about how students learn, as discussed below.

5.5.2 Teaching Against Principles

Despite Reagan's orientation towards hands-on learning and communication of thinking processes, in the fourth week of this research she directed a step-by-step long division procedure. I was perplexed; I wondered what Reagan's motive was for teaching this lesson when she had often expressed a belief that students needed to learn mathematics with understanding. She explained her motive:

> I would probably say that in my own mind I thought it was just a natural progression towards an understanding of division and towards short division. I felt that repeated subtraction was one level of the foundation of the identification of division, which is to look for how many groups of something are in a particular number. And long division, because it has a formula that they are to follow, you know, divide, multiply, subtract, bring down, that long division actually allows the student to see that happening, before they actually have to do short division where a lot of the thinking is mental. They are still seeing the repeated subtraction; they are still seeing that division is a process and that will help them when they have to go to short division, which is just mental computation. They can visualise, 'Oh yes, that is the process that I followed'. (IV4, p. 1)

Reagan's explanation revealed that in her view developing conceptual understanding was not linked to any particular method of teaching. For example, when she taught repeated subtraction, her intention was to develop conceptual understanding about division that would assist procedural understanding of long division. In short, she believed that the repeated subtraction method would help students to visualise the division concept when they were required to work with numbers in more abstract and mental ways, such as with the long division procedure. She believed that if the students could visualise that division involved repeatedly subtracting groups, they would be better prepared to learn long division. Then she explained that if the students could visualise how the long division procedure worked they would be better prepared to learn about short division, where more 'mental' thinking was required. Hence, her reference to the term 'process,' in the statement, 'Oh yes, that is the process that I followed,' implied a 'thinking' process. To elaborate, her use of the term related to the cognitive schema she hoped the students would have developed about division. She envisioned that cognitive schema would become the existing ideas students could draw on to make sense of the long division procedure. Then she envisaged that as the students drew upon their understandings of both the repeated subtraction concept and the long division procedure, they would be ready to devise thinking strategies to solve short division calculations. In Reagan's view, short division involved the use of mental computation strategies that have no visible, external form of representation to assist understanding.

Reagan noticed when she was teaching the long division lesson that the students seemed to grasp the procedure quickly. In her opinion, they were able to understand the long division procedure because they recognised the connection with the concept of repeatedly subtracting groups, as she revealed:

> I was surprised how quickly the students actually grasped the concept. Particularly because they had found, some of them had found, repeated subtraction quite difficult. I didn't expect them to pick up the long division as quickly as they did because, with repeated subtraction, there was a process that they had to follow, and ask themselves questions. Long division again is a similar thing; the students have to ask themselves questions, but follow, step-by-step, the procedure. The formula gave them like a rule that they could follow. (IV4, p. 1)

Reagan’s description indicated that she thought the two methods of division were similar yet different. Both required mathematical questioning and thinking about repeatedly subtracting groups; however, long division required following a rule to think through the procedural steps, where repeated subtraction enabled more flexible thinking about numbers.

However, Reagan’s final statement about why she taught the long division procedure revealed another motive. She had felt pressured by the parents, and therefore, by including the long division, she would be addressing the parents’ concerns:

> Also, I’d had parents who were finding the concept of repeated subtraction difficult, saying, ‘They should just be learning long division,’ so I was allowing the parents to see that long division was going to be included, that their kids weren’t missing out. So it was part of parent satisfaction as well. (IV4, p. 1)

In fact, that particular week she had asked me not to visit the classroom because she was having a difficult situation with a parent. Thus Reagan’s rationale for teaching the long division procedure was twofold; she wanted to please the parents as well as to prepare the students to work in more abstract ways.

At this point, I was interested to find out how Reagan perceived the students’ willingness to participate in the more teacher-directed lesson. Prior to teaching this long division lesson, she had been encouraging a student-centered approach to the classroom learning environment. She believed that the students had responded well, and some were enthusiastic. She attributed this enthusiasm to the students’ sense that their learning and understanding had progressed. She described the students’ experience from her perspective, and then spoke about her reflective thinking:

> I thought that they were actually quite upbeat at the start of the lesson, and once again I think it was that we were moving onto the next part of it [division] and they were excited how quickly they were progressing. I do think overall they prefer hands-on learning, constructivist learning, but there are times when they want that teacher instruction. I thought to myself, ‘Oh, it feels as though I’m the one controlling the learning here’, but I really think they just needed to, we needed to, just keep going through that process with me saying, ‘This is how I’m thinking about it,’ or asking, ‘What do we do now, what is the next step?’ (IV4, p. 3)

There were many interesting points in this statement. First, her explanation indicated a consistency between her interpretation and application of the term ‘process’. Here she used the term ‘process’ to describe how she modeled her idiosyncratic self talk, or in Vygotsky’s (1896-1934) words, her inner speech. Second, her response indicated that she believed the students were motivated when they sensed their learning was progressing. Her view aligned with Kilpatrick and colleagues’ (2006) opinion that when students sense that they are in a process of learning, they are less likely to fear failure and more inclined to participate. Third, her thoughts also suggested that she had reservations about the teaching approach because she felt as though she was in control of the learning, as will be elaborated on below.

5.5.2.1 Who Controls the Learning?

Even though Reagan believed that the students made the switch easily because they were familiar with the teaching approach, she was concerned about the fact that she had resumed control of the learning process. For instance, she reflected, ‘it feels as though I’m the one controlling the learning here’ (IV4, p. 3). She was concerned about how the students had been positioned as receivers rather than creators of knowledge in the learning environment. In this lesson, the students had been expected to switch from a student-centred approach, where they were in control of the learning, to a teacher-directed approach, where they were passive recipients of information. Reagan reflected:

> I think they do [make the switch] because they are used to a lot more teacher-centred learning, and at times I think how I feel about hands-on learning; they probably also feel a bit like that. Even though they enjoy it, there are some who are really scared to make mistakes. I’ve got a few people in my class whose self confidence is down because they think that if they make a mistake they are not good at maths. So when they are doing it [taking charge of their learning] they don't know whether they are going in the right direction. They don't like that. I also notice that when it is heavily teacher centred, they just start to wane after a while. Like that lesson, I felt by the end of it, it was just like. ‘Oh they are losing their actual engagement here because they were over listening to my voice straining on.’ It was almost a bit rote. (IV4, p. 3)

Reagan linked the students’ confidence levels with their desire to be correct. Her comments allude to a fear she believed was experienced by the students and herself. Both she and the students were apprehensive about what direction was the ‘right’ direction when a student-centred approach had been implemented. She sensed that the students were also dependent on her confirmation that their ideas were correct.

The need to get things right also impacted on Reagan. When she taught in ways that encouraged thinking and reasoning, she feared losing control. For instance, she described a hands-on learning experience where her students were outside investigating and comparing units of measure. Reagan was concerned in this lesson that she could not monitor whether or not the students were developing accurate knowledge and understandings, as she explained:

> I do prefer hands-on learning, but there is also a sense of fear, a fear of not being in control. I am there to oversee that they are actually learning this [converting measurements], but I fear that they are going down a path where they have thought that this is the right way to go and they are actually teaching themselves incorrect procedures. So there is that fear factor, whereas with a more teacher-directed approach, such as, 'This is what we are doing, has everybody got that, swap your books, let's see if …' I have more control. I think to myself, 'Yes, now I know everybody, or eighty to ninety percent of the class knows how to do this,' but I think you need a mixture of both, and it is just finding the right combination. (IV4, p. 3)

Reagan's thoughts about adopting a balanced approached aligned with the NCTM (2007) and Askew and colleagues' (1997) connectionist teachers' viewpoints. They all valued a holistic approach towards teaching, where both teacher-directed and student-centered methods were included. However, at this midway point in the research, Reagan appeared to be apprehensive about adopting a student-centred approach as a way to encourage thinking and reasoning. The main reason was because she feared not being in control over the learning, and subsequently, she did not trust that the students were learning. Her apprehension aligned with Farmer, Gerretson and Lassak's (2003) concern that unless teachers feel a sense of control over the new curriculum ideals, they may be reluctant to change.

5.5.3 Conflicting Viewpoints

In the context of Reagan's working environment, she experienced conflicting viewpoints, both interpersonal and intrapersonal. For instance, she did not believe that her colleagues focussed on aligning the classroom curriculum with the QSA (2004) mathematics syllabus document. Nor did she believe that her colleagues focussed on planning to ensure that the mathematics learning was relevant to the students' lives. The teachers planned the term's schedule together. In the first interview, Reagan described her collaborative planning experience:

> Some of the teachers I work with hate learning outcomes, but that is what I love … the QSA allows that concept of outcomes, where you can do what is real and appropriate with your class. Where I am working, they sit down and they unit-plan; maths does not even get spoken about in the unit plan because it is just assumed that you will do the *Maths at School*, it comes nowhere into it. They never unpack an outcome, they just pull out their folders, this is the group of teachers I am working with, and say, "Well I did this two years ago," and so on. Then we try and find an outcome that fits with what we are doing. (IV1, p. 4)

She believed that her willingness to plan according to the syllabus outcomes and according to the students' lives differed from her colleagues' intentions.

However, once inside her classroom, Reagan experienced freedom. She made it clear in the first interview that even though she listened to her colleagues' planning suggestions, she implemented the plan in ways that she perceived would suit her students' learning needs. She reflected:

> This is where teaching is interesting too, you are quite isolated in your classroom; nobody knows what is going on in the classroom really, so I will sit there [when planning with colleagues], and take on board what is being said about what we are doing, but then I will implement it with how it suits my actual students. (IV1, p. 4)

Reagan's thoughts reflected Cuban's (1993) opinion. He believed that teachers are at liberty to do as they choose within the context of their own classroom environment, which is why he encouraged researchers to investigate teaching practices. At this point, one wonders if her description of being isolated involved more than physical isolation. There also appeared to be a professional isolation, even alienation, which did not support Reagan in her attempt to teach mathematics effectively.

5.5.3.1 Conflicting Values

At the beginning of this research, Reagan expressed the feeling that her ideas were not valued by others in the school. She stated that, "My ideas get squashed down by other people telling me to do things a certain way." (IV1, p. 2) She described a conflict between what she valued and what she perceived her teaching colleagues to value. She reflected on a staffroom discussion after she had described a learning experience about a repeated subtraction method of division:

> When I went into the staff room and talked about the repeated subtraction concept for division that I had just shown the students, some of the teachers told me that I was confusing the students. They believe that the students just need to learn one way, which is short division. (IV2, p.1)

Reagan was taken by surprise by the comments from her colleagues after she had expressed her excitement about one student's thinking strategy. She was describing an example where the student was repeatedly subtracting multiples of fifteen to solve a division problem:

> Actually today one student came up with a strategy that was beyond my thinking about maths. The repeated subtraction enabled her to use multiples of 15 to subtract from a problem I had set, she knew that 6 x 15 = 90, because 3 x 30 = 90, and 2 x 15 = 30. And her reasoning went on from there. It was really interesting to see how she had thought about the problem; she had drawn from what she already knew in creative ways that enabled her to solve a complex math problem. (IV2, p. 1)

Reagan was excited about the student's ability to break the number into more manageable bits to assist mental computation. However, in the classroom she also explained that the other teachers had told her she was confusing her students by allowing them to use a variety of mental strategies (T3WK1D1).

Initially, Reagan did not believe that her colleagues viewed her as a competent teacher. Her perceptions caused her to set aside her own beliefs about how to teach to follow the lead of her colleagues. In the last interview she lamented:

> When I left university I had pictured the way I was going to teach, and then I got sidetracked. Being a new teacher and having all these ideals of what I wanted to achieve in my first year, and then having to actually shelve them. So I just kind of followed the path of what the school was doing, or what I thought they were doing. (IV5, p. 2)

However, Reagan's comments also suggested that she believed that, because she was a beginning teacher, she had not trusted her ideas; thus, she willingly relinquished her ideas to follow what she perceived others to be 'doing'.

5.5.3.2 Conflicting Pressures

There was another pressure that Reagan experienced throughout this research. She felt compelled to ensure that the students completed the units of work prescribed in the *Maths at School* journal because of the parents' financial outlay. Her compulsion militated against her aspirations gleaned from university about planning learning activities that were informed by the syllabus and the students' interests, as she described in the first interview:

> When I came out of university, I thought, 'I'll have this syllabus to plan from, and kind of go with what the children are interested in, and then apply it in the classroom'. But I just find I'm locked into this book of having to get the 16 units done to be prepared for the mid semester test…I am already three weeks behind. (IV1, p. 3)

However, as time progressed, Reagan altered her perception about the implementation of the *Maths at School* program. In the first half of this research, she often questioned herself because of how she thought she was viewed by the parents and her colleagues. Yet, as time moved on, her confidence to follow her teaching ideas appeared to grow, as depicted in this statement:

> I feel a lot more positive about teaching, and I guess empowered. Today we did use the program; however, we have spent less time working with the journal. We are probably about three or four units behind, but I really couldn't care less at this point in time because they are actually learning a lot. I have got a lot of the students knowing how to do division, so part of me wants to reject the program. (IV3, p. 4)

She had decided that her teaching decisions no longer needed to be controlled by the program (IV3). At this point in the research, Reagan believed that in order to follow her beliefs about how students learned, she would actually need to reject the program.

In the final interview, Reagan completely changed her view about the intention and implementation of the *Maths at School* program. She now perceived the program as a teaching tool to reinforce learning rather than a prescriptive plan to develop understanding (IV5, p. 2). When asked why she had changed her view, she responded:

> Towards the end of the term, because we were behind in our *Maths at School* journal, we went through and just marked the pages, not every page, but at least

two or three pages per unit, that we needed to complete. Then I allowed the children to actually complete those pages at their pace, and then choose what they wanted to work on. I held little conferences with the kids who were misunderstanding the concepts. They did that really, really well. A lot of them felt confident enough to talk through the mathematical ideas themselves. Some of them, who I thought would be with me all the time, I noticed, were actually grasping what was required. (IV5, p. 6)

In the classroom Reagan had explained to me that she felt pressured to ensure that the students had completed their journal activities. Thus, she sat down with them during a lesson (I was not present), and together they marked pages to complete. The students then worked at their own pace through the activities. She explained to me in the classroom that the students discussed ideas confidently. In her view they had developed conceptual understanding, and therefore, they could apply their knowledge to the units in ways that enabled them to successfully complete the tasks (T4WK3D1).

In summary, Reagan's initial thoughts appeared to be in a state of flux. There were many conflicting ideas about how to teach mathematics. In particular, she felt a conflict between her willingness to implement the syllabus and the perceived pressure to implement the *Maths at School* program from the school, her colleagues and the parents. As time progressed though, Reagan believed that her students were learning and understanding more. Consequently, she was feeling more confident to initiate her own teaching ideas, even though at times she was in a quandary about what to do with the *Maths at School* program. The next section describes the types of understandings that she believed were important.

5.6 Developing Conceptual Understanding

Skemp (1986) asserted that students need to develop cognitive schemas necessary to understand more complex tasks. Reagan also believed that her students needed to develop understandings in ways that would assist future learning. However, when I asked her in the first interview how much time she allocated towards developing conceptual understanding she responded:

> This is one of the issues I have with the *Maths at School*, the book that we are doing. They will have a concept to teach over five days, but sometimes those concepts really jump, they jump all over the place, but also within the concept,

> the actual structure of it, they expect in a day that the students will get it. I am starting to become brave enough to go, ‘Well, we might learn that another way,’ or we might do a couple of pages and then spend a couple of days actually building on that, and then go back to the book when they have a lot more knowledge. They then often can get through it really quickly. (IV1, p. 6)

Reagan believed that the journal activities did not help the students to secure conceptual understandings of mathematical ideas. It appeared that she felt the students were not being given sufficient time to develop robust foundational knowledge. Rather the learning, from what she described, appeared to be shallow. Shallow learning, or surface learning, is common in Queensland mathematics classrooms, according to a report by Hollingsworth, Lokan and McCrae (2003).

Reagan believed that for understanding to be developed, the students required more than the journal activities prescribed. For example, she was alarmed that many of her students lacked conceptual understanding of multiplication and place value. Prior to the research, the students had completed a half year test, and Reagan was concerned about the outcome, as she explained:

> When I went over the half year tests, I realised that some students don’t know basic mathematical concepts, like place value, times tables and decimal placement. I would like to spend some time going over these basic concepts, even if it means we have to hold up the learning for a while. (IV2, p. 1)

The expression about ‘holding up the learning’ meant Reagan was prepared to disregard the *Maths at School* program for a period of time. She believed that it was important for the students to relearn both multiplication and place value concepts before they moved on to learning more complex ideas. Her rationale for revisiting these concepts was captured in this statement:

> Because the students won’t have to keep going back to first base each time another concept is introduced. (IV2, p.1)

Reagan’s statement about ensuring the students’ understandings were sufficient to avoid having to revisit foundation ideas each time a new idea was introduced, reflected the views of many researchers and mathematicians (Anthony & Knight, 1999; Battista, 1999; Booker et al., 1998; Madell, 1985; Sun & Zhang, 2001; Van de Walle, 2004; Van de Walle & Lovin 2006; Westwood, 2003; Willingham, 2003; Mathematicians: Howe, 1997; Klein et al., 2005; MSSG, 2004; Quirk, 2005; Wu, 1999). These mathematicians and mathematics education researchers believe that for students to move on to solving more complex mathematics, they must have developed sound knowledge of basic number facts and operations. Reagan's thoughts revealed that she also believed that if the students did not understand how and why concepts worked, such as multiplication and place value, then they would struggle with further mathematics learning. The next sections describe how Reagan addressed the students' lack of conceptual understanding over the first three weeks of this research.

5.6.1 Developing Conceptual Understanding of Multiplication

Reagan's decision to teach division as repeated subtraction was an attempt to make explicit the relationship between division and multiplication that underpinned it. She envisaged that by teaching the students repeated subtraction she would be assisting the development of multiplicative thinking. Hence, the learning experience was intended to assist those who were not confident or competent with the concept of multiplication, as she pointed out:

> Repeated subtraction, especially for the kids who struggle with multiplication, is really beneficial. This is because they use multiplication and think, 'If I know that 3 x 6 is 18, well, then I know that 3 x 60 is 180, and then I know that 3 x 600 is 1800'. (IV3, p. 3)

However, Reagan asserted that when she was teaching the students division, using the repeated subtraction method, she realised that the lack of both conceptual and procedural understanding about multiplication for her students was worse than she had initially thought (IV3). She said:

> Today they were doing four times tables, the fours and threes, and they weren't automatically responding to even 3 x 4, which just frightens me at year six level. (IV3, p.1)

Reagan decided that the students needed to revisit the multiplication tables to complement their number fact recall strategies. Her decision was because she wanted to free up the students' mind so that they could concentrate on the division concept, rather than solving the number fact, as she described:

> I do think they have to have that recall, so they can work quickly, so they can think, 'Alright, well 4 x 3, that is 12, I know it.' It frees up their minds so that they are not struggling with that [recalling a number fact] because when they

> struggled with that, with division, they just almost froze, and were not able to work out the patterns in their division. (IV3, p. 3)

She believed that because some of the students struggled to recall basic number facts, they found it difficult to think about number patterns and relationships, and thus they feared learning division. At this point, Reagan used counters to help the students create visual representations in their minds, as discussed previously. Developing conceptual understanding of multiplication was so important to Reagan that she offered her time to the students over lunch periods to provide extra support (T3WK2D1). Many students had taken advantage of this opportunity, and in this time they used counters to help make sense of multiplication as repeated addition (T3WK2D1).

Reagan's urgency to re-teach the concept of multiplication reflected Booker and colleagues' (2004) and Van de Walle and Lovin's (2006) views that students must develop conceptual understanding of number facts and operations before they can start to develop recall thinking strategies. Once conceptual understandings about number and operations are achieved, students recognise relationships between numbers and operations and thereby develop thinking strategies to assist number fact recall. Consequently, because many of her students could not recognise number patterns, Reagan had decided to use the hundreds chart to talk about the patterns with numbers (IV2). She also focussed on pointing out number relationships, and used rote chanting of the tables, as she described:

> I was really surprised in year six just how many of them did not know their times tables, and couldn't see patterns in maths. It is important for students to be able to visually see patterns in the 100s charts and times tables because that is the foundation of so much of the mathematics we are doing now. I've gone back to even chanting the tables. I heard the year threes the other day chanting their five times tables and then the teacher asked randomly what they were and the kids were just picking them up. But even those little patterns within the nine times tables, that if you add the last digits together they have to add up to nine, just finding all of those things makes it so much easier when you start doing your division. Even with fractions, knowing what a half is. (IV1, p. 6)

Clearly, Reagan believed that it was important for her students to be able to automatically recall number facts. Thus, she valued both the development of conceptual understanding by helping the students recognise number patterns and relationships and with memorisation to enhance fluent recall of number facts. Her explanation about the multiples of nine indicated a desire to help the students develop thinking strategies, as was her link to division and halving. Reagan's view aligned with Skemp's (1986) view that teachers must make explicit the connections between mathematical ideas for relational understanding to be developed. She believed that the students had not previously been guided towards developing thinking strategies through recognising number relationships. She attributed this to the way the *Maths at School* program was implemented in the mathematics classrooms at her school (IV1).

Towards the end of these three weeks, Reagan focussed on developing the students' multiplicative thinking. Her intention was to help the students recognise that numbers could be broken down into more manageable bits, as reflected in the following statement:

> Today we looked at dividing three digits with a single digit number and breaking down those three digits into workable things. One of the examples was 372 divided by 6. I asked, 'What are the two parts of 372 that you could break down to make it easier for you?' So like 360 and then 12. They struggled with that and I was taking them back to their multiplication table and said, 'Well, you know it is being divided by 6, so have a look at your 6 times table, go down'. I said, 'If you know this, you will start recognising, 'Okay, well, 36, if I add a zero on, that is 360, and I can take that away'. (IV2, p. 2)

Reagan's intention aligned with Anghileri's (2000) view. Anghileri believes that breaking numbers down into more manageable bits helps students recognise number relationships, and thereby develop understanding about the logic and structure of mathematics.

Reagan was beginning to value the students' idiosyncratic thinking strategies. For instance, she stated, "I think they need to know that there is more than one way to figure something out" (IV2, p. 1). She reflected on an article she had read while at university:

> I read an article which suggested that a strategies board is a good idea. You can record the various strategies that students come up with on a *class strategies board*. Many children have different strategies and ways to think about the maths. (IV2, p. 1)

The classroom 'strategies board,' was an idea that she had gleaned from Van de Walle (2007). She believed that the students needed to recognise that she did not always have the 'best method' to solve a problem, as she indicated:

> I think they need to know that there is more than one way to figure something out. I may not always have the best method, and another student may clear something up for others in the classroom. I was taught the traditional way, and now I find it interesting to see the strategies that students are capable of coming up with. (IV2, p. 1)

Her intention was to use the strategies board to represent the students' ideas and to remind the students of the many ways that they had devised to approach tasks. However, even though she thought the strategies board was a 'good idea,' this idea had not been implemented whilst this research was being conducted.

5.6.2 Developing Conceptual Understanding of Place Value

The other conceptual area Reagan wanted to revisit was place value. This was part of her motivation for teaching the repeated subtraction method of division, as was discovered in an interview midway through this research. She was responding to a question I had asked about why she decided to include the game, 'I am the Greatest,' after the long division lesson:

> 'I am the Greatest'. It is a game that I played with some of the students last year, in year five, and they really liked that game. In one of our previous lessons we looked at place value. So I was acknowledging their previous learning. I was also acknowledging the reason why I wanted to do repeated subtraction first, which was so that they [could] grasp the idea that division was still using the correct place values; whereas, long division throws place value out the window because they start talking about, 'Does six go into one?' and the one is really 100, and you ask, 'Oh, well, does six go into 17?' and really, that is 170; it is not 17. So 'I am the Greatest' was a way to get back into that thinking of place value again, and also they really like the game, and after such a dry session of maths, it was good for them to do something that was still maths, but enjoyable and fun. (IV4, p. 4)

The game involved rolling three dice and the students were to arrange the three digits to make the 'greatest' number, in other words, the largest number. Hence, Reagan wanted to ensure that the students' prior learning about place value was not undone by the long division lesson. Her concern was related to the classroom talk about numbers, such as dividing 17 by six, when the digit ‘one’ actually represented one hundred or ten tens, and the digit ‘seven’ actually represented seven tens or seventy ones. She wanted to maintain the place value relationship of numbers as opposed to seeing numbers as a series of digits. Interestingly, also, she wanted to restore the students’ beliefs that mathematics can be fun, after she had taught what she described as a ‘dry session’ of mathematics.

Reagan’s concern about the demise of place value knowledge was also her motivation for an outdoors measurement activity. In this activity, the students worked at two different stations; one involved marking lines then measuring these with tapes or trundle wheels, and the second involved weighing water and sand. The students were required to convert their measurements from metres to centimetres and millimetres, and kilograms to grams, as a way to investigate place value. Her intention, as she explained to me through the lesson (T3WK2D2), was to follow up the outdoor activity with a whole class discussion, to link the activity to place value ideas and language. Reagan reflected on the week’s learning:

> I really feel like with place value, what we have done this week, a lot more hands-on activities, while for some it may be a bit boring because they have already grasped that understanding, but many found it really helpful. The actual language of place value as well. Yesterday they really enjoyed what they were doing with measurement and thinking of place value, especially when they were converting the metres to centimeters and millimeters. They were excited and were saying, ‘Oh we are going to get a go at that one, are we going to come back and do it again?’ So they really enjoyed that yesterday. (IV3, p. 1)

Reagan believed that the lesson was successful and that the students enjoyed it. However, I wondered whether she related the students’ excitement specifically to the outdoors investigative nature of the activity or to their beliefs about their learning of mathematics.

In the last interview, when Reagan reflected on the measurement activity, she summed up the learning as being, “a lot freer learning” because the students were outdoors and, “They could actually make a mess.” (IV5, p. 7) She believed that the investigative nature of the activity, where the students were asking themselves questions about the relationships between units of measure, facilitated intellectual development, as she revealed:

> The fact that they were there, reading it this way or measuring it and seeing, ‘Okay, well it is that many, and how much is it in kilos, and how much is it in…,’ actually helped their mental development. (IV5, p. 8)

She continued:

> It was interesting that some students who were, are, still having difficulty with conversion found it easier when we went over it again later, and we put it in the context of not just length, but volume and mass… By going outdoors and measuring and realising that a gram is smaller than a kilogram, and so on helped. They could actually explain the difference between a gram and a kilogram. (IV5, p. 10)

Hence, in Reagan’s view, the students constructed conceptual understanding about units of measure by representing and then communicating their ideas. The students had constructed mental representations to draw on and assist sense making in later learning, as Skemp (1986) frequently emphasised in his writing about how conceptual understanding is developed. However, in this later reflection, Reagan focussed on the relationship between units of measure and did not mention the relationship between this activity and the development of place value knowledge.

In summary, developing conceptual understanding of basic number facts and place value was of utmost importance to Reagan. She was concerned that without this foundational knowledge the students would find it difficult when they were being encouraged to think and reason about more complex mathematical ideas. Hence, the first three weeks of this research were devoted to remediation of the students’ understandings. Her goal was to equip the students with the capabilities that would enable and advance learning of more complex ideas. Reagan’s view aligned with Booker and colleagues’ (2004) opinion that the ability to reason about and with basic number facts is pivotal towards developing the confidence and ability to think and reason mathematically. However, during these three weeks, Reagan realised that the students were not familiar with the process of mathematical thinking, as will be discussed in the next section.

5.7 Scaffolding Investigative Thinking Processes

After the three week period where Reagan focussed on developing conceptual understanding of place value and multiplication, she believed that they were all faced with a challenge. She recognised that the students were unfamiliar with mathematical thinking and that she needed to teach the students how to think, as she revealed:

> I think what they are realising is that they do have to think, that that is expected of them. Today I had three boys finish early and I got them to go around and help others. I listened to them, they were peer tutoring, but were telling them [the other students] what it was rather than getting them to think about it. So even with them, it is a process of, 'Yes you know how to do it, but what I'm asking you to do is help other people to actually think how to do it, to inquire into it.' So I think in a lot of ways, they need to learn to think. (IV3, p.3)

Upon reflection, Reagan inferred that her students, 'Needed to learn to think'. Hence, there was a correlation between her thoughts and Kilpatrick and colleagues' (2006) notion that students need to learn how to think mathematically, as well as think mathematically to learn.

Reagan recognised that a key enabling characteristic to advance learning involved mathematical thinking. However, she doubted some of the students' ability to think, because they were not accustomed to thinking for themselves, as she described:

> This is probably an area that I feel generally, as a class, some of them don't have that thinking capacity, even outside of maths. They will come and ask me things, that if they had taken a moment to actually sit down and try to work it out, they could have done it themselves. But it is almost a learned helplessness, 'I will just ask,' or 'someone else will do it for me or tell me.' I mean, thinking is just so holistic; it has got to encompass everything. I do feel, though, that they are saying to themselves, 'Okay, well, yeah, and one day I'm going to know, I'm starting to get those skills to be able to do this myself, to think about it and work out how I'm going to find out the solution and what process to use.' (IV3, p. 3)

It was now midway in the research and Reagan sensed that the students were beginning to realise their mode of learning had changed. They were being encouraged to think for themselves. It was at this point that Reagan and I collaboratively planned a sequence of activities based on Van de Walle and Lovin's (2006) advice about developing algebraic reasoning. Together we indicated opportunities where the students could represent, justify or make generalisations in each activity. Regan took the sequence home and then utilised the activities in the latter part of this research. The following section describes Reagan's thoughts about scaffolding thinking and reasoning processes through representing and justifying mathematical ideas.

5.7.1 Reagan's New Impression of the Process Approach

As this research progressed, so did Reagan's impression of what encouraging a thinking and reasoning process approach involved, although for the entire period of this research, she encouraged the students to create their own representations. The students were never given a worksheet with a pre-drawn diagram. Even when she taught an introductory lesson about describing patterns symbolically, she drew a simple pattern on the board and the students had to replicate the pattern in their maths books (T4WK1D2). Reagan explained in the final interview that her rationale for doing this was twofold. First, she was concerned about the environment, and was conscious of unnecessary usage of paper. Second, and important to this research, she believed that the physical action of creating, drawing or writing representations helped the students to construct understanding:

> I just think that if I give them something with it on, they are not actually constructing it themselves. I do think there is something in that, of actually having to manipulate what they are putting into their books. So, recognising that, 'I'm drawing a square', 'This is what a square is.' If you give them a work sheet and there are things on it [problems to solve], they will just think, 'There is the blank space; I've got to go to there for the answer, and put the answer in.' They might not read all the information or the instructions, so sometimes it is better to verbalise it or to put it up on the board for visual impact. (IV5, p. 12)

Hence, Reagan's view related again to Battista's (1999) idea that learning involved a cycle of action, reflection and abstraction, and Vygotsky's (1896-1934) notion that inner speech helps students clarify their thinking. For instance, as she explained, when the student draws a square he or she may also be describing to themselves the properties of the square.

In the final interview, Reagan's own understanding of the mathematics practices as a way to scaffold mathematical thinking had altered significantly. This change was obvious by the length of her reply and the enthusiasm expressed through the tone of her voice and the sparkle in her eyes. When I asked her what she understood about these practices now that we had finished the practical part of the research, she responded:

> A lot more than when I first started. After the unit of work we did, the patterning and algebra, I am now actually thinking about that as a process for teaching maths. So when they were doing their representation, which, for example, was with the worm, where they were actually constructing the pattern, we had them out in the undercover area with chalk. They were creating their patterns on the concrete using chalk; there was all different coloured chalk too, so that was really good. Some of them had a square and two triangles, and then they recorded it in their books. But for them not just to do that, but to actually justify what was happening in that process, recording their data and then looking at their data and actually having to think about what connections were between those bits of data, to make their general rules. Now looking back on their work, what was interesting was while there were rules I could easily apply in an abstract context, for instance, if I wanted to know how big the worm was on day 57, I could just apply their rule and be able to find out how long the worm was. There was a group of children, a significant group of children, who could do that. But then there was the group of kids who still represented their pattern, recorded their data, so justifying it. Then they did generalise, and created a rule that fitted their data, but not necessarily a rule that you could apply. You couldn't use prediction. But for me, that was still really good because this is the first time they have been doing that, so they actually understand those three processes and what is necessary, and they are on the pathway to understanding. (IV5, p. 9)

The lesson Reagan referred to was a culminating activity after a series of lessons on patterning. The students were required to create a pattern on the concrete with chalk, number the repeating part of the pattern, each element, and record the steps of their pattern in their books, using pictorial and symbolic representations. They were encouraged to use a table to record the pattern. Then, as a group, they had to create a generalisation that would enable them to determine the twentieth element. She explained that what she was looking for in this lesson was the students' ability to recognise the 'repeating part' of the pattern and their ability to competently justify the relationship between the concrete, pictorial and symbolic representations (T4WK2D1). She was enthusiastic as she recounted this lesson. She felt that the lesson had helped the students to recognise the relationship between the pattern they had drawn and the data table they had recorded in their books. Even though she believed that some students were not able to make a general rule for their pattern, she was delighted that many had reached a level of abstract thinking and understanding. The emphasis in this lesson was placed on the students' construction of understanding through communicating about and reflecting on the relationship between the concrete, pictorial and symbolic representations. Reagan had encouraged reflective thinking and communication with mathematical ideas, as recommended by DETA (2008) and Van de Walle (2007).

However, Reagan pointed out that it was difficult to provide evidence of the thinking and reasoning dexterity inherent in the students' learning experiences:

> I was looking at their maths books at the end of term and I thought some parents might look at these maths books and think, 'Gee they haven't done much this term.' Which some of them will do, and will be quite cranky about it. But when they [the parents] look at a page of their [the students] maths book, they won't have the ability to think about what thinking and reasoning went into getting there, that is, what is recorded on the paper, because we had lots of discussion time and some of our lessons were two hours long. (IV5, p. 5)

She was concerned that what had been recorded in the students' workbooks could not adequately reflect the sum or significance of the thinking and reasoning processes that occurred throughout the lesson. Reagan was referring to the students' level of engagement with learning processes of reflective thinking and discussion. Reflective thinking and communication are important aspects of the learning process (DETA, 2008; Hiebert et al., 2000; Kilpatrick et al., 2006; QSA, 2004; Van de Walle & Lovin, 2006; Van de Walle, 2007). However, as she explained, these processes are difficult to record in ways that accurately represent the comprehensive learning or thinking processes involved.

Reagan believed that the mathematics practices helped her scaffold mathematical thinking and reasoning learning processes. For instance, when I asked her about what effect she believed the practices had on the students' learning for understanding or their developing dispositions, she responded by comparing the students' experience with her own:

> I think definitely it does, because what it does for me, as a teacher, is gives a framework to the actual learning, I guess a pathway to learn a concept. Also, with these children, it would be nice to take them for another year in maths, but next year I'll have a different year six, where I'll start to apply that [learning process] straight away. So it teaches them an actual process for understanding. I think they were getting that, slowly for some of them, but they were getting it; that there is an actual way to grasp understanding. There is a way of thinking; there are actual steps towards their thinking. (IV5, p. 10)

She appeared to find the practices helpful to scaffold her own thinking processes about teaching mathematics. Yet, this statement also revealed that she had noticed that the students were learning how to think. Hence, there was a change in Reagan's beliefs about the students' thinking capacity from her doubts expressed in the first interview.

Reagan believed that the three mathematical practices framework helped her to become more conscious of the students' mathematical thinking. She believed that this framework structured her teaching and structured the students' reasoning process, as she explained:

> It [the framework] gives me structure for my lessons. It has actually made me stop and think, where prior to doing this with you, I was teaching and not sure whether it was beneficial. Okay, looking at it this way, I think, yes, I actually like the framework better, that we have been doing. And it helps with assessing where a child is up to or where they might be becoming a little bit stuck. They can do that representation and actually record it and justify why they are doing it, or maybe they have got the first two steps, but not the next, so in that way it is really good. (IV5, p. 11)

She had acknowledged the collaborative nature of this research, and appeared to perceive that the time taken to co-reflect had helped her to adapt her teaching practice. She was beginning to feel more confident in being flexible in her approach to teaching:

> A lot of it is trial and error. A lot of it is thinking, 'Well I'll have a go at this,' and sometimes being flexible to see where the interest is going, and being comfortable enough to think, 'Well, no, we don't have to stick to my path here.' I am now allowing myself to deviate and be aware that, 'This actually isn't working, so let's try another way.' (IV5, p. 6)

This statement revealed that Reagan had started to think and plan according to the situation at hand.

When asked about the possibility of continuing her approach to mathematics teaching and learning in this way, her response was emotional. Reagan's eyes misted up as she responded:

> Definitely, it gives me structure for my lessons. It has actually made me stop and think about how I was teaching prior to doing this with you, and whether it was beneficial. Okay, looking at it this way, I think, yes, I actually like the framework better. It allows me to assess where a child is up to or where they might be a little bit stuck. They can do that representation, and then they can take that and actually record it, and justify why they were doing it, or maybe they have got the first two steps but not the next, so in that way it is really good. (IV5, p. 11)

The fact that Reagan felt emotional implied that her confidence about her ability to teach effectively had changed, at a heartfelt level.

5.8 Establishing a Positive Mathematical Community and Identity

Prior to this research, Reagan believed that for students to be successful in life, they need to become successful users of mathematics. She made it clear from the beginning that her goal was to ensure that the students left her classroom feeling competent and confident in their mathematical abilities:

> Because it [mathematics] is such an essential part of learning and of life, I want these kids to feel confident and competent when they are out there in the real world, in their everyday life, and that they will be able to… work out their change and do it really quickly, so just getting them to think that they are good at maths. (IV1, p. 6)

She frequently mentioned that she wanted the students to sense that they were 'good' at mathematics. Her aspirations for her students were threefold. The first reason was because, as she explained, the discipline of mathematics was an essential life skill. However, the second reason, not mentioned here, but mentioned several times over the period of this research, was that she wanted her students to feel more positive about their mathematical ability than her schooling had purveyed. The third reason was because she was particularly concerned about the students approaching their final year in primary school.

5.8.1 Becoming Mathematically Smart

The students were in year six and were moving on to year seven, the last year before high school. The end of the year was drawing closer as was the end of the research. At this point Reagan had become more specific about her ambitions for her students. She was still focussed on the students developing positive beliefs about themselves as mathematics users, yet she extended her focus. Now she wanted her students to leave her classroom also believing that they had a mathematical 'thinking process' to follow and apply, as she explained:

> The students for me, the goal is by the end of term four, that they feel competent about going into year seven because it is a really big step for them, that they are going to be school leaders. It is their final year before they go to high school, so that they feel that they are maths smart, that they are mathematicians and so that is what I want for them. And also that they are not going to get everything right all the time, but they actually have the confidence to go, 'Well, I got that wrong, but this is where I made an error, and I just have to do this bit or know my multiplication tables better.' Like that was in the division test, a couple of kids made just that error, on their multiplication, but knew the division process, but got the answer wrong, so, yeah, just being able to have that thinking process, but believe that they can do it. (IV4, p. 5)

Reagan's aim was to develop in the students a sense that they were in a process of learning, and that with perseverance they had the ability and knowhow to figure things out. Her view aligned with Kilpatrick and colleagues' (2006) advice that students need to perceive achievement as a product of effort rather than ability. Hence, students need to sense that they are in a process of learning; in other words that their mathematical ability is developing, and not fixed.

When I asked Reagan what she considered to be the most important attribute she hoped her students would take from her mathematics lessons, she concluded:

> That they have the basic skills there, they do have those skills to apply to any maths situation. They have now gone through year six, and they have now got just those simple basic number facts: their addition, subtraction, multiplication,

divisions, which are the foundation of a lot of maths. They have even started doing their algebra, but they think [to themselves], ‘I’ve got this toolbox of skills, I know that there will be times that I will get it wrong, but I’ll try something else to get it right, that I actually feel confident enough to have a go.’ (IV5, p. 8)

Hence, she wanted the students to believe that they could draw upon their mathematical and strategic knowhow to ‘have a go.’ Yet, she also mentioned that she wanted her students to understand how to meaningfully apply mathematical concepts to a variety of contexts. She believed that unless the students could apply previously learned knowledge to other situations, then understanding had not been developed. For instance, she explained:

> Real, true understanding is where students can go back to a concept, and they are able to re-engage with that concept, and still have those skills that they learnt earlier. So, even if it is like a month or so down the track, they then go back to measurement, and actually think, ‘Well I have this foundational knowledge, and now I’m going to apply that to my new situation.’ (IV5, p. 9)

Her view related to what it means to learn mathematics for numeracy. As Willis (1990) explained, a numerate person can transfer and apply mathematical understandings and strategies across contexts.

5.8.2 Building Mathematical Identity

On several occasions throughout this research, Reagan referred to her students as ‘little mathematicians.’ She spoke about the students in this way during the interviews and I observed her referring to them as ‘little mathematicians’ within the classroom learning environment. In the first interview, she explained that the one thing she wanted her students to take from their mathematics lessons was to identify themselves as ‘mathematical beings.’ She wanted her students to take away a personal belief that they had the capacity and capabilities to work mathematically. Reagan wanted her students to recognise that to ‘struggle,’ which sometimes involved making mistakes, was part of the learning process, as she described:

> I want them to believe they are mathematical beings, so even if they have struggled throughout the lesson, that they haven’t felt that they should just give up. I work a lot on that; it is a part of learning, and it is okay to make mistakes. I guess too, because I make mistakes on the board and they love that, the fact that

I do. Or I say to them, ‘Oh, I find this really difficult. Can somebody give me a way to remember this.’ (IV1, p. 6)

Reagan encouraged and valued the students’ thinking, and she let them know this by inviting their ideas to assist her thinking processes. Her intentions also related to Palmer’s (1999) opinion that students often fear making mistakes, and it was important for them to sense that mistakes were an integral part of the learning process. Her words also reflected Kilpatrick and colleagues’ (2006) perspective that a productive disposition involved a willingness to persist and persevere.

5.8.3 Building Mathematical Stamina

Reagan believed that the students’ willingness to engage in the learning had increased. She measured this increase in three distinct ways. First, she noticed the tone of the classroom changed when mathematics was mentioned. Second, Reagan noticed a more even dispersion of discussion across the classroom. Third, she noted the surprised comments that the students expressed when they realised that they had been working mathematically for over an hour:

> Sometimes it was just the initial response to ‘Get out your maths book’, and what the general feel was in the class when you say that, because there has been children that when you say that, or you can see them almost shut down, and some of those kids were more engaged. Also, the contribution to conversation in the class; sometimes some of those conversations were really quite evenly distributed across the class, rather than by the particular few who will always contribute. So those type of things. Yeah, and at times they were surprised and said, ‘I’ve been doing this for an hour and a half.’(IV5, p. 5)

Reagan explained that prior to this research her mathematics lessons rarely extended over forty minutes; hence, the surprise when the students suddenly realised the amount of time they had been working mathematically. However, it is important to note that Reagan had not intentionally extended the lesson times; this happened inadvertently.

Reagan was also well aware that not all students were engaged at all times. She was concerned about one student in particular who drew throughout most lessons. However, at times this student appeared to be paying more attention, as she described:

> Occasionally I will get one hundred percent engagement, but not very often.

> You will get your percentage, for whatever reason, that day will not engage. In particular, there is a child who really, I guess I would consider her probably ADHD, where she is difficult to engage in any subject. She just wants to be drawing, but I think she was a little bit more engaged when she had to actually be constructing a pattern, drawing it and colouring it in. (IV5, p. 5)

Hence, one wonders if this student was engaged mentally, even though she had not appeared to be actively engaged in the participatory aspects of the learning. This required further discussion with Reagan to determine her thoughts about what being engaged does involve.

5.8.4 Building a Mathematical Working Community

Reagan noticed that the classroom learning environment had altered during this research period. For instance, she described in the last interview how the students were becoming more confident to work together, and seek help from each other. In the statement below, she indicated that the students were taking a more active role in the learning, and were less dependent on teacher directives. In her opinion, the students appeared to be more confident. When I asked her why she believed the students' confidence had grown, she responded:

> I think it has risen because I believe they are feeling that instead of the teacher having to direct their learning, that they are actually competent mathematicians. We talked about that, saying that you are mathematicians, and they actually think, 'Oh I do have those skills, and if I don't, it is okay, I can source other knowledge,' and I also asked for volunteers, for children who wanted to share their knowledge, to put up their hands. I told everyone to have a look around to see others they can go to for help as well. (IV5, p. 6)

Reagan's intention to bring to the fore the supportive network available in the classroom was a characteristic reminiscent of Palmer's (1999) view of a hospitable environment.

Each lesson Reagan taught involved whole class discussion. However, she discovered that the students were starting to contribute more to discussion, and when I asked her why she thought this change had occurred, she responded:

> They are actually doing something, and then discussing it. They have got something to talk about because they have been actually manipulating materials, and actually constructing some foundational knowledge, but then being able to

> talk about it and explain their reasoning and justify and listen to others and agree or disagree and think, ‘Oh, I didn’t think of it that way.’(IV5, p. 13)

Reagan believed that when the students had actively engaged in hands-on learning experiences, they had something to discuss. Thus in her view, as the students reflected on the learning experience, they could explain and justify their mathematical thinking about how and why things worked. She also believed that sense was made about mathematical ideas, through listening to each other’s ideas, and in some cases, challenging others’ ideas.

This instructional strategy was also a strategy that the connectionist teachers in Askew and colleagues’ (1997) study employed. Whole class discussion was used by these teachers to demonstrate that they valued the students’ thinking, to help students clarify understanding and to make explicit links between mathematical ideas. Reagan also believed that whole class discussion gave the students an opportunity to show what they knew and to assist their peers, as she expressed:

> I might be trying to get a concept across, but I look at it from adult’s eyes, and then, when they hear their peers saying, ‘Well, this is the way I do it, or think,’ they say, ‘That makes more sense to me than what Miss had said.’ I really do believe that whole thing of actually talking about things to make the concepts fall into place in your head. (IV5, p. 13)

Active participation in discussion was also a way Reagan determined the students’ level of understanding (IV1; IV5).

However, Reagan also mentioned that teaching children how to discuss ideas was a whole new learning process in itself. She reflected that she needed to facilitate discussion ‘etiquette.’ She suggested that next year she might introduce strategies such as talking sticks to avoid the students talking over each other (IV5). At this point though, Reagan was satisfied that the students were beginning to discuss ideas, as she concluded:

> I mean, ideally what I would like in a classroom is where we have those social etiquette rules in place, that you don’t interrupt somebody, or you wait your turn to speak, but they are also 11-year-old children. That is a learning process.

However, the nature of discussion changed as the research progressed. From the beginning, Reagan believed it was important to develop a productive classroom atmosphere to promote discussion. For instance, in the first interview, she explained that if the students made a negative comment about themselves or others, even if they said, "I can't…," they were expected to diffuse this by following up with something positive (IV1, p. 6). In the final interview, she appeared to be focussed on how the students were being positioned within the learning environment. She was still aware of how the students spoke about their own mathematical abilities, although in an attempt to evoke discussion, she was also wary not to cause unnecessary angst for students:

> I used to; I would just call on people, whereas now I realise that some of the kids just freeze up. So now I don't ask unless I know that they have got the answers, unless I know that they have got that moment during a maths lesson to show others. I think that a lot of it is being exposed to ridicule or, 'Don't you know that;' but letting them get it right. (IV5, p. 13)

Palmer (1999) warned that when students fear ridicule they will not contribute, and it appeared that Reagan shared a similar viewpoint. Therefore, she felt it was important for the students to have an opportunity to show what they did know. She also believed that listening to others in discerning ways was something her students needed to learn, as she explained:

> What I have been trying with the students too, is to say, 'You need to be discerning; you need to actually be thinking when you are in discussion. If somebody says something, not just going, 'Oh yeah, that must be right.'' (IV5, p. 13)

Whole class discussion presented implications for Reagan and the students. Motivating students to participate in discussion was challenging because some of the students were uncomfortable expressing their ideas for various reasons, as she explained:

> There are some children, if they have their way, they would never contribute, in any class. Not whether it is maths, they would never contribute in class because they are not comfortable with it, but also they need to. Sometimes I will randomly select kids I know have actually switched off, and are thinking about what they are having for lunch, and that brings them back into the conversation. But there are still the ones who will sit there quietly, and not say anything. Sometimes there is a bit of laziness, or it is not cool to show that you actually know something. (IV5, p.14)

Hence, there were four factors that Reagan perceived impacted on the students' willingness to discuss ideas. The first and second factors she described were intertwined because the students were unaccustomed to discussion as a classroom learning strategy, which meant that some were reluctant to contribute. She related the third and fourth factors to students' personal dispositions, for instance, she believed that either they were lazy or they were image conscious. Despite the challenge of getting students to participate actively in the learning or discussion, Reagan was still inspired, as she explained below.

5.8.5 In Summary: Adopting a New Approach

The collaborative nature of reflection throughout the interviews, and Reagan's process of reflection, in and on action, has altered her thinking about mathematics teaching and learning. For instance, she now believes that learning mathematics involves more than acquiring mathematical knowhow; it also involves contextual and strategic knowhow, as suggested by Willis (1998). Mathematics learning, as Reagan explained below, involves knowing how to identify mathematical problems and the mathematical concepts needed to devise and solve those problems:

> There is a lot more literacy skills and learning to identify the problem, and I guess in real life, that is what you do, you problem solve; and finding the mathematical concepts that you need to solve that problem. (IV5, p. 1)

Reagan's first comment, that there is 'more literacy skills' involved now in mathematics learning because students need to be able to identify the problem, reflected the OECD (2006) definition of becoming mathematically literate. Her description related to what it means to learn mathematics for numeracy, as described in the literature review (e.g., Coben, 2000; Cockcroft, 1982; NCB, 2008; 2009; OECD, 2006; Thornton & Hogan, 2004; Willis, 1998). For instance, as the National Curriculum Board (2009) has recently reiterated, the quest to become numerate includes the capacity, confidence and disposition to apply mathematics, and recognise its utility for continued learning.

In this final interview, Reagan discussed what inspired her most about teaching, and what she perceived to be her past, present and future challenges. She described two key factors that kept her motivated. The first was to witness changes in the students, and the second was her own quest to become a more effective teacher:

> I would have to say the actual kids themselves, when you see someone, either the child who has struggled with the concept that she is now grasping and feeling good about it, or the child who maybe is very competent in maths, but sits back, but then actually getting involved and trying to push the thinking further. So I'd have to say the actual kids themselves, and just my own quest of trying to teach better. (IV5, p. 6)

Here Reagan revealed that she now felt more prepared to advance learning for both the competent and struggling learners, whereas at the start of this project she was concerned because the mathematically competent students were bored, while others revisited concepts. The varying levels of understanding were just one challenge of many for her throughout this research.

Reagan had pointed out various challenges over the practical part of this study. For example, she had experienced conflicting viewpoints where she did not perceive the school to be implementing the syllabus as it was supposed to be intended (IV1). She also felt that her planning decisions had been compromised because of perceived parent and peer pressure (IV1; IV3). In addition, she found teaching using hands-on materials a challenge at first (IV2), as was enticing the students to participate in whole class discussion (IV5). At about the midpoint in the research she also pointed out that the constant interruption that extracurricular activities imposed on the classroom learning was a challenge (IV4). Whilst I was in the classroom, I noticed students being called to music lessons and learning support. This frequent interruption was a challenge for Reagan because she believed that the timing often meant that the students missed important learning opportunities where mathematical ideas were discussed explicitly (IV4). Furthermore, she explained that having sufficient time for planning was a challenge when she had reports to complete, a school camp to plan and religious studies to complete to secure her position on the school staff (IV5).

However, Reagan's biggest challenge appeared to be futures oriented. Whilst she was concerned about being organised, she was more worried about sustaining her new teaching approach. She reflected on her experience over this research period:

> The biggest challenge for me, in all of my teaching, is my organisational skills, and not slipping into old habits, I guess. This, next year will mean thinking, 'Oh that is right, this is my framework that I'm working with now,' and, 'How am I

> going to put that in place?’ I have found it most rewarding to have the opportunity to actually have to think, and change the way I’m teaching, because of what you are doing. Because if that hadn’t come along, I might have just stayed in my little rut and that is one thing with teaching that I have found is that it is very, you are very isolated, very isolated as a new teacher, you don’t get that opportunity to go out and have a look at how other people are doing it, so you can just get into this rut until you are challenged to change. For example, on one thing we did early in the year … even what I consider my brightest children in maths got a low score. I had to think, ‘well, that is not them, that is you [referring to herself]; that is how you taught that.’ It’s just been nice to actually have to think about why I am teaching the way I do and what is best, what is working best for everybody else in the classroom as well, that’s the rewarding part. (IV5, p. 18)

In this statement, where Reagan stated ‘because of what you are doing’ she was referring to the research project. She believed that this project had enabled her to reflect on her teaching in ways that have challenged her beliefs and her taken for granted ways of doing things. For instance, the co-planning of the activity sequence was a contributory factor towards raising Reagan’s awareness of how to incorporate the mathematics practices to scaffold investigative thinking. Reagan’s reflection revealed that the opportunity to inquire into the specific things that help to maximise learning, had been rewarding. She acknowledged that the collaboration between her the teacher and me the researcher, throughout this study helped to focus her attention. As Cavanagh (2008, p. 123) also found, there was value in using the model of “researcher as sounding-board” to facilitate Reagan’s attempt to adopt new curriculum ideals and her subsequent pedagogical inquiry.

5.9 In Conclusion

Reagan was purposively selected to participate in this research because of her expressed desire to implement innovative teaching ideas. She was willing to collaboratively investigate the practical application of those ideas. As she described in the last interview, she is, and was, motivated by her “own quest of trying to teach better” (IV5, p. 6). Clearly, the data above has revealed that Reagan is an intelligent, conscientious and hard working teacher who is keen to implement only the best in teaching practices for her students. Her keenness to do the best for her students was initially attributed to her own schooling experience. She left school not trusting her own mathematical ability throughout her adult life, and was consequently determined to ensure that her students would leave her classroom feeling mathematically competent and confident. As the research progressed, it became apparent that when she witnessed the unfolding of her students' mathematical understandings, she was even more inspired to persevere with her pedagogical inquiry.

However, Reagan's initial interpretation of the curriculum was equivocal. The *Years 1-10 Mathematics Syllabus* (QSA, 2004) was designed to provide opportunities for students to develop attributes for lifelong learning through activities that promote thinking, reasoning and working mathematically. She had not mentioned either 'lifelong learning' or 'thinking, reasoning and working mathematically' in the first interview. Rather, she inferred that the main emphases of the syllabus were to make the learning relevant and to ensure that the students were active in the learning process. The usage of the terms 'relevance' and 'active' were not clearly defined by Reagan or the QSA (2004) syllabus document, as the next two paragraphs reveal.

The Queensland Studies Authority (2004, p. 3) described 'active investigators' as learners who value an inquiring habit of mind to "promote reasoning and thinking about possible ways of resolving mathematical problems." Even though the QSA does not delineate what an active investigation involves, their description implies adopting an inquisitive mind. Reagan's use of the term 'active' often related to physical action. For example, in the HPE lesson, the students were taking their pulse and then graphing the results, and then in the multiplication and patterns lessons, they were manipulating concrete objects. Reagan appeared to be focussed on the physical nature of the task as a way to stimulate active inquiry. The three lessons discussed here had the potential to promote active investigation, where students could inquire into the how and why of mathematical ideas to make sense of things for themselves. However, investigative thinking did not always occur. The students manipulated counters in the multiplication lesson, and recorded their own pulses to graph purposeful data, yet Reagan's description of the lessons indicated that she directed the inquiry and the students followed. By contrast, the patterns lesson promoted more opportunities for the students to ask how and why questions because the students were required to justify attributes that enabled pattern reconstruction. Thus the patterns lesson appeared to foster more investigative thinking opportunities than the other lessons. However, without knowing what happened in the students' minds, the amount of investigative thinking inherent in any activity is difficult to determine.

The second term in question was 'relevance'. Initially, Reagan believed that her role was to make the learning 'relevant' by connecting the learning situation to the students' lives (IV1). Her interpretation of the term seemed to relate to how the term was used in the syllabus. The document stated that: "Learning is enhanced when the context is relevant and motivates learners" (QSA, 2004, p. 9). To clarify, she believed that when students learn with understanding the mathematics itself becomes relevant and learning is enhanced. For instance, Reagan's learning was enhanced, and she appeared more motivated, when the concept of multiplication made sense to her for the first time, and when she discovered the relationship between naming numbers and place value. By the end of this research, she was intent on ensuring that the students were able to recognise that the discipline of mathematics was relevant; it existed all around them and could be used to make sense of the world in which they lived. Further investigation is required, however, to qualify Reagan's altered view and to clarify whether 'relevance' in mathematics learning necessitates learning with understanding.

Reagan's beliefs about the importance of learning with understanding reflected the QSA syllabus, eventually. I use the term 'eventually' because, at first, she had not emphasised the importance of developing students' abilities to meaningfully apply mathematical understandings. However, by the end of this research she believed that 'true understanding' was evident when students could use and apply mathematical concepts in various situations (IV5). Reagan's thoughts correlated with the QSA (2004, p. 3) statement that the "selection and effective application of mathematical knowledge, procedures and strategies is evidence of deep understanding." Her beliefs about understanding also related to the psychological view of learning whereby mathematical competence depends on the availability of mental structures that can be drawn upon, and utilised, to create further mental structures (Anderson, Reder & Simon, 1998; Battista, 1999; Hiebert & Carpenter, 1992; Skemp, 1986). In fact, Reagan refused to advance learning about division until she was sure the students understood multiplication (IV2). Her actions aligned with Skemp's (1986) advice that if students do not have the available cognitive schema to apply to new learning, they will need to go back to the beginning to understand the how and why of the concept. Reagan had some preconceived ideas about how understanding was developed, yet she was willing to investigate further.

Reagan's initial thoughts about how understanding was developed largely aligned with policy documents and research, yet not explicitly. For example, she believed that students needed to represent mathematical ideas, often through manipulating objects, and then discuss their thinking about these representations to create abstract ideas in their minds. Her thoughts reflected Battista's (1999) cycle of action, reflection and abstraction, and the emphasis she placed on communicating one's thinking related to policy (DETA, 2008; QSA, 2004) and research (Hiebert et al, 2000; Van de Walle, 2007, Van de Walle & Lovin, 2006; Skemp, 1986; Vygotsky, 1896-1934). Reagan was particularly drawn to this cycle because she had validated her own mathematical ideas through investigating multiplication with concrete objects and communicating her thinking about place value. However, Reagan's beliefs about developing understanding were not explicitly aligned with the literature in the sense that she did not clearly point out whether she intentionally highlighted the connections between mathematical ideas, as stressed by the QSA (2004) and researchers (Anghileri, 2000; Hiebert & Carpenter, 1992; Hiebert et al, 2000; Kilpatrick et al, 2006; Van de Walle, 2007, Skemp, 1986). For example, when Reagan taught the measurement activity her goal was to assist students' conceptual understanding of place value, yet she did not clearly indicate whether she highlighted the relationship in the follow-up whole class discussion.

Helping students to see the link between mathematical ideas was not made clear by Reagan throughout the interview sessions, although she appeared to unintentionally connect mathematical ideas. For instance, when she approached the learning of division, she began by helping the students to construct cognitive schema of division as repeatedly subtracting groups. She then diverted the learning to readdress multiplication and focussed on multiplication as repeatedly adding groups, hence illustrating a connection between the operations. After a subsequent repeated subtraction lesson, Reagan taught the long division procedure. In the interview, she spoke about her intended learning sequence as a 'natural progression.' Her use of the term 'progression,' as I interpreted it, involved a sequence of developing conceptual understanding (repeated subtraction) and procedural fluency (long division), and then she focussed on developing thinking strategies (for short division). I believe she was helping the students to recognise the connection between repeated subtraction, long division and short division, although this was unintentional at the time. As Reagan described, the students grasped the long division procedure because they had constructed cognitive schema about repeated subtraction. Consequently, because they understood how and why division worked, she envisaged that the students would be better prepared to develop mental computation strategies to apply to short division. However, these thoughts were generated after she had reflected on both lessons. These lessons all occurred at the beginning of this research, and as the data revealed, Reagan's intentions seemed to be clarified after the experience. Her motives about how to structure the learning process at that stage were unclear.

Reagan's preliminary thoughts about how students learn were inconsistent with the mathematics syllabus, simply because she had not mentioned that the learning process involved thinking, reasoning and working mathematically. However, she often talked about her students as 'little mathematicians.' It appeared that she wanted her students to believe that they had the capabilities to work like mathematicians. Whilst Reagan's aspirations were clear, she seemed to be in a quandary because she was uncertain what working mathematically looked like in practice. For instance, at midpoint in this research, she indicated that both she and the students were faced with a challenge. The students needed to learn how to think mathematically and she needed to know how to guide them. It was at this point that Reagan expressed more interest in the mathematics practices, as described by the RAND Study Panel (Ball, 2003). She then indicated opportunities in her lesson plans for students to represent ideas and justify their thoughts about mathematical ideas, and in some places, where they could work towards making a generalisation.

In the last interview, it became apparent that Reagan had already valued the practice of representation to assist students' learning. For instance, she preferred not to use pre-drawn worksheets because she believed students' thinking about how mathematical ideas were represented, symbolically or pictorially, was enhanced when they created their own representations. However, the thinking process was taken for granted; she assumed that as the students constructed the images, their inner dialogue would enable them to justify how and why ideas were represented as they were. Nevertheless, as the research progressed, Reagan focussed more explicitly on the students’ communication of their reflective thinking. As revealed in the patterns lesson, the students were encouraged to justify their own ideas about how and why they could, or could not, reconstruct patterns. What also became apparent in the last interview was that Reagan believed that she now had a structure for her lessons; in other words, she believed that she could structure a ‘thinking and reasoning’ learning process. In short, she explained that the mathematics practices gave her a framework that provided “steps towards mathematical thinking” (IV5, p. 10). Reagan believed that framework enabled her to be more aware of the students’ thinking processes. However, she added that enticing students into contributing their thoughts and ideas in discussion was a learning process in itself. She believed that the students needed to learn how to listen discerningly and respond thoughtfully. The next chapter discusses the observed data and the students’ self-reported data, to discern whether Reagan’s spoken interpretations about changes to the learning process and environment reflected the classroom lived experience.

CHAPTER SIX
INTERPRETING THE LIVED EXPERIENCE:
WAYS OF KNOWING AND DOING MATHEMATICS

"If you use it, you must understand why it works and be able to explain it"

(Van de Walle, 2007, p. 220)

6.1 Introduction

The most important influence on students' performance and achievement is determined by what teachers do in the mathematics classroom (BTR, 2005). However, Walshaw and Anthony (2008, p. 520) pointed out that often research describing best practice has "failed to tell the whole pedagogical story." Heeding their advice, I intend to tell as much of the pedagogical story as is possible within the confines of this research project. I do not claim to tell the 'whole' story because there are issues that have been overlooked to maintain focus on the research aim. For instance, this research did not investigate social equity issues that may have arisen in the classroom, nor did it investigate practical issues such as desk arrangement or classroom grouping strategies. Rather, the focus was on how Reagan adopted a 'process approach,' implementing thinking, reasoning and working mathematically, and the subsequent implications for mathematics teaching and learning. In each section, I have included data, in the form of vignettes from the lessons observed, to illustrate the lived experience. Reagan's speech is represented by a 'T' and the students' speech is represented by 'S.' I have used numbering to indicate when a different student spoke, although the numbers are not coded to any specific child. The numbering will restart for each lesson. Names have been used in some places for ease of discussion, and the names are pseudonyms.

The progression towards a process approach was gradual. Prior to the commencement of this research, Reagan's teaching practice involved following the *Maths at School* program, and once this research began, she feared that her students were not equipped with mathematical understandings that would enable investigative mathematical thinking. Hence, the initial stages of this research involved developing conceptual understanding of place value and multiplication. The previous chapter revealed that, even though she had beliefs about how to develop understanding, she had not yet put her beliefs into practice. As the research progressed though, so did Reagan's teaching practice. Her teaching aspirations became actualised as she implemented change and engaged in reflective dialogue with me, the researcher. This chapter discusses the ‘ways of knowing,’ which are the conceptual/relational understandings that Reagan envisaged her students would be developing. It then looks at the ‘ways of doing,’ which describes how she incorporated the mathematics practices into the classroom learning environment to encourage active communal investigation and validation of mathematical ideas.

6.2 Ways of Knowing

This section discusses the first five lessons observed, and reveals Reagan’s gradual implementation of her own beliefs about how students learn. She introduced each activity with a discussion about the previous lesson or about key terms the students had been exploring. Her intentions linked with Van de Walle and Lovin’s (2006) view that it is important to get mathematical ideas up and running in the students’ minds before starting a new investigation. Even though the interview data did not clearly depict whether she explicitly highlighted the connections between mathematical ideas, the observations indicated that she did indeed do this. At first, Reagan pointed out the mathematical connections that existed between ideas herself; however, later in the research, she helped the students to discover the connections between ideas, as will be revealed in this chapter and the chapter following.

The first lesson I observed Reagan teach was not a lesson she had planned. In this lesson, the students were working though division problems that were written in their *Maths at School* journals. They had been working through their journals, and some had sought help because they were confused about how to solve ’18 ÷ 5.’ She decided to stop the class to discuss some division ideas. She asked them what terms could describe division, and they responded with terms such as ‘sharing’ and ‘dividing’ (T3WK1D2). She then explained how she would approach this problem mentally. She suggested that she would think about multiplication, and said, “If I know that 5 x 3 = 15, then I could take 15 from 18 and I would have 3 left over, so the answer would be 3 with 3 remaining.” Whilst Reagan was encouraging mental computation strategies, she may have inhibited the development of the students’ thinking strategies by imposing her own, as Van de Walle (2007) warned. Van de Walle believes that the most important part of the lesson is when the students have an opportunity to explain their solution methods, and, in this example, it was Reagan who was explaining her solution methods.

Reagan appeared unsure of herself in this lesson as she taught from the *Maths at School* program. The journal required the students to represent the answer to the problem ‘18÷5,’ as ‘3r3,’ meaning three with a remainder of three. She asked the students if any of them could represent the remainder in a different way, and Louisa raised her hand. Reagan invited her to share her thoughts and Louisa explained:

> I did it like this: I thought of 18 as $18.00. Then I took away 3 dollars to make it easier: 18 take away 3 is 15, so 15 divided by 5 is 3. Then I had 3 dollars remaining, so I added a 0 to the 3 to make it 30. I knew that 6 x 5 = 30, so 3 dollars divided by five was 60 cents. Then, 3 dollars plus 60 cents makes 3 dollars and 60 cents. (T3WK1D2)

After the lesson, Reagan told me that Louisa was further advanced than most of the students in the class; hence the reason why the majority of the students looked lost, and could not see the relationship that ‘18 ÷ 5’ could be broken down to be ‘15 ÷ 5’ and ‘3 ÷ 5’ (or 30 ÷ 5). She pointed out to the other students that Louisa had broken the numbers to make them easier to manage. However, Reagan seemed deep in thought, and paused for a moment.

Reagan then moved on to the next question in the journal, and read aloud, “We have 16 pieces of pizza to share between 5 people, so how many pieces will each person get?” (T3WK1D2). She asked the students to write down the corresponding number sentence. I noticed that some wrote, ‘16 ÷ 5’, and others represented this as a short division algorithm. She then asked for another volunteer who was willing to share his or her thinking. The girl who volunteered had been working with the short division method. She explained that, “You divide 5 into 1; it won’t go, so you divide 5 into 16, which is 3 with 1remaining.” A quick scan of the room, and Reagan, looking concerned, decided to show the students a repeated subtraction method. She drew a T-table on the whiteboard, and talked them through the method:

5	16	16÷5
2	10	Two groups of 5 makes 10,
	6	10 from 16 gives us 6,
1	5	There is 1 group of 5 in 6.

3	1	6 take away 5 leaves 1remaining, and therefore, we have 2 groups plus 1 group makes 3 groups of 5, and 1remainder.

A student called out claiming that the answer was three with one left over. Another student interjected and suggested that the answer could be three and one fifth. Reagan asked the class if they agreed with this student, that the answer could be represented as three and one fifth. Many of the students appeared to be confused, and consequently, she did not investigate the student's idea any further.

Reagan appeared to be perplexed about how to manage the two students' insightful mathematical thinking, where the first student represented the remainder of the division problem with a decimal, and the second used a fraction. She had not extended these two students' thinking nor did she enter into any further discussion about these ideas with the class. She commented in the interview that she was concerned about the varying levels of understanding in the classroom, and therefore, it was possible in this lesson that she was wondering how to move the whole class forward. It was after this lesson that she decided to teach repeated subtraction because she wanted the students to develop conceptual understanding of division. However, as Reagan reflected on the lesson after class, she explained that she was uncertain how to go about helping the students to clarify their thoughts for themselves and for each other. I suggested that she let the students come to the whiteboard to represent and justify their thoughts. The follow up repeated subtraction lesson, along with a lesson to revisit multiplication, occurred whilst I was not in the classroom. The next lesson observed was a lesson Reagan had planned to help the students to relearn place value, as follows.

6.2.1 Reflective Thinking to Connect Mathematical Ideas

In this lesson, Reagan focussed on highlighting the connections between mathematical ideas and their applications, to help reinforce the students' place value ideas. She asked the students to think about and write down at least five different ways they could represent '4.95,' which helped to evoke the students' existing ideas about place value. She asked for a volunteer to share one idea, and a student responded with, "4 dollars and 95 cents" (T3WK2D1). After ten minutes recording their ideas, Reagan discussed their thoughts as below:

S1 I used a number expander to represent 495 hundredths

S2 I drew a square and divided it into 10 columns and then 10 rows to show 100 pieces.

T [Reagan held up a 100s *Multi Base Arithmetic Block* (MAB) square]

S2 [The student glanced at it and continued.] Then I shaded 4 wholes [columns] and 9 squares and 5 small bits [½ squares].

T [Reagan held up the 100s square again]. If this square represents 1 whole, what does each piece represent?

S2 One hundredth [1/100].

T If I shade 4 columns, would that represent 4 parts of 1 whole?

S3 No, it represents 40.

S2 Oh, so I need 4 whole squares and 1more with 95 little squares shaded.

T Well done (T3WK2D1).

The students listened intently as this discussion progressed. Reagan had used concrete materials (hundreds squares, tens sticks and single unit squares) to prompt the students' thoughts. As she pointed out the relationship between the numerals and their place value through helping the student to recognise the number as a whole number and then a part of a whole, the student altered her misconception about representing place value. Reagan's focus was on thinking about part-whole number relationships, which, as Owens and Perry (2001, p. 83) pointed out, "is an important issue for coordination of number knowledge."

Reagan invited another volunteer to come to the whiteboard and share her ideas. Lisa volunteered and represented her ideas with a number line:

Lisa I used a number line using money

.____________________________________.___.

$0.00 $4.95 $5.00

T What else could you put on the number line to show us that $4.95 fits there? What would be the first mark?

Lisa Half way.

T What would that be?

Lisa $2.50

T Put that in.

Lisa Now I can work out where to fit the $4.95.

T Good work (T3WK2D1)

In this dialogue, Reagan's focus was on helping the students to recognise the relationship between 4.95 and the whole number of five, as well as the relationship between 4.95 and half of five. Another student explained that he thought of 4.95%, and she asked the student where they might see a representation like that in their daily lives. There were many ideas expressed such as, on tests, at sales, on pie graphs, as bank interest and on television advertisements. Continuing, she asked if the students had thought of other ideas related to 4.95. They gave various measurement examples and the last student said "4.95 represented 4 wholes and 95 parts of 100." (T3WK2D1) Reagan spent several minutes getting the students' ideas about place value up and running before she introduced the next activity, as recommended by Van de Walle and Lovin (2006). The activity involved their using place value charts. They were instructed to write a number on the chart, verbalise the number, and state the place value of the numerals that made up that number. They were also required to write the number in their books, both symbolically and with words. They worked in pairs for ten minutes, followed by a synthesis discussion about place value and naming numbers.

It was through the synthesis discussion that Reagan realised the true value of communicating one's thinking about mathematical ideas, as discovered in the previous chapter. As she wrote the students' ideas on the whiteboard, she realised that numbers could be named differently, such as 4.375 could be named as, 'four and three hundred and seventy five thousandths' or 'four thousand three hundred and seventy five thousandths' (IV2). It was through her own reflective thinking as she represented the numbers that she clarified her thoughts and understanding about place value and naming numbers. She then held up a MAB thousand block and said, "If we have 375 thousandths, how many more do we need to add to get 1 whole?" (T3WK2D1). Reagan then asked a student to come to the whiteboard to share his ideas.

S3 The answer is 625 thousandths.

$$\begin{array}{r} 0.375 \\ \underline{+0.625} \\ 1.000 \end{array}$$

T Good thinking I would do this in my head and visualise a number line to represent counting back. If 1 000 take away 300 is 700, and then 700 take away 50 is 650, then I can say that 650 take away 25 would be 625. (T3WK2D1)

In this lesson, Reagan encouraged the students to come to the whiteboard to represent and explain their ideas. This was a new strategy for her and the students, which later became an integral feature of the learning environment. However, it seemed that the investigative thinking at this stage occurred in Reagan's mind more so than the students. For instance, she stepped the students through her mental thinking processes and pointed out that in her mind she pictured a number line, and visualised counting back. Also, she frequently referred to concrete materials to guide them towards constructing visual representations in their minds. She believed that concrete objects helped students' knowledge construction; yet, at this early stage in the research, it appeared that she manipulated the concrete materials to represent and communicate ideas more than they did. Clearly Reagan's intentions were to assist the students to create visual representations in their minds to connect to in later learning; however, at this point, they were not yet active in their own construction of mental representations.

Indeed, in this lesson, because of her own clarification of a mathematical idea, Reagan substantiated the value of providing opportunities for students to represent and communicate their thinking and ideas. She explained in the interview that using the whiteboard helped her to clarify the students' thinking processes, and thereby she believed it was becoming easier to pose probing questions. However, at this point Reagan was doing most of the communicating and representation of ideas. She did, however, maintain focus on helping the students to recognise number and place value relationships, and thus develop relational/conceptual understandings of place value. For example, she reinforced the relationship between whole numbers and parts of a whole, and further encouraged this thinking by asking the students what needed to be added to 375 thousandths to make a whole number.

6.2.2 Introducing Resources to Develop Understanding

To further enhance the reinforcement of place value understanding, Reagan planned a measurement activity. Despite her beliefs that conceptual understanding was developed through a process of action, reflection and abstraction, aligning with Battista (1999), this lesson was only the second time she used resources to encourage active investigative thinking. The first time was with counters in a multiplication lesson, discussed in chapter four. Reagan oriented the students' thinking with a review of the key ideas from the previous place value lesson:

T What are some words we use when we talk about place value?

S2 Hundredths.

T Where do we find hundredths, to the right or the left of the decimal place?

S2 Right.

T What about tenths? [Reagan randomly selected students, using eye contact, to elicit responses.]

S3 Right.

T What about thousandths?

S4 Right.

T Are thousandths before or after the tenths?

S5 After.

T Where would the hundredths be?

S6 Right of the decimal, before the thousands and after the tenths.

T What is left of the decimal that we don't have to the right?

S7 Ones [meaning whole numbers].

T Where would I find the thousands, how would you describe it?

S8 After the hundreds to the left.

T What words could I use to describe the numbers before or after the decimal place?

S9 Decimal fractions, whole numbers, and parts of a whole.

T Could I write one and one tenth as a decimal?

S10 Yes, one decimal one [1.1].

T How else could I write it, all have a go [she paused to ensure all of the students recorded something in their books].

T What ideas have you written down?

S13 One and one tenth, 1 1/10 [he held up his symbolic representation] (T3WK2D2)

In this discussion, Reagan randomly selected students with eye contact. This was a strategy she believed helped to maintain their focus (IV3). She and the students were familiar with this type of discussion, where she led and the students responded. Throughout the revision, Reagan was focussed on the mathematical language used to describe the value of numeral placement. When she paused to provide an opportunity for them to record something in their books, I wondered how she could determine whether all of them had recorded something, and whether they were in fact on the right track.

Reagan then discussed units of measure to prepare the students for the outdoors activity, where they used a measuring wheel and some kitchen and bathroom scales. But first they were asked what unit of measure would be applicable for each instrument:

T We have this measuring wheel. What would the unit of measure be?

S1 Centimetres and metres.

T Are we going to have thousandths?

S2 No.

T Why not?

S3 Because there are no millimetre markings on the wheel.

T What about bathroom scales. What would be the unit of measure?

S4 Kilograms.

T We also have 'stones' which were the unit of measure before we had the metric system. Some countries still use this unit of measure. However in Australia we use the metric system. What is the unit of measure for the kitchen scales?

S5 Grams.

T Where would you use these scales? [The students came up with many ideas, such as hospitals, science laboratories and so on.] (T3WK2D2)

The class went outside and used the kitchen scales to weigh sand, the bathroom scales to weigh themselves, and the measuring wheel to measure lines. They were required to convert metres to centimetres and then to millimetres and kilograms to grams. They were enthusiastic in this lesson, yet Reagan approached me during the lesson and showed me her sweaty palms. It was obvious that she felt anxious. As revealed in the interview, she was concerned about maintaining control of the students. The outdoor area was in close proximity to the Principal's office, and hence, I wondered if this added to her angst. She was also concerned that she was not in control of their learning; for instance, she feared that the students may not be accurately converting measurements.

Despite Reagan's angst, I noticed that the students were actively engaged mentally, physically and socially in this lesson. For example, they were challenging each other to estimate how much something would weigh or how long a line might be, and then they would measure the item in question to see who was the closest. After each measurement, they studiously converted the measurements from kilograms to grams, centimeters to millimetres and the reverse. Their curiosity and a sense of ‘I wonder’ had been evoked in this lesson, and hence, they were intrinsically motivated. In addition the estimation game helped them to develop measurement sense. As Van de Walle and Lovin (2006) pointed out, estimation helps the student to focus on the attribute and develop familiarity with the unit of measure as well as provide intrinsic motivation.

When they returned to the classroom, the synthesis discussion was short. Reagan asked the students to raise their hands if they felt more confident to convert measurement, and almost the entire class raised their hands. Four boys did not; however, these boys had been asked to sit out the lesson because they had misused the scales. Reagan explained to me the following day that one of the boy’s parents had mentioned how her son was disappointed because he had missed out on the activity. As a consequence, the boy promised his mother he would be more respectful next time. Upon reflection in the interview, Reagan explained that these boys had simply overacted because they were excited about the opportunity to learn mathematics in an outdoors’ environment; they had not done this before. However, she was pleased that this parent was taking an interest in her son’s learning and was supportive of Reagan’s new approach.

Before the students went to lunch, she wound up the lesson with a brief discussion:

T How many centimetres in a metre?

S1 100.

T How many millimetres in a metre?

S2 1 000.

T How many grams in a kilogram?

S3 1 000.

T If I had 500 grams, how many kilograms do I have?

S4 Half a kilogram.

S5 Or 0.5kg (T3WK2D2).

It appeared that Reagan intended to highlight the relationship between common fractions and decimal fractions, although it was not done explicitly. When the students left the classroom, she breathed a sigh of relief. Following this lesson, she spent more time with the students exploring multiplication with counters, and she believed that their confidence in their understanding about multiplication was growing (IV3). I detected that Reagan's confidence was growing too; she appeared to feel more confident about using resources, as well as facilitating discussion to draw out the students' ideas, as will be revealed below.

6.2.3 Knowing How and Why Ideas Work to Develop Understanding

In this next lesson, Reagan maintained a focus on helping the students to create visual representations in their minds; however, she also found that unless they understood how and why ideas worked, knowledge application was difficult. The division lesson occurred in three stages: first, Reagan helped the students to develop a strategy to estimate an answer; second, she revised the repeated subtraction method of division, and then she taught the long division procedure. Her intention was to help the students to visualise multiples of numbers being repeatedly subtracted as they divided numbers:

T Raise your hand if you understand repeated subtraction. [80% of the class raised their hands.]

T Write this in your books. [Reagan pointed to an equation on the whiteboard: '136 ÷ 4 = ?']

T When we use repeated subtraction we look for multiples of 10, and then we take these away from the number we are dividing. What are some easy ways you think about if you are dividing a number by 4?

S1 100.

T Will this [100] divide by 4, will it make 4 equal parts?

S1 400

T We are looking at 136 though. (T3WK3D1)

Reagan explained to me throughout the lesson that she had been helping the students to recognise that a multiple of ten would be easier to work with; hence, she moved on from the student's suggestion that 100 was divisible by 4. When this student was asked to elaborate on her thinking, she became confused about whether she was supposed to be dividing 100 by 4 or multiplying 100 by 4.

The discussion continued and a second student responded, in what seemed to be an attempt to help the first student:

S2 100 divided by 4 is 25 and 120 divided by 4 is 30.
[Reagan wrote '120 ÷ 4 = 30' above the '136 ÷ 4' on the whiteboard]

What about the other side of 136?

Tad 140 divided by 4 is 35.

T What else?

S4 160 divided by 4 is 40.

T So what is the nearest to 136?

S5 120 divided by 4 is 30, and 160 divided by 4 is 40.

T So the answer for 136 divided by 4 has to be between 30 and 40. What do you think the answer might be?

S6 It is 34 because 34 multiplied by 4 is 136. 30 multiplied by 4 is 120, and 4 multiplied by 4 is 16, so the answer is 30 plus 4, which is 34. (T3WK3D1)

Reagan was guiding the students towards thinking about easier numbers to work with, which may have been the reason she skipped over Tad's response that, '140 divided by 4 is 35'. Her intention, it seemed, was to encourage the students to think about single digit multiples of four, which could then be multiplied by ten. For example, if 12 divided by 4 was 3, then 120 (12 by 10) divided by 4 would be 30. However, she had not made this relationship explicit through discussion. Also, Reagan had been investigating strategies such as double and halving when working with multiplication, and maybe Tad had applied a similar thinking strategy. I wondered whether in his mind he had halved both 140 and 4, to work with 70 divided by 2. His thinking was not investigated.

The students were then asked to apply the estimation strategy to an equation Reagan had written on the whiteboard:

T Now it is your turn. Do this one in your books, and then check your answer with repeated subtraction. [The students were given five minutes to apply the strategy to the equations written below.]

? ÷ 6 =

258 ÷ 6 =

? ÷ 6 =

T Okay, do we have a volunteer to come up to the board to show how they worked this out?

S7 [Wrote on the whiteboard]: 240 ÷ 6 = 40

258 ÷ 6 =

$300 \div 6 = 50$

T Can you explain your thinking?

S7 I looked at the multiplication chart; I looked for a number less than 258, so I didn't worry about the 8. I didn't look for 250; I looked for less than 25. I saw 24, so 6 times 4 is 24, so 6 times 40 is 240. Then 30, 6 times 5 is 30, so 6 times 50 is 300.

T So the answer is between 40 and 50, good. So now, who can show us the repeated subtraction method to solve the problem?

S8 [This student wrote on the whiteboard]:

	258
10	- 60
	198
10	- 60
	138
20	-120
	18
3	-18
43	0

T Who did it differently? (T3WK3D1)

There was a brief discussion about the different ways that students approached the repeated subtraction method, whereby some students started at 30 groups of 6, and others started at 20 groups of 6. Reagan's intention was to demonstrate an estimation strategy, aligned with Anghileri (2000), who believes estimation helps students to develop number sense and an intuitive feel for numbers. Reagan was preparing the students with a strategy to justify the reasonableness of their answer through being able to check whether the answer came between 40 and 50.

Reagan then introduced the long division procedure, which she had not previously taught to these students:

T Okay, now we are going to look at long division. Who knows long division?

T Who can show me how to solve this using long division? [She pointed to the algorithm written on the whiteboard: '258 ÷ 6 =?'].

Seon [Seon came to the whiteboard and wrote the symbols below]:

÷ ×–↓

T Can you explain that?

Seon First you divide, multiply, subtract and bring down.

T Okay, can you do it?

Seon I don't know how many 6s 'goes into' 25.

Lisa [Lisa, who also knew the rule, came to assist, and wrote on the whiteboard]:

$$\begin{array}{r} 258 \\ \underline{\div\ 6} \end{array}$$ 6 goes into 8 one time, and now I am lost.

Lisa did not use the typical long division format to represent the equation; rather, she set the algorithm out as a traditional multiplication algorithm. Seon and Lisa had memorised the rule associated with the long division procedure, yet they could not apply the rule.

In an attempt to make things easier for the students, Reagan asked Lisa to choose something easier to work with, and Lisa wrote an algorithm on the whiteboard:

Lisa $$6\overline{\smash{)}80}^{\ 20}$$

Then, 20 times 6 is [she paused]. No, that's not right either. Can you help? [She looked at Reagan who picked up the marker and wrote]:

T What about $$5\overline{\smash{)}130}^{\ ??}$$

T [Reagan addressed the whole class] Lisa can do this in her head easily. (T3WK3D1).

Lisa nodded to agree with Reagan, but was confused. She did not know where to start with this procedure, although I noted that when she selected an easier algorithm, she set it out as a repeated subtraction format. As Lisa tried to apply the 'goes into' rule, she became lost, and picked up her own misconception by multiplying 20 by 6. Furthermore, when Reagan gave her an easier algorithm, Lisa was still confused, even though she could solve 130 divided by 5 mentally. Clearly, she could not make sense of this long division procedure. Hence, both students had memorised the formula, yet did not understand how to apply it. This situation confirmed Skemp's (1986) viewpoint, that unless students understand conceptually how a procedure works, they will need to relearn the procedure every time it is encountered. Reagan then stepped the students through the procedure following the rule, 'divide, multiply, subtract and bring down.' She positioned the students in a group on the floor below the whiteboard; they had their books on their laps and wrote down each step as she demonstrated. Reagan informed me after this lesson that from here on, her goal was to ensure that the students develop mathematical understandings that would sustain further mathematical investigations (T3WK3D1).

At the midpoint in this research, Reagan felt confident that the students were prepared to move on in their learning. Her lessons changed, although she still began each lesson with a brief discussion, either to evoke the students' existing ideas, or to revise the previous learning. The lesson format generally involved an activity where they manipulated concrete materials, either in groups or individually, and then Reagan facilitated a whole class discussion. She started to incorporate more hands-on activities, whereby the students could manipulate concrete materials to help construct mathematical understandings. At this point in the research Reagan started to utilise the activities we had co-planned. The activities enabled the students to investigate mathematical ideas and make sense of things for themselves, and Reagan used pre- and post-discussion phases to distil the key mathematical ideas, as revealed below.

6.2.4 Using Hands-on Learning to Develop Understanding

Reagan and I had planned a sequence of activities to help the students develop algebraic thinking during the break between school terms. Reagan adapted and implemented these activities from this point forward. The sequence started with an investigation of growing patterns. This investigation involved two parts: first, the students constructed and recorded patterns, and second, they reconstructed the same pattern based on their recorded information. Reagan's intention in these activities was to assist the students to recognise for themselves that patterns consisted of repeating parts, although I was not to learn this intention until the second lesson. She had various resources that the students could select from, including Lego, pencils, counters and coloured cubes, and they were instructed to construct a pattern and then record it in their books. Her instructions were brief, and I wondered why. The students busily constructed patterns, except for one group of boys who appeared to be overwhelmed by the large bucket of Lego; Reagan redirected them several times. These boys spent a long time selecting materials to make a pattern. Two girls worked together and made a pattern with colours; the pattern was layered, and each layer had one less block. When asked where the repeating part was, they said, "Here, we take away one part each time" (T4WK1D1). They remarked that it was a shrinking pattern. Another student collected coloured pencils, and drew a complex pattern that had repeated parts within repeated parts, and furthermore, she had circled the repeated parts. All of the students made a pattern; some continued to make and record patterns while others continued making patterns, but only recorded the initial pattern.

After twenty minutes, Reagan circled the classroom to ensure that every student had created and recorded a pattern, and then stopped the class to discuss their patterns. She focussed on highlighting the mathematical language they could use to describe patterns. This focus, I was to learn later, was also to prepare them for the next activity:

T Think about your pattern and whether it would be an easy pattern to recreate. How many steps would I have to make if I made your pattern?

S1 12.

T Can you explain your thinking, which part is the step?

S1 The red and white part 2 reds and 1white.
[Reagan nodded, and then glanced at Tom, inviting him to describe his pattern; Tom had created a pattern that resembled a toy car.]

Tom I have red, red, red then blue, blue, blue then green on the bottom.

T If I had to repeat a part how could I?

Tom [He shrugged his shoulders and appeared deep in thought. In the following lesson, his partner managed to reconstruct the pattern.]

Tom It goes red, blue, white, and the blocks criss-cross. All the reds go this way [pointed vertically], and all the blues go this way [horizontally]; the pattern grows up.

T So what language could I use to explain, 'this way' and, 'that way'? What words could I use? [pause]

T I could use compass points, what is the opposite of east?

Tom West.

Tom Or I could say, 'north and south', or I could use degrees because it turns 90 degrees.
[Other students offered suggestions]:

S2 Or left to right.

S3 Or towards and away.

S4 Or horizontal and vertical. (T4WK1D1)

The students had been investigating direction in a previous lesson, hence Reagan's link to compass points. As she closed the lesson, she said to them, "Now you have some words that you could use to describe your pattern" (T4WK1D1). Before they left the classroom for lunch, she ensured that each student had recorded a pattern. This pictorial representation was important because it was the key feature to guide the next investigation.

The next lesson involved representing and communicating ideas about patterns. The lesson began with a brief recount of the previous activity:

T What did we do yesterday?

Ss Patterns.

T What did we learn about patterns?

S5 That they repeat.

T Can you explain what you mean by, 'they repeat'?

S6 You can have a pattern on a pattern.

S7 It can also be called algebra.

T Well, it is the foundation of algebra.

S8 You can make it out of anything.

T Okay [holding up three pieces of paper], is there anything that I need to do to make a pattern? What would I do with this?

S9 You could cut it.

S10 Or put one up and one down.

T And what is another way that we could describe that? Remember our use of language.

Tom Vertical and horizontal (T4WK1D1).

Tom remembered the discussion from the previous lesson, where he was asked about language to describe direction. It was interesting to note that Reagan had not mentioned the term 'algebra' previously, yet this student (S7) had related their activities to algebra. After this revision, and a reminder about using appropriate mathematical language, Reagan paired the students. She instructed them to refer to the pattern that they had constructed and recorded in their books, and to collect the necessary materials that would help their partners to reconstruct the pattern. While they were collecting their pieces, Reagan told me that she hoped the activities would help the students to understand that after they had constructed the first step of the pattern, the pattern could then be repeated easily (T4WK1D1). At that moment I understood why they were given brief instructions the day before, and why Reagan had focussed on mathematical language that described patterns. She had envisaged that if the students had not made a pattern with repeating parts, they would find it difficult to describe their pattern. Her intentions reflected the views of Van de Walle (2007), QSA (2004) and DETA (2008), who recommend that students need to investigate ideas in ways that they can make sense of things for themselves.

The students sat back to back and instructed their partners to recreate the patterns. By sitting back to back, the partner could not see the pictorial representation, and also the describer could not see the concrete representation being constructed. Once finished, they faced each other, and compared the pictorial with the concrete representation. The aim was for the students to discuss why or why not the pattern resembled the pictorial representation. They took twenty minutes to reconstruct the patterns, and two pairs (four students) managed to swap partners to reconstruct two more patterns each. The students were more focussed in this lesson than they were in the preceding lesson. Reagan spoke to each pair as she circled the classroom, and asked questions such as: ‘How long before you could see the pattern forming?’ ‘What did you find difficult?’ ‘How many steps were in your pattern?’ ‘How many parts were in each step?’ (T4WK1D1). I had not seen her address the students as individual groups previously. The students seemed to enjoy the interaction with each other and with Reagan. After she had spoken to each pair, she briefly revised the activity by asking the students to explain what they found easy or challenging when recreating or describing patterns. She was particularly focussed on them being able to recognise that if a pattern had a repeating part it would be easy to describe and reconstruct.

After the lesson, Reagan explained that she felt comfortable using resources now, despite the boys’ excitement with the Lego. She believed that the manipulation of materials helped the students to construct abstract ideas about patterns having repeated parts (T4WK1D1). It was these abstract ideas that Reagan envisaged would help them as they worked through the algebraic sequence of lessons to follow. Also, I noted that the classroom atmosphere was changing. The students appeared to be participating more in discussion, which then promoted intellectual engagement. It was possible that as they were physically manipulating materials, they were also becoming more mentally active. The next activity was intended to build on the understanding that some patterns grow.

In summary, in the first lesson, Reagan appeared unsure, and the lesson seemed to lack direction. This may have been because she had not planned the lesson herself because she was still working from the *Maths at School* program. This was the last time she used the program while I was in the classroom. The reason she seemed unsure may have been because she was not teaching according to her beliefs, as she had described in the interview. For instance, Reagan said she valued the students' thinking strategies, yet in this lesson they had little opportunity to express their ideas; rather, Reagan expressed her problem solving strategies. However, she did invite two students to share their thoughts, which presented her with a new challenge. She was uncertain how to embrace these students' insightful ideas to advance learning for all. This uncertainty however was the catalyst for Reagan to alter her practice, to promote opportunities for the students to demonstrate their thinking and reasoning processes. Once she enabled the students to use the whiteboard to help them to discuss their ideas, both the student sharing and the others in the class seemed to make more sense of the mathematical ideas. By doing this, Reagan was also adhering to the syllabus goals (DETA, 2008; QSA, 2004) that students must be given opportunities to communicate their ideas in ways that allow them to make sense for themselves and others.

In addition, Reagan slowly became more comfortable using resources. Despite believing that conceptual understanding was developed through hands-on learning, up until this research began, the actual implementation of hands-on learning had been a stumbling block for Reagan. Of course, there could be many possible reasons why she found it difficult to follow her beliefs, but two that became apparent were that she felt obligated to follow the *Maths at School* program, and she lacked confidence in using resources. However, in the place value measurement activity, when the students were provided with the opportunity to become more hands-on, there seemed to be increased levels of student enthusiasm and curiosity. Lastly, the long division lesson confirmed to Reagan the importance of learning with understanding. Thus, she planned the first two algebra lessons to ensure that the students would understand how and why patterns had repeating parts. The activities were hands-on and the students were encouraged to manipulate concrete materials, reflect on the activity, and communicate their ideas. Hence, Reagan promoted Battista's (1999) cycle of action, reflection, and abstraction.

Reagan's desire to help the students to learn mathematics with understanding, in other words, to ensure that the students understood how and why mathematical ideas worked aligned with Skemp's (1986) viewpoint. Skemp asserted that if students are to recognise how ideas, facts or procedures are related, they need to learn mathematics in ways that they can discover and understand how and why mathematical ideas work. Skemp was adamant about this way of learning because he believed that cognitive construction of a conceptual system is an internal process that occurs inside the students' minds. Yet he also stressed that the construction process can be assisted with the *right* guidance. I have added emphasis to the word 'right' because for some time researchers, policymakers, teachers, mathematics educators and mathematicians have been investigating and posing ideas about what constitutes the 'right' guidance. The guidance investigated in this research was the recommendation by DETA (2008) and QSA (2004), to promote working mathematically through processes of thinking and reasoning, communicating and reflecting. Reagan was willing to alter and investigate her teaching practice to implement the curriculum goals. She was also beginning to explore the RAND Study Panel's (Ball, 2003) mathematics practices of representation, justification and generalisation to scaffold these processes of working mathematically, as will be discussed in the next section.

6.3 Ways of Doing

The progression towards implementing the 'mathematics practices' (Ball, 2003) as a way to scaffold thinking, reasoning and working mathematically was slow, yet by the end of this research the practices were a distinct feature of Reagan's daily teaching practice. As she explained in the first interview, she had not heard of the mathematics practices prior to this study; neither had the students. For instance, when Reagan asked the students to write down various 'representations' for the number '4.95,' they were not sure what the term meant, as the vignette below indicates:

T	Think of how many different ways this number could be represented and write down at least five ways. [Reagan pointed to the number '4.95' on the whiteboard]
S1	What do you mean by that?
T	How can it be shown, represented? Let's look it up in the dictionary.

[She asked a student to look up the word and choose the most appropriate meaning for this particular of the context.]

S1 Symbolised.

T Okay, let's think about it. What is one way I could say this number?

S2 4 decimal 95.

S3 Or 4 dollars and 95 cents.

T That's it. How can this number be used? You should be able to come up with at least three representations. (T3WK2D1)

This was the first time Reagan had used the term 'represent' with the students, and instead of explaining the term herself, she helped them to clarify meaning by asking a student to locate the word in the dictionary. She then asked for an example to help illustrate the meaning of 'representation.' She did not explicitly use the term 'justify' throughout the research; she referred to the term, 'explain' instead. The term, 'generalisation' was not used either; however, these practices were incorporated into the lessons.

As the students worked through the place value lesson, and discussed their ideas, it became clear that some of them had used representations before to clarify their place value understandings. For instance, one student spoke about using a 'number expander' to think about 4.95 as 495 hundredths. Another student drew arrays symbolising 'MAB' blocks, although incorrectly. Reagan redirected this student's thinking by holding up the correlating MAB blocks to represent the arrays the student described. Her goal was to help the student alter her misconception for herself. Reagan also facilitated the students' sense making processes by enabling them to create concrete and pictorial representations of patterns, and then instruct others to reconstruct the patterns. At this point Reagan envisaged, although it was not explicitly stated, that as the students attempted to instruct to reconstruct the patterns, they would justify how or why the patterns could, or could not, be recreated. The lesson to follow was aimed at developing their understanding that patterns can be represented pictorially and numerically.

6.3.1 Representing Pictorial and Numeric Relationships

In this lesson, Reagan helped the students to investigate the connection between pictorial and numerical representations of the circles pattern (see Figure 6.1). They had just drawn the pattern, and were instructed to "continue the pattern another four times."

(T4WK1D2) I observed one student replicate the pattern four more times. Reagan noticed this too, and once she had physically distanced herself from this student, she said, “Continue the pattern on, and do not just copy what is on the board four times” (T4WK1D2). The atmosphere of the classroom was changing. The students appeared to start work immediately, and were eager to hear the next instruction, apart from Jay, who would go to the front of the class to use the pencil sharpener, or go outside to get her books, or she would stare at the furry tail on the end of the pencil that she twirled. Reagan often used close proximity as a strategy to redirect Jay.

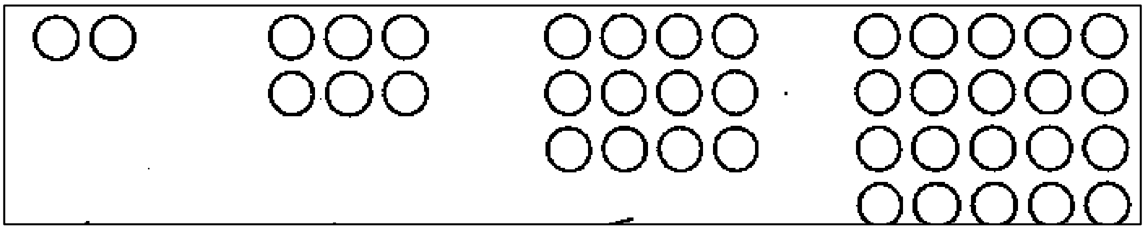


Figure 6.1. Pattern used for lesson on Growing Patterns, taken from Van de Walle and Lovin (2006, p. 267)

Once Reagan was certain that all of the students had drawn and extended the pattern, she asked them to describe how the pattern grew. She was guiding them towards being able to record each step of the pattern as a pictorial representation and as a symbolic/numeric representation. She was now beginning to focus on the mathematics practices as well as evoke their investigative thinking, even though she closely guided their thinking:

T	How could you explain this in a mathematical way? What ways could you represent this pattern? How could you write it so somebody else could see what this pattern was doing without having to draw all of the circles? [She paused, giving the students time to think.]

S1	Every time you add on one row and one vertical [this student looked at her pictorial representation to explain what was occurring].

T	[She nodded.] Could you record it on a table?
[Pause].

T	I will give you an idea. [She drew a table on the whiteboard]:

?				
?				

T What could you write in the boxes? [She pointed to the question marks.]

S2 Oh, right. (T4WK1D2)

This student was deep in thought, as were many others. Reagan asked the students to copy the table next to the pictorial representation of the circles pattern on their page. She then asked them to think about what headings they might be able to use for the table. Two girls looked confused, and turned to each other to chat, although then they looked up and asked for help, and Reagan walked over to attend to their question. At this point, I wondered if the confused look was because the students were unfamiliar with having to think about what headings to use. As Reagan had pointed out previously, they were accustomed to teachers giving them the information, instead of sourcing their own ideas.

By now the students were more familiar with my being in their classroom, and consequently, I felt more at ease to walk around the classroom to observe the students' work. Previously, I had positioned myself at the rear of the classroom to avoid the students feeling as though I was an intruder, as warned by Lankshear and Knobel (2005). As I moved around the classroom, I noticed that Renee had not only written the headings in the appropriate boxes, but she had completed her table, as shown:

Rows across	2	3	4	5
Columns	1	2	3	4

Reagan had not yet mentioned that the students needed to number each step in the pattern sequence. Renee had described how the pattern grew by looking at the columns and rows separately. When I looked at her work, she justified her numerical representations, indicating with her finger which step in the sequence belonged to which cells in the table. This was a teachable moment that unfortunately was missed. At that moment, Renee was called out for extracurricular activities; hence, she missed the discussion to follow and the opportunity to receive her own accolades for successful learning. Had she been able to show her ideas, Reagan may have been able to point out the importance of numbering each step, and then recording the correlating step number on the table.

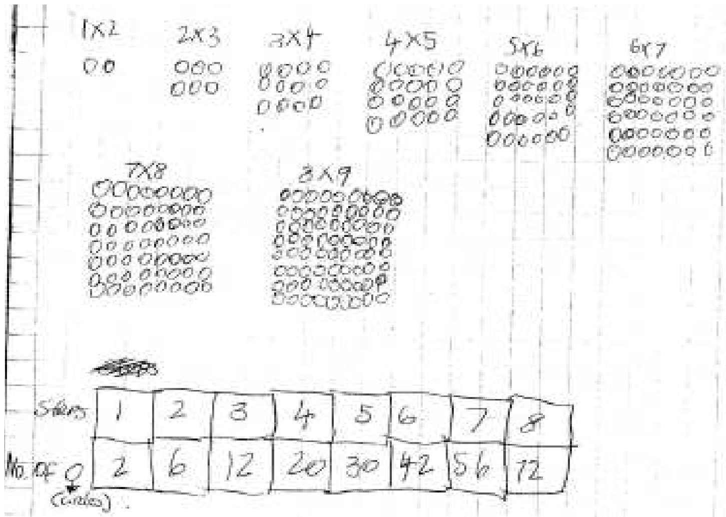


Figure 6.2. Rob's pictorial and numerical representation of the circles pattern.

Rob, another student, had also thought about the pattern as an array (see Figure 6.2). He had labeled each step of the element with a multiplication to explain the array of circles. When I asked him to explain his thinking, he reasoned that the multiplicand was the step number, and the multiplier increased by one more than the step number each time. Clearly, he was able to see a relationship between the pictorial and numerical representation of the pattern. I continued observing the students' work for about 10 minutes, until Reagan asked them to put down their pencils and look to the whiteboard. She pointed out that they must record the step number of the pattern sequence first. Maybe she had observed that many of the students had not numbered the pattern sequence, and thus had not recorded the step number on the table. Reagan numbered the pattern and filled in the table to demonstrate, as below.

Steps	1	2	3	4
?				

Reagan continued the discussion:

T So what would go in the bottom row?

S1 The number of circles.

T Okay, record the step number on your table and the number of circles that is in the pattern for each step, and look to see if there is a pattern.

Can you see a pattern happening to the numbers? [The students followed the instructions, and after a few minutes she stopped them.]

T Any ideas?

S2 They are all going up? [The tone was more a question than a statement.]

T Put you hands up please. [Reagan scanned the room to ensure all students heard her request, and then returned to this student and asked him to elaborate.]

S2 The number you add is going up in twos.

T Can you explain your thinking?

S2 Step 2 is 2 plus 4 to equal 6 [2 + 4 = 6].

Step 3 is 3 times 2 to equal 6 [3 x 2 = 6], then 6 plus 6 to make 12 [6 + 6 = 12].

Step 4 is 4 times 2 to equal 8 [4 x 2 = 8], then 12 plus 8 to make 20 [12+8=20]. (T4WK1D2)

This student was quick to respond to Reagan's question, yet she pointed out that he needed to raise his hand if he wanted to speak, and then he must wait until he was invited to share his ideas. The boy was thinking mathematically. He had figured out that each step of the pattern was the element number doubled, and then added on to the previous step. Reagan did not comment on his ideas, although she paused for a moment after he spoke. I wondered what she was thinking, and I also wondered why she did not allow him to come to the whiteboard to justify his reasoning. By doing this, he may have clarified his own thoughts, and also enabled the other students to visualise his thoughts. Hence, this instance revealed the complex decisions teachers are required to make in the moment. After the brief pause, Reagan turned to the whiteboard and filled in the first four steps of the pattern.

As Reagan completed the table, she explained the connection between the pictorial and numerical representation at each cell. She used a marker to point back and forth between the two representations, to highlight the relationship between the two representations:

Steps	1	2	3	4
Number of circles	2	6	12	20

Reagan looked at the students and asked them to think about how they could make a general rule for this pattern. They had been working for over an hour and were starting to appear tired. Some students got up and walked out of the classroom, and I must have look astonished, because Reagan turned to me and said, “They are going to get some brain food” (T4WK1D2). Apparently, this was a classroom rule that she and the students had agreed upon. The students could eat a healthy snack at their desks if they were feeling tired. They came back, and appeared to be willing to persevere and solve the puzzle, even though they were tired.

Reagan posed the next question, “If I had to work out how many circles would be in step ten, what could I do?” (T4WK1D2). Jay asked for permission to come to the whiteboard to explain her thinking (see Figure 6.3). Reagan nodded, and Jay picked up the marker, mimicking the way Reagan used the marker, and pointed to the pictorial representation on the whiteboard. She stated: “3 across, 2 down, then 4 across and 3 down, then 5 across and 4 down” (T4WK1D2). She glanced at Reagan with a proud look on her face. Satisfied with her contribution, she sat down and proceeded to work on a drawing she had stored under her desk. Jay had also thought about the pattern as an array. Scanning the room to see if any other students were willing to explain their ideas, Reagan noticed Jay drawing, and discreetly confiscated the picture. Jay often drew throughout the class; it appeared as though she needed to keep moving or doing something. I believe she was still listening because she contributed sensible comments or ideas to the discussion. For instance, her numeric representation depicted a row for the element number, as well as two separate rows to record the circles as an array (see Figure 6.3). Many of the other students had not recorded a step number. I wondered if there was any correlation between Jay’s physical and mental activity.

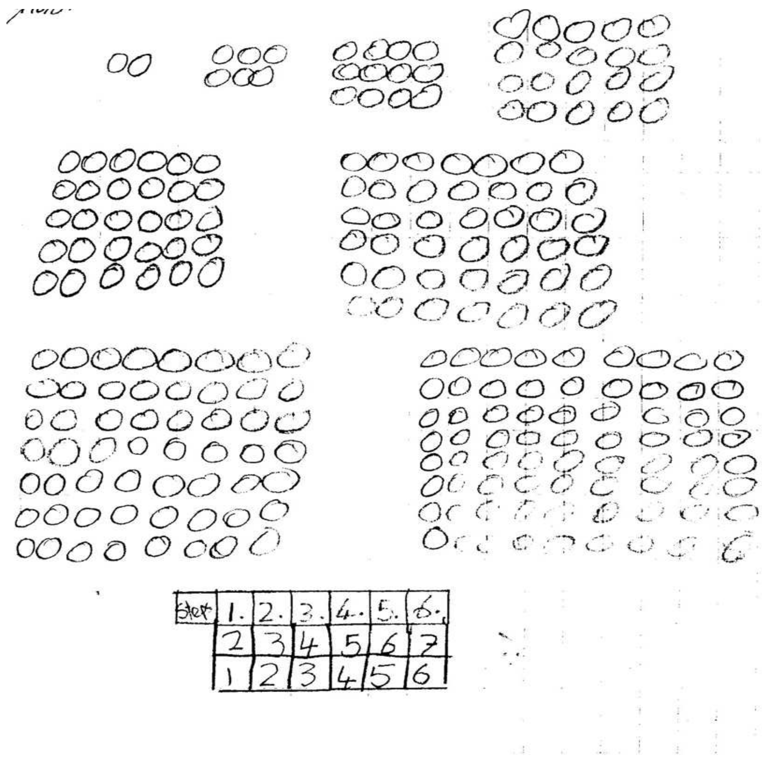


Figure 6.3. Jay’s pictorial and numeric representation of the circles pattern.

Reagan then invited another volunteer to come up to the whiteboard to share their ideas. Previously, she had selected students, either randomly or purposefully, to contribute their ideas. This was the first time that I had observed such willingness from the students to share their ideas. Many had their hands raised, and as each student shared, the others listened intently. The discussion continued:

T Okay anyone else willing to have a go?

Ned [Ned had delight on his face; clearly he was proud, and wanted to share what he had understood about what was occurring in the pattern.]
Well, 2 goes into 6 3times [he pointed to the corresponding numbers in the table with the marker].
3 goes into 12 4 times, 4 goes into 20 5 times, and 5 goes into 30 6 times (T4WK1D2).

Reagan congratulated him on his thinking and he sat down. She invited one more volunteer to share his or her thinking with the class. Interestingly, as Ned justified his ideas, he used a marker to point back and forth to the numbers on the table. He could see a relationship between the step number and the number of objects in the pattern. Reagan informed me after the lesson that Ned was not confident with his mathematical ability and she was proud that he was thinking in a multiplicative way. She believed that the whole class discussion was helping Ned to make sense of mathematics, and thus, as he was feeling more competent, he was simultaneously developing confidence in his ability to contribute knowingly to discussion (T4WK1D2).

Rob, whose work I had observed (see Figure 6.2), raised his hand eagerly. He came to the whiteboard and also picked up the marker to point to the pattern as he spoke:

Rob So we have got 1 by 2 equals 2 [1 x 2 = 2]; then 2 by 3 equals 6 [2 x 3 = 6]; then 3 by 4 equals 12 [3 x 4 = 12]; then 4 by 5 equals 20 [4 x 5 = 20]. So the times [the multiplier] goes up in ones.

T So if I say [Reagan wrote on the board the equation '36 x ? = ?'], what would the 36 be multiplied by?

Rob 37.

Rob was starting to use language that signified he was justifying his ideas. For example, he said, 'So we have got', and then continued by saying, 'Then…'; hence, Rob was indicating that one step related to the next step in a functional way. Reagan extended his thinking further:

T So are we able to make a rule? What does 36 represent? Or if we look here [she pointed to the table], what does the 2 represent?
[Two more students offered their ideas.]

S3 The number of rows.

S4 The number of steps.

S5 So the number of steps multiplied by the next number up.

T Or, what could we call the next number up [She scanned the room and offered a hint]. It starts with an 'o' and sounds like order?

Rob Ordinal.

T [Reagan nodded.] Therefore, could we write it like this?
'The number of steps multiplied by the next ordinal number = the total number of circles.'

T Then if we have 74 steps, what would we do, what we have to multiply 74 by ?
[She paused to enable all of the students to comprehend the question]

T Okay, all of you [Reagan gestured for all to speak].

Ss 75. (T4WK1D2)

By this stage, the students were exhausted; they had worked on this task over the whole block between break periods. Generally, the mathematics lessons were approximately forty minutes, yet this lesson spanned almost two hours. Reagan rewarded the students with a game, a mathematics game called, *Elevens*.

In this lesson, Reagan used a combination of explicit instruction and student investigation. For instance, she posed a question, the students investigated either individually or with their partner, and then she called them in to communicate, represent and justify their thoughts. This vacillating movement back and forth, from demonstration to investigation, happened several times throughout the lesson. For instance, Reagan demonstrated how to draw a table to record the pattern numerically and then she asked the students to think about and investigate appropriate heading labels to apply to the table to record their data. As the syllabus explained, an active investigator can "manipulate concrete materials and make a variety of representations and displays …, to assist their mathematical thinking and reasoning" (QSA, 2004, p. 3). Reagan gave the students time to discuss their ideas with their peers, which aligned with the syllabus statement that: "Learners operate as active investigators when they contribute to, and share ideas about, mathematical knowledge, understandings, procedures and strategies" (p. 3). The students looked at the pictorial representation and then the table, and collaboratively or individually puzzled over an effective way to record what was happening in the pattern.

However, attending to all of the students' investigative thinking was challenging for Reagan. For instance, some engaged in robust thinking, as the examples depicted, yet opportunities to build on these well thought-out ideas were missed. For example, Renee had labeled her table, and immediately filled out the cells of the table. Her thinking and representation presented an opportune moment to point out the importance of recording step numbers. However, at that moment Renee was called out of the room. Reagan sensed that the students had not recorded the step numbers, and she stopped them working for the next explicit instructions. She demonstrated how to record the step numbers on the pattern and on the table. She then asked them what would be recorded in the bottom row and she gave them time to fill in each step of the table. At this point, some had not recorded the circles as a whole group. Instead, they had recorded the circles as arrays (see Figure 6.3) and Reagan did not pick this up in this lesson.

Reagan then asked the students to investigate the numbers that were recorded in the table, and look for a pattern or relationship. She was leading the students to forming a generalisation, although this was not the key feature of the lesson. The key understanding she had focussed on was for the students to understand and recognise that the pictorial and numerical representations of a pattern coincided. However, she gave them the opportunity to search for a generalisation, which I believe she perceived as an opportunity to orient their thinking in preparation for the following lessons. For instance, Reagan finished the lesson by writing the following generalisation on the whiteboard: 'The number of steps multiplied by the next ordinal number = the total number of circles.' (T4WK1D2) She finished the lesson by asking the students to apply the rule to her question about how to determine the 74^{th} step of the pattern sequence, asking what they would need to multiply 74 by to work out how many circles were in the 74^{th} step. During the discussion, she invited the students to the whiteboard to use the representations to justify their ideas. Interestingly, they picked up the marker and used it in the same fashion that Reagan had during her explanations. They were willing to share their thinking, and simultaneously appeared willing to listen to each other's ideas.

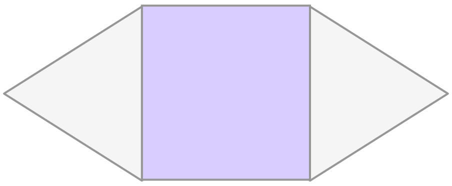
6.3.2 Justifying Functional Relationships

The aim of the next lesson was to develop an understanding that a numeric equation could be written for each step of a pattern sequence. This lesson was a preliminary lesson for the one to follow. I missed the lead in to the lesson because Reagan had asked me to help a student understand maps as part of a test that was to occur the following day. The students had been working on coordinates and directions of maps and this particular student had missed the lesson due to extracurricular activities. I was willing to assist because this was one way I could return the favour for the good will of Reagan and her students' willing participation in this research project, as recommended by Lankshear and Knobel (2005). Nevertheless, as we returned to the classroom she was talking about 'ordered pairs' and highlighted the connection between ordered pairs on the numeric representations recorded on the table from the previous lesson and ordered pairs of latitude and longitude on maps. She had pre-drawn a pattern on the whiteboard that the students were to copy and extend.

Reagan explained to the students that the goal of this lesson was to recognise a relationship between the ordered pairs which were the step number and the number of objects in that step:

T In this activity I want you to see the relationship between two things, the number of steps and the number of objects in that step.

T How many blocks are in this pattern? [Reagan pointed to the pattern on the board as shown]



S2 3.

T [Reagan nodded.] This is a worm.

S3 A book worm?

T [Reagan smiled.] It eats a lot each day and it grows a lot each day, so where could I extend him?

S4 The second block.

T Which is?

S4 The square.

T Yes, I can make him longer. In your books make him grow, show how he extends each day and record the steps on the T-table.

Reagan had drawn the T-table on the whiteboard, yet without headings. The students were to apply what they had learned from the previous lesson to draw a pictorial representation of the growing pattern and to record the numeric representation in a table. She did not provide them with headings for the table.

Instead, the students were to devise their own headings, based on what they had understood from the previous lesson. Two students wanted their ‘worm’ patterns to represent a worm:

S5 It would look more like a worm if it had semicircles instead of triangles for the head and tail.

T [In a playful tone] Oh, how can you discriminate against my worm who is from a minority group.

T Remember, he eats lots every day. Record how he grows and use a table to record it; it is up to you how much he grows. I need to see a pattern. You are to create a pattern. This is individual work today.

S6 Can we make him look more like a worm? [This student also mentioned using colour in the next lesson.]

T If you wish, but you are to focus on the pattern.

S6 The maths.

T Yes, this is not art. (T4WK1D3)

Reagan encouraged individual work in the lesson to assess the students’ understanding as they applied previously learned knowledge. Two of them wanted to make their worm look ‘pretty,’ although, they had also developed a sense that they were to focus on the mathematics. Worksheets could have been prepared for this activity; however, as Reagan pointed out in the interview, she believed that the physical action of drawing representations aided cognitive development and the internal construction of webs of related ideas. She also focussed on helping the students to recognise connections between ideas by highlighting other applications of ordered pairs of numbers, there again helping their cognitive construction of related ideas, as recommended by Skemp (1986) and Hiebert and Carpenter (1992).

The students were given twenty minutes to work individually, while Reagan circulated the room observing their work. She did not interject, other than to redirect the occasional student who was distracted. Then she called for their attention and asked them to share any challenges they had come across as they drew and recorded their growing patterns:

T What challenges did you have?

Liam How to draw it.

T So what was your solution?

Liam Can I draw it on the board [Liam drew day 7 (see Figure 6.4)].

T What day is this?

Liam I don’t know. This is how he is growing.

T What do we need to think about?

Liam The day.

T What about the headings? [Reagan pointed to the T-table.]

S2 I have days.

T Who else has days? [Majority of the class raised their hands.]

S2 You write 1, 2, 3, and 4 in the column under days.

T Yes, what other headings can you put? What are some of the things you have written in your books?

S3 How much he grew.

S4	How many blocks he grew each day.

S5	The worm’s growth.

T	Yes, but can we be more specific?

Liam	I had 1 for the days, 1 for the triangles and 1 for the squares.

T	Oh that sounds tricky, so this was a complex pattern that included triangles and squares. (T4WK1D3)

Reagan acknowledged Liam’s ideas; however, she did not elaborate or work towards clarifying his thinking. In the following lesson, I realised that she was providing Liam with an opportunity to pick up and clarify his own misconception, which did occur, as will be explained later.

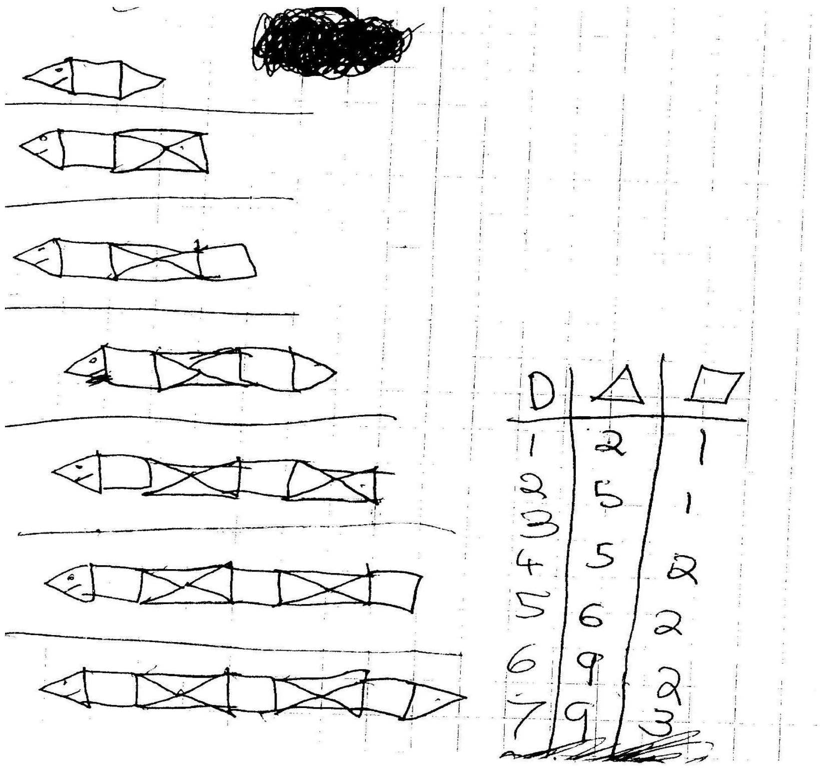


Figure 6.4. Liam’s pattern which was a challenge to record numerically

Reagan asked for another volunteer to share what they had discovered, and a student explained that she had drawn circles and squares. At that moment, the music teacher walked in, and Reagan said, “Oh, here is our music teacher so soon. Let’s finish off quickly. Did anyone manage to come up with a rule?” With that, Tad raised his hand and said, “Yes, my pattern was 3, 7,15,31,63,127, and the rule was, times 2 plus 1 block from the day before = the total amount of growth for that day.” (T4WK1D3) Tad had formulated a generalisation based on the representations he had created, and then justified his generalisation to the class by describing the growing number sequence first. It was unfortunate that the lesson finished so soon because Tad did not get the opportunity to justify his thoughts by using a representation. Hence, the other students missed an opportunity to discuss his ideas and to make sense of Tad's generalisation for themselves.

6.3.3 Formulating Generalisations

The students' pictorial representations of the growing worm were used in this next activity, where they worked collaboratively to formulate generalisations. They organised themselves into groups of five, and then they were taken outdoors. They were required to copy one group member's growing pattern onto the concrete with chalk. Then they were to record the T-table to depict the relationship between the pictorial and numerical representations, yet before going outdoors, Reagan briefly revised the previous lesson:

T What were we doing last time in maths?

S1 We made a worm; at day 1 it had 2 blocks, and it makes a pattern.

T [Reagan selected students, by name, to answer her questions.] How do we make a pattern_____?

S2 I added 2 squares each day.

T How did you record that____?

S2 By doing a [paused]; oh, I forgot what it was called; we recorded the days and how much it grew.

T How did you record it____?

S3 On grid paper.

S4 [A student called out.] On a T-table.

T What were we trying to find by using a T-table, Ned?

Ned It was the number of steps divided by the…[Reagan interrupted, Ned had remembered his reasoning for the circles pattern].

T So we found a rule, so we can see from that what comes next in our sequence and what came____?

S6 Before. (T4WK2D1)

Reagan explained to me that her motive for going outdoors was to add some variety for the students (T4WK2D1); however, this choice also seemed to provide them with some space to think. They were instructed to put the nose of each step of the pattern against a line marked on the concrete to ensure that they could see the pattern growing. Then, as a group, they were to discuss a way to explain (justify) to the class how they could determine the number of objects the worm pattern would require on the 20th day, without drawing twenty steps of the pattern. Hence, together they were required to formulate a generalisation. They were given time to represent their pattern and formulate a generalisation.

After twenty minutes, the class moved around to each group's pattern to listen as they justified their generalisations. There were five groups in total, and each group had the opportunity to show what they had discovered:

T Tell us about your pattern and your rule [Reagan spoke to group one].

S1 Each day it grows by 2, so, we double the day and take 1 [$2n - 1$].

T What does this equal? What are we trying to find out? What is your rule telling us?

S2 The total number of blocks.

T Okay, let's pretend it is day 45 [Reagan directed the question to the class]. Using the rule how big would the worm be on day 45?

S3 89 blocks.

T Do you all agree? Hands up who agrees. [Approximately 70%]

T Why is it 89? Explain your thinking.

S3 Because I doubled the 45 to get 90 and took away 1, so it was 89.

T Is it a good rule? Does it apply to the pattern?

Ss [All students nodded in agreement.] (T4WK2D1)

However, the students in this group were counting the squares only, not the total, which would have included the head and tail. Reagan did not point this out at this early stage of the discussion.

The class moved on to group two, who had used Jannay's pattern (see Figure 6.5):

T Next, tell us about your pattern and then tell us your rule.

S4 Day 1, there was 1 square and 2 circles.
Day 2, there were 2 squares and 3 circles.
Day 3, there were 3 squares and 4 circles.

S4 So, on any day the day number is the same as the amount of squares, and the circles is greater by 1 [$n + (n + 1)$].

T So, on day 27, what would we have? [Reagan directed the question to the whole class.]

S5 We would have 27 squares and 28 circles. [A student from group two answered.] (T4WK2D1)

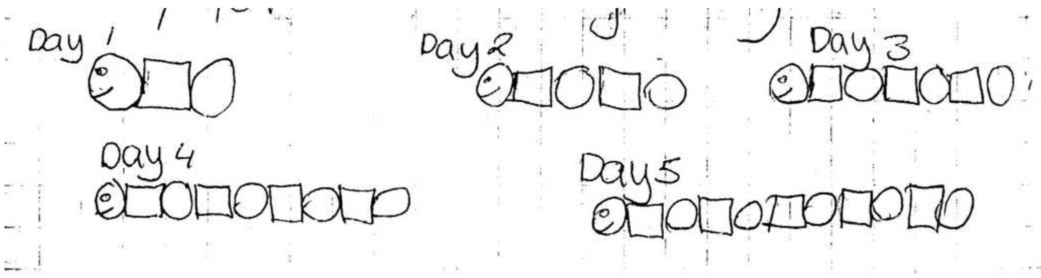


Figure 6.5. Jannay's 'growing worm' pattern inclusive of squares and circles

As group two described their pattern, it was clear that they too had not counted the total number of objects in each step of the pattern. Reagan still did not point this out to the students.

Instead, Reagan moved on to group three:

T Next, tell us about your pattern, and then tell us your rule.

S6 On day 1 it was 3, on day 2 it was 8, then 13, 18, 23.

T [Reagan turned to the whole class.] What is the pattern here? What are they going up in?

S7 Fives.

T So what was your rule?

S8 The number of days, times 5, minus 2, equals the total number of blocks [$5n – 2$].

T If it was day 10, how many blocks would there be?

S9 48 blocks.

T Okay, let's check. [She then applied the rule to each step of the sequence to prove that the rule worked.]

T Well done. (T4WK2D1)

By applying the rule to each day in the sequence, Reagan was demonstrating to all of the students how to prove the generalisation; thus, she modeled how to validate the truth of their ideas.

The class then moved to group four:

T Next group, tell us about your pattern, and then tell us your rule.

S10 Our pattern is 1, 6, 11, 16, and our rule is times the number of days by 5, and take away 4 [5*n* – 4].

T [Reagan turned to the class.] So whose rule is this one like?

Two boys started to get distracted here, and Reagan separated them. It appeared that when the students were being asked to think, some became distracted, while others were inspired; they wanted to solve the problem, as is revealed in this next vignette:

S11 Group 3's.

T What was your rule again group 3?

S6 The number of days, times 5, minus 2, equals the total number of blocks.

T So what is the difference? [Reagan paused and gave the students time to think about the question posed.]

[Several students raised their hands; some waved their arms about enthusiastically.]

S12 I think I know: it is because they counted the middle and left off the head and tail.

T [Reagan nodded.] Okay, on day 15, what would it be with this rule?

S13 71 blocks.

T Do you all agree? [Reagan selected one of the quieter students and gestured for him to respond.]

S14 Yes, well, 15 times five is 65.

T [Reagan quietly corrected the student.] Seventy five; I think in my head 10 x 5 = 50, and 5 x 5 = 25, so 50 + 25 = 75.

S14 So 75 take away 4 is 71. (T4WK2D1)

In this last discussion, the issue of not counting the whole group of objects was addressed. The students had made sense of this idea for themselves, which was clearly depicted by the way they waved their hands about, wanting to answer the question. Reagan had provided an opportunity for them to be "complex thinkers" (QSA, 2004, p. 3). Complex thinking is an attribute of lifelong learning outlined by the QSA, which proposed that by "reflecting on their thinking, reasoning and generalisations about mathematics, learners build on prior knowledge and incorporate new information into existing conceptual structures" (p. 3). The open space gave the students the opportunity to move around, visualise the representations, listen to other students' justifications, and then reflect on these instances. This experience enabled them to recognise the difference between the two generalisations that described the same pattern. This discussion also helped the students when exploring the next pattern.

The following pattern was a pattern with the triangles and squares, similar to Liam's pattern (see Figure 6.4). Liam was in this group, and I observed him listen intently.

T Next group, tell us about your pattern and then tell us your rule.

S15 The pattern on day 1 had 2 triangles, on day 2, it had 9 triangles, day 3, it had 13 triangles, day 4, it had 17, then 21 and 25.

T [Reagan addressed the class.] What is the pattern going up by?

Liam 7.

T Yes on the second day it does. What about on days 3, 4 and 5?

S16 4.

T So somewhere there has been a jumble; there is something wrong with day 2. What is it? What was your rule?

Liam Times the day by itself, and add 1 [$2n + 1$].

T Who can see what has happened here; what is the number sentence? [Reagan persisted until all of the students' hands were raised]

S18 2 times 2 plus 1.

Liam The answer is 5 for day 2.

T Yes, and what would day 4 be? [Again she persisted until all of the students' hands were raised. She then turned to Liam.]

Liam 17 blocks.

T Okay, so look at the pattern. Is that what is on your pattern?

Liam No.

T So the pattern did not repeat itself. (T4WK2D1)

At this stage, one of the students in this group backed away, and said that it was not his pattern. Reagan explained that it *was* his pattern because they all belonged to a group, and, as a group, they were to think about and discuss the pictorial and numeric representations of the pattern together. She continued:

T Your group obviously did not discuss this; you were all responsible to see that the pattern did not repeat itself. (T4WK2D1)

Liam realised his error, and after this instance, he seemed to contribute much more to the classroom discussion. I wondered if he had thought to himself that it was okay to make a mistake because he was in a process of learning, as suggested by Kilpatrick and colleagues (2006).

Through discussion, the students managed to clarify where they went wrong. Reagan was determined that the whole class would follow the discussion, and that they would all become aware of the misconception. As they discussed each group's generalisation, they appeared to make sense of some key mathematical ideas involved in algebraic reasoning. For instance, they realised that all of the objects in a pattern must be included and that patterns repeat themselves or grow in an ordered and logical sequence. Reagan concluded the lesson by asking the students to signal if they believed they had developed, "a good understanding of patterning and creating a rule." (T4WK2D1) Out of the 25 students present, 24 raised their hands. She then asked, "Raise your hand if you are on your way to understanding." Three raised their hands, the one who had not previously, and two others.

6.4 In Summary

This chapter has detailed some of the classroom experiences indicating a gradual take up of thinking and reasoning learning processes. To summarise, the first lesson in this sequence of lessons to develop algebraic thinking focussed on the repeating part of patterns. The students used concrete materials, and constructed a pattern which was then transferred to a pictorial representation in their books. Then, through collaboration with a partner, they reconstructed the patterns and justified how or why their patterns did or did not work. Reagan allowed them to make mistakes, in the hope that these mistakes would provide valuable learning experiences as they justified their representations. Her intentions aligned with the connectionist teachers in Askew and colleagues' (1997) study, who valued students' misconceptions. Like these teachers, Reagan valued the misconceptions as a way to help the students engage in further investigative thinking and discussion to determine what did, or did not, help to reconstruct patterns.

The next activity targeted the development of foundational knowledge that a relationship existed between pictorial and numerical representations of patterns. In this lesson Reagan combined explicit instruction with students' investigative thinking.

However, she closely monitored the thinking. Then the last two lessons built on the understandings from the first three lessons, and also included numeric equations to represent growing patterns. She applied a similar teaching strategy for these two lessons to the one she had used previously. For instance, she provided the students with an opportunity to represent a pattern pictorially, and left misconceptions to be addressed in the following lesson. The follow up lesson was more hands-on, and engaged them in investigative thinking, where collaboratively they used their representations to justify their thoughts and formulate generalisations. It was through the whole class discussion that Reagan posed guiding questions that enabled the students to share ideas and help each other to clarify their own misconceptions. This type of group discussion required persistence and concentration, from both Reagan and the students. However, for Reagan, her role was to ensure that all of them stayed focussed, and to pose thought provoking questions.

The mathematics practices had helped Reagan to facilitate investigative thinking. The data revealed that in the first couple of lessons she was unsure how to advance the students' thinking about mathematical ideas. However, as they became more accustomed to using representations to communicate their ideas, their thinking became more obvious. This practice enabled Reagan to pose thought provoking questions to redirect or extend the students' thinking about mathematical ideas. Thus they were beginning to take up a more active role in their own knowledge construction. Reagan's role was changing too; she was relinquishing her role as the giver of knowledge, and taking up a role as a facilitator of investigative thinking. Her changing role aligned with Killen's (2003, p. 213) advice that the teacher's primary role should be "to help students to learn how to think, rather than teaching them how to remember." There were two more activities in her algebraic sequence of activities and an assessment activity, as will be explained in the following chapter followed by a section on the students' developing dispositions.

CHAPTER SEVEN

INTERPRETING THE LIVED EXPERIENCE:

WAYS OF THINKING AND BEING MATHEMATICALLY

"Successful learning facilitates more successful learning."

(Killen, 2003, p. 43)

7.1 Introduction

The previous chapter revealed that Reagan valued communication and reflective thinking about and with mathematical ideas. She believed that as students reflected on their thinking and communicated their ideas, they were able to abstract ideas and develop cognitive schema that would enable further learning. Her beliefs aligned with the syllabus documents (DETA, 2008; QSA, 2004) and various researchers (Battista, 1999; Hiebert & Carpenter, 1992; Hiebert et al., 2000; Skemp, 1986; Van de Walle & Lovin, 2006). However, as pointed out in the literature review, Van de Walle and Lovin (2006, p. 5) suggest: "We can't just hold up a big THINK sign and expect children to ponder new thought." Even though the syllabus documents (DETA, 2008; QSA, 2004) are adamant that students need to think mathematically to learn, as Kilpatrick and colleagues (2006) suggested, students also need to learn how to think mathematically. This chapter reveals that as the interactions in the classroom assumed the qualities of the RAND Study Panel's (Ball, 2003) mathematics practices, the quality of Reagan's and the students' thinking and reasoning processes improved. The chapter is divided into two parts. The first part reveals how the mathematics practices' framework supported Reagan's attempt to encourage students' thinking and reasoning processes. The second part looks at the changes to the students' disposition from their point of view. As the students participated more in the thinking and reasoning processes, it appeared that their dispositions towards mathematics learning became more productive, as will be discussed below.

7.2 Ways of Thinking

Initially, while Reagan valued the students' thinking, it appeared that she was uncertain how to draw out, or draw upon, their mathematical thinking and complex ideas to further enhance learning. For instance, in the first lesson (T3WK1D2), she modeled her own mental computation thinking strategy: “If I know that 5x3=15, then I could take 15 from 18, and I would have three left over, so the answer would be three, with three remaining.” Continuing, she said, “If you did it different, you can come up and explain it to me.” Whilst Reagan’s intentions were to invite the students to share their thinking strategies, their thinking may have been thwarted, or possibly overshadowed by her ideas, as Van de Walle (2007) warned. Later in this lesson, when two students expressed their ideas about representing division remainders as decimals or fractions, she appeared to be unsure what to do with these complex ideas.

However, as the research progressed, and Reagan altered her teaching practice, by slowly implementing the mathematics practices, she started to purposefully guide the students’ communication and thinking about mathematical ideas. For instance, in the place value lesson, she used concrete and abstract representations to help the students think about and communicate their ideas. She held up corresponding MAB blocks to prompt a student who was justifying her ideas about place value (T3WK2D1). Her prompts helped the student to alter her own misconception. Even though Reagan imposed her ideas, for instance when she pointed out that she visualised a number line as a way to solve mental computations, she was also highlighting that a number line was one way to visualise Menne’s (2001) notion of ten-ness. Helping the students to create visual images in their minds through relating concrete, pictorial and numeric representations became an integral feature of Reagan’s teaching practice, as will be revealed in this chapter.

7.3 Algebraic Reasoning Activity Sequence

The algebra activities were implemented by Reagan to sequentially develop foundational understandings that would enable the students to think and reason algebraically. For instance, as discussed in the previous chapter, the first activity focussed on growing patterns. The students constructed a pattern and transferred it to a pictorial representation to be reconstructed by a partner. Reagan enabled them to make mistakes that were later clarified when they justified their representations. The next outdoors activity was intended to develop relational understandings about pictorial and numerical representations of patterns. In that lesson, she expected them to construct a ‘growing worm’ pattern pictorially (on the concrete), record the pattern on a large T-table for the class to see, and then as a group justify how their pattern grew at each step.

This lesson helped prepare the students to formulate generalisations. Then in the circles pattern lesson to follow, she combined explicit instruction with the students' investigative thinking to help them describe patterns numerically. The last two activities, which will be discussed below, were intended to assist them to formulate generalisations, and apply variable notation.

7.4 Combining Explicit Instruction and Investigative Thinking

Throughout this next lesson, Reagan combined explicit instruction and investigative thinking. The orientation of the lesson involved a revision of what the students had learned up to this point. They were given five minutes to ensure that they had completed the previous task, and during these five minutes, she handed out sticks to the students for the activity to follow. As Reagan moved around the classroom, she was also checking on the students' progress. Then she spent a couple of minutes explaining an activity to Shaun, a student with an intellectual impairment. This was the first time he was in the classroom while I was present; usually a mathematics learning support teacher took Shaun to another room. After another scan of the room to look over the students' work, Reagan instructed them to copy the matchstick pattern, and number each element (see Figure 7.1). While she returned to Shaun, two boys disturbed other students as they played with the sticks. She left Shaun to speak with the boys, and realised, when she looked at their books, that they were waiting for the next instruction. She spoke quietly to them, and then instructed the whole class to draw a T-table, and record the pattern. Once this was completed, they were asked to explain to each other what was changing about the pattern and how it was growing (T4WK2D2).

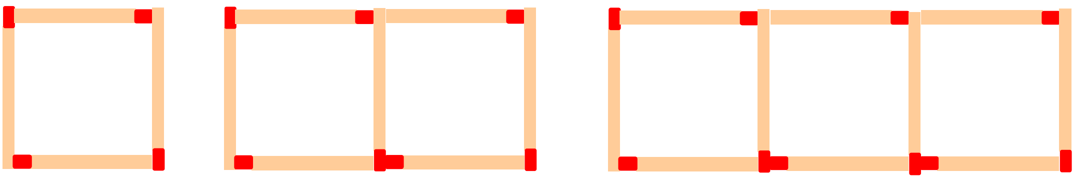


Figure 7.1. Matchstick Pattern (as cited in Klein, 2006)

The students collaboratively discussed ideas, while Reagan worked with Shaun. I observed some students use the matchsticks, while others used their own T-tables. Some worked in pairs, and others worked as a group. I also noticed that they were helping each other to draw the pattern, and then record the steps. In a couple of the groups, one student seemed to take the lead, and others questioned and followed. After five minutes, Reagan returned to the whole class to discuss what they had achieved up to this point, throughout this discussion she selected students to respond by using eye contact:

T Okay, so what happens?

S2 We add 3 each time.

T Who agrees?

T Alright, so how many do we have in step one?

S3 4.

T What about step two?

S4 7.

T And step three?

S5 10.

T What about step four?

S6 13.

T So how is the pattern changing?

S7 It grows horizontally.

T Good language and use of words. What direction?

S7 Right.

T Now, can someone give me their description? (T4WK2D2)

Whilst most of the students raised their hands, when Reagan asked the students who agreed that the pattern grew in threes, she looked concerned. I assumed her concern was because some students had not raised their hands, and she was curious about why they could not see the pattern's growth. Hence, I believe that she posed the step by step questions to ensure that every student had recorded the pictorial representation of the pattern, and numbered the steps. Then, when she asked for a description of the pattern, there was no response. It seemed that the students did not know what she meant.

Hence, Reagan drew a blank T-table on the whiteboard, and explicitly guided the students to record each step of the pattern numerically. After each question, she paused until many students had raised their hands. Reagan purposefully selected students who did not have their hands raised to respond:

T What could I put as a heading on this side of the T-table?

S8 Number of sticks.

T What about here?

S9 Number of days.

T No, not days today; we are not recording days.

Nick Circumnavigate.

T Oh, you are about two hours behind; welcome to this lesson though.

T Who knows? [At this point, several students had their hands raised, waving them about to capture Reagan's attention.]

S10 Elements or steps.

T Yes, elements and steps are the same thing here. Could I say that when I go home I am going to walk up the elements? [The students laughed.]

T Okay, individually fill in your table, and then, together, see if you can write a rule. If you can do that, I want you to write a numeric equation for each step. What is a numeric equation? What does this mean?

T What does it look like? [She wrote the phrase, 'numeric equation' on the whiteboard.]

S11 Number.

S12 Is it like a number story?

T Yes, it is a number sentence. (T4WK2D2)

Reagan circulated the classroom to ensure that each student had both the pictorial and numeric representations recorded. She then encouraged the students to use the sticks on their desks to represent the pattern, and to assist communication of their ideas. However, maintaining their attention was a challenge for Reagan. For instance, Nick's blurting out of the word 'circumnavigate' indicated that he had not yet switched his mind to mathematics; he had still been searching for the word from the previous lesson. Also, in this activity, I noticed Jay draw pictures next to a story she had written, totally unrelated to mathematics.

Reagan often let the students choose how they wanted to investigate ideas, either individually, in pairs, or in groups. I observed an industrious group of five boys who constructed the pattern physically, and then individually recorded each step. These boys often worked together, and Tad, who was one of the more advanced students, according to Reagan, often led the group. After twelve minutes, it was time to discuss what the students had discovered:

T Let's share what you have found. What have you started with in the T-table?

Tom [Tom asked to come to the whiteboard, and he drew his T-table as below.]

Steps	Sticks
1	4
2	7
3	10
4	13

T Have you all got this in your books? [The students nodded.]

T Alright, what have you done to get from four to seven?

Tom Added three.

T Okay, give me the numeric equation for step four. [She directed the question at a student, who had been part of the group of five boys.]

Nick I did not do that.

T Should you have done it?

Nick Yes.

T Have the others in your group done it?

Nick Yes.

T The rest of the group should be making sure that each student does the task. (T4WK2D2)

Reagan had spoken to Nick previously, yet he appeared to be struggling to stay focussed in this lesson. From the outset, his mind was preoccupied with the earlier mapping lesson.

Nonetheless, Reagan then walked around the classroom to determine whether anyone had written a numeric equation next to each step on the table. Up to this point, she had explicitly instructed the students to record the pattern numerically and pictorially. She now wanted them to think about how to write a corresponding numeric equation for each step. Reagan had not demonstrated how to write a correlating numeric equation with each step of the pattern because, as she revealed to me after the lesson, she intended to encourage investigative thinking. She envisaged that through collaboration with a peer, the students would think and reason about how they could represent what was happening at each step with a numeric equation. However, it appeared that they had focussed on formulating a generalisation; they seemed to be inspired by puzzling over what the generalisation might be.

7.4.1 Using Representations to 'Puzzle It Out'

Sensing the students' curiosity, Reagan altered the focus of discussion to align with what they were interested in investigating at this juncture. She encouraged them to use their representations to help them justify and communicate their ideas:

T Did anyone think about a rule?

Guy The number of elements plus one, times three, equals the total number of sticks. [Reagan wrote the number sentence on the whiteboard.]

T So, I should be able to work out step 3. Let's see. [She wrote on the whiteboard.]

3+1 x 3= ?

So, 4 x 3 = 12, is this correct?

Guy No [self correcting], it should be 13.

T So, what is wrong with this numeric equation?

Tad It should be times 3 plus 1.

T Let's check, if it was step 25 what would it be? [Reagan invited Tad to come to the whiteboard and talk the class through his thinking. He wrote the equations as he spoke.]

Tad 25 x 3 + 1, if 20 x 3 = 60, and 5 + 3 = 15, then 60 + 15 = 75, and so 75 + 1= 76. So the total number of sticks would be 76.

T Who agrees? (T4WK2D2)

They all raised their hands, except Shaun, who was still working on his activity Reagan had given him; he was matching clock faces to time. Reagan wrote the rule on the whiteboard: "Rule: the number of elements x 3 + 1 = the total number of sticks" (T4WK2D2). The students were to write this in their books if they had not already written a generalisation.

The complexity of teaching was evident in this short discussion. The teaching decisions that are to be made in the moment can be challenging. For instance, Guy, who was one of Reagan's more timid students, had drawn on his courage to come to the whiteboard and explain his ideas. Reagan used Guy's example to model how to justify and prove rules. Guy realised that something was not quite right about his rule when he self corrected. He was puzzled, and appeared deep in thought when Tad, who could see what was happening, interjected. Guy, therefore, missed the opportunity to clarify his own misconception. Further thought provoking questions may have enabled Guy to see that he needed to add the constant ‘one’ after he had multiplied the element number by three.

Nevertheless, the students’ communication of their ideas was changing. For instance, Tad used an ‘if…, then…’ statement. He used symbolic representations as he justified his thinking, and finished off his reasoning with a concluding statement. Hence, he used the mathematics practices to communicate his thinking and reasoning processes. Reagan looked at me with a smile, and then directed a question to the students. Her aim, it seemed, was to evoke more investigative thinking. She asked, “What about the reverse? If I had to work out the step number, which is the element number, what would I have to do?” (T4WK2D2). Three students raised their hands, and Reagan acknowledged them with a smile, although she encouraged everyone to investigate and record their thoughts in their books. She said, “If you can do this, you will be doing grade seven work. Think about what information you will need.” (T4WK2D2) She gave them a few minutes to discuss their ideas and then asked if they were ready to share. No hands rose, which indicated the students were still intent on figuring out the problem.

Hence, Reagan gave the students more time to puzzle it out. She assisted their inquiry by highlighting the connection of this problem with their previously discovered rule and with reverse operations. She also modeled investigative thinking processes by using an ‘if…, then…’ statement. She said, “If this is our current rule, then for the reverse could I write: ‘the number of sticks______ = the _______ number’?” (T4WK2D2). Reagan encouraged the students by saying:

> I know that you can work this out. There is enough information on the board or in your books for you to be able to work this out. Remember, to think about the reverse, you will need to think about the reverse of operations. What is the reverse of multiplication? What is the reverse of addition? (T4WK2D2)

Reagan’s invitation to extend the students’ thinking indicated that she was beginning to feel more comfortable discussing the students’ thinking strategies than in the initial stages of this research. Her statement, ‘I know you can work this out,’ revealed that she valued and respected the students’ participation. She walked around the classroom observing the students as they worked and making herself available if they needed assistance. Some students discussed their ideas while others worked individually.

7.4.2 Using Justification to Test Ideas

In this next part of the lesson, Reagan's intention was to help the students to understand how to apply and test rules. She had sensed they were ready to move on (they were getting restless), and continued with the discussion:

T What is the rule? [Reagan selected a student who did not usually raise his hand. He came to the whiteboard, and wrote his rule, as below.]

S18 Rule: the total number of sticks, minus 1÷3 = the element number.

T Who believes this is the rule? [Some students raised their hands.]
How can we test it? Let's apply the rule. If we have a total of 13 sticks, thirteen minus one is twelve, then twelve divided by three is four.
Check the table, did the rule work? [They all agreed.]

T Now I will use a letter to replace a phrase, where could a letter be?
What are we trying to find out?

S19 The element number.

T Can I do this then? [Reagan wrote on the board '*n* – 1 ÷3 = 4']
What would the symbol '*n*' represent?
Choose the rule this number sentence applies to and then do the sum.
(T4WK2D2).

Here Reagan was guiding the students' thinking about how to apply a generalisation, in other words, how to solve an algebraic equation. Some of them could apply the rule to solve the equation. However, others struggled, even though Reagan had just given them the answer. Thus to help those who were still struggling, she invited the ones who believed they understood how to apply the equation to peer tutor.

The students who did understand were eager to help their friends. I believe they were proud of what they had learned, and wanted to show others what they understood. Gian approached me, and asked if she could tell me about her rule. I sat with her while she used the T-table and the pictorial representation to guide her through her thinking and reasoning process. She appeared proud that she had made personal sense of the algebraic equation, and subsequently, hurried off to peer tutor. Then she returned, and asked, "How do I explain it?" (T4WK2D2). She found it more difficult to explain her thinking to a fellow student who was confused than to someone who understood what she was talking about. It became apparent that Gian was being provided with an opportunity to clarify her own understanding, as Van de Walle (2007, p. 220) states: "*If you use it, you must understand why it works and be able to explain it.*"

After ten minutes, Reagan facilitated the final discussion to distill the key ideas of this lesson. She had been focussed on helping the students to recognise the relationship between the algebraic equations and the numeric or pictorial pattern representation:

T We have 2 rules, rule one will work out the total number of sticks, and rule two will work out the element number.
So, in this number sentence [$n - 1 \div 3 = 4$], what did the '*n*' represent? [Reagan waited for all hands to be raised, and then acknowledged Emma, who did not usually contribute to the discussion.]

Emma The number of sticks.

T Yes, so what rule could we use to work this out if we had to reverse it?

Emma We would use rule number one, four times three is twelve, plus one is thirteen, so *n* is thirteen. [Reagan smiled at Emma and nodded.]

By recognising this relationship, Reagan was preparing the students for the next activity, where they would be required to formulate their own numeric expressions.

Reagan then concluded the lesson with a reflective moment to determine how the students believed their own learning was progressing. She asked:

T Who feels pretty confident when working with this, not fully, but close to halfway there? [Approximately half the class raised their hands.] Who feels fairly confident? [The remainder of the class raised their hands.]

T Okay, raise your right hand, place it down your back, now pat yourself on the back; you have been doing grade seven work. (T4WK2D2)

After the lesson, Reagan approached me, and she looked pleased. She said to me that the lesson had 'gone well.' She believed that the students were taking a more active role in the classroom discussion, and were becoming more intellectually engaged with the activities. She attributed the increased participation and engagement to their developing confidence in their own mathematical abilities (T4WK2D2). In other words, she believed that they were sensing that they were capable of thinking and reasoning mathematically. Her thoughts reflected Kilpatrick and colleagues' (2006) perspective, that when students view themselves as being in a process of learning mathematics, whereby their mathematical ability is not fixed, they are more willing to engage, and less likely to be discouraged by failure.

7.4.3 Formulating Generalisations

Reagan had now reached a point in her learning sequence where her goal was to assist the students to understand ‘functional relationships.’ Her aim was to help them record a numeric equation for each step of the pattern, and then introduce a “functional variable expression” (Van de Walle & Lovin, 2006, p. 269). As Reagan revised the previous activity, she remembered that most of the students had not written a numeric equation beside each step, which had been her initial intention in that lesson. Instead, they became engrossed in puzzling over the generalisation. This activity was intended to help them symbolically record the generalisation. Again, Reagan combined two methods of instruction where she explicitly instructed, and then encouraged investigative thinking, similar to her approach in the circles pattern lesson (T4WK1D2).

However, this time Reagan used the students’ thinking and reasoning to guide her explicit instruction. She was about to get started with the discussion when she noticed Jay fidgeting and disrupting others. Jay had many things on her desk besides her mathematics book, which had to be collected from her bag. After attending to Jay, Reagan asked a student to demonstrate how to write a numeric equation for each step, and through this demonstration, Reagan explicitly pointed out the relationship between the equations and each step of the pattern:

T	Now we are settled, how did you describe your equation?
S1	To figure it out that ‘*n*’ minus one, divided by three was four.
T	Yes, we did that yesterday, but remember what were we looking for?
S2	The total number of sticks.
T	Everyone look at your books, look at the table, and look at the headings. What are they?
S3	Element number.
S2	Total number of sticks.
T	So what was the rule to work out the total number of sticks?
S4	The number of elements, times 3, plus 1.
T	So what would the equation look like for that rule? [Reagan selected a student to come to record his thinking on the whiteboard.]
S4	1 x 3 + 1 = 4 [He wrote next to step one on the T-table.]
T	Yes, that is for the first step. What would the second step look like?
S4	2 x 3 + 1 = 7 [He wrote next to step two.]

T	And the third?
S4	3 x 3 + 1 = 10 [Reagan nodded, he sat down, and she turned to the class.]
T	Check your table to make sure you have this written beside the table. (T4WK2D3).

The students were required to write the correlating numeric equation for each step, and then discuss their ideas about how the equations related to the pictorial representation of the pattern. As I moved around the classroom I noticed that Lou had already written an equation next to each step of her T-table, yet had not related to the generalisation (see Figure 7.2). She appeared to be still thinking about what happened to the element number to get to the total number of objects. After the demonstration, she wrote the correlating equations, and then 'well done' above the equations. Note that the brackets were added after the next discussion. The cognitive step for Lou to understand the relationship between her first row of equations to the second required deep analytical thought, and I wondered if she understood the relationship. I also wondered how Reagan could assess each student's cognitive development; in other words, how could she ensure that each student in her classroom understood the relationship between '1 + 3 = 4' and '1 x 3 + 1 = 4', or '2 + 5 = 7' and '2 x 3 + 1 = 7.'

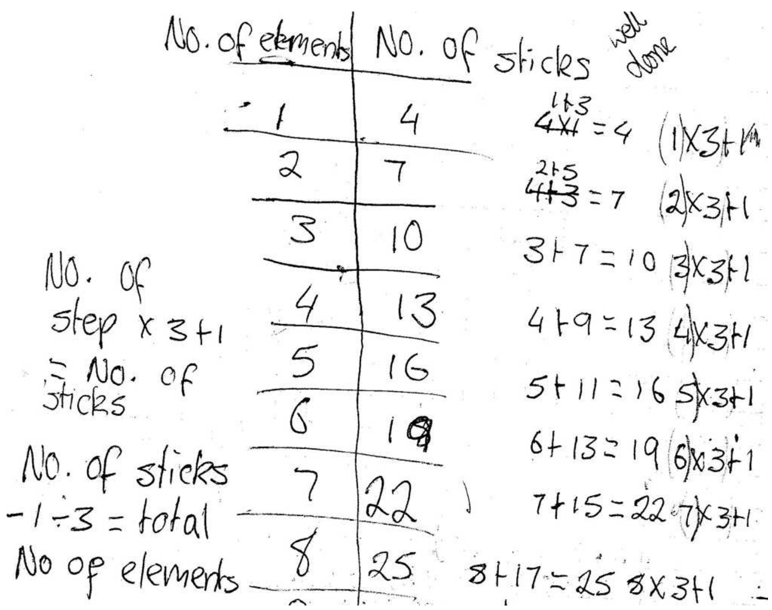


Figure 7.2. Lou's numeric equations to describe the match stick pattern.

Whilst the students wrote down the numeric equations, Reagan attended each group to check on their progress. She sat with each group and listened to their thoughts about what they could see occurring in the pattern. Jay at this stage was still busy doing other things besides mathematics; however, she did not distract the other students. Reagan quietly redirected her many times, although, for some reason, Jay persisted with her own activities. After ten minutes, Reagan resumed the class discussion:

T Now tell me, what do you see in these equations? [She paused for a minute and then nodded at a student to respond.]

S4 The answer goes up in threes.

S5 The times 3 plus 1 is the same.

S6 The ordinal numbers [the element numbers] change each time.

T Good. What are two parts of the rule that change each time? [Reagan selected a student who rarely contributed to discussion.]

S7 The number of elements. [Reagan smiled, and then asked another student who also rarely contributed to discussion]

S8 The total number of sticks. [Again she smiled and nodded.]

T Okay, I want all of you to put a bracket around the part that changes; discuss this with your partner. (T4WK2D3)

Reagan had given the students an opportunity to think about the bracket placement. As I moved around the classroom, sitting in different areas so as to avoid intrusion, I noted that many students had correctly placed the brackets (see Figures 7.2 & 7.3). It was possible that when they communicated their ideas with their peers, they clarified their thoughts. For example, I noticed in Gian's work (see Figure 7.3), that she had accurately represented the pattern with words, pictorially and numerically; she had also recorded an accurate general rule. Her work was clear, which may have resulted from the previous day when she had to explain her ideas as she peer tutored. I also noted that Jay, even though she appeared to be distracted, had written the numeric equations and correctly placed the brackets around the 'part that changes.' On the other hand, two more attentive students were slightly confused; they had bracketed the rule that was written in words, such as: 'Rule: **(**the number of elements**)** x 3 + 1 = **(**the total number of sticks**)'** (T4WK2D3). The two boys did put the brackets around the parts that change, although these were not the numerical parts. Reagan missed this misconception, although I observed the boys place the brackets around the numerals during the discussion. Then another student, Jon, placed the brackets around the whole numeric equation, but this was not picked up in the lesson (see Figure 7.4).

At this stage in the research, I realised I was in a special position compared to Reagan. I had the opportunity to observe all of the students' work, whereas she on the other hand, did not have this opportunity. She had a multitude of tasks to focus on, such as observing and assessing whether all students were keeping up with the discussion, were socially and mentally engaged and were on task. She also had a multitude of things to think about, such as what the students were understanding or not understanding and how, where and when to probe students' investigative thinking, as well as what learning might need to be addressed in follow up lessons, and how and why. Hence, Reagan did not have the overview that I did, and therefore, picking up students' misconceptions as they occurred seemed to be a challenge.

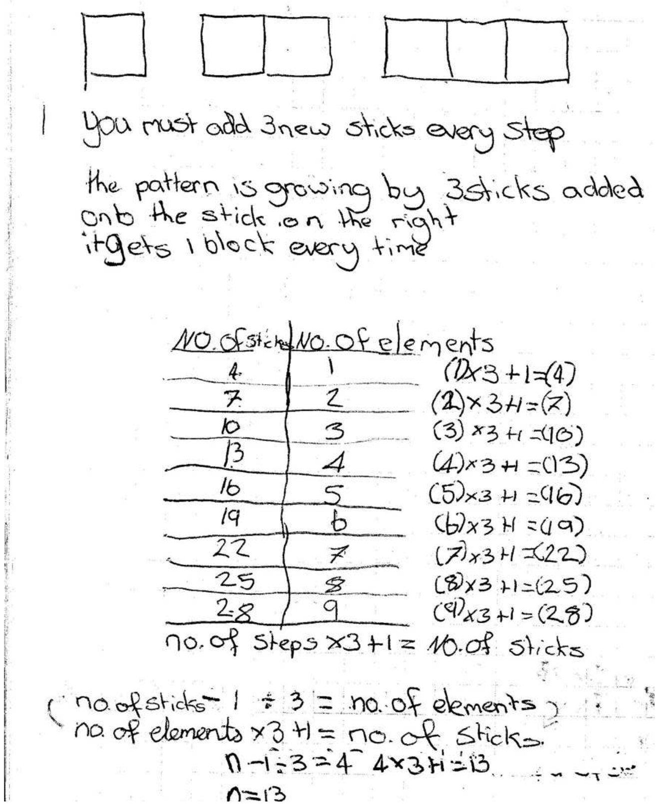


Figure 7.3. Gian's indication of the functional variable.

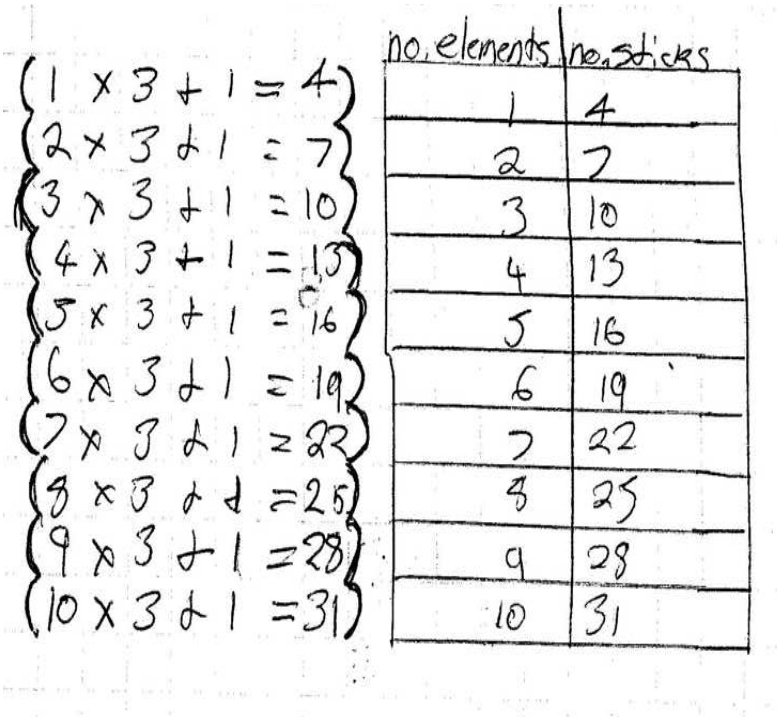


Figure 7.4. Jon's misconception about bracket placement.

7.5 Group Discussion to Investigate and Clarify Ideas

This section reveals the value of discussion. First, an impromptu group discussion helped Craig clarify his understandings, and second, the whole class discussion assisted Ryan. Reagan introduced and defined the term ‘pronumeral.’ She then asked the students to indicate where they thought a pronumeral, or variable, could be used in the equations:

S9 The number of sticks and the element number.

T What letter could I use?

At this stage Reagan saw Jay cutting pictures under her desk and Reagan discreetly confiscated her scissors and put the paper in the bin. It appeared that when Jay was expected to think, her hands became busier. Interestingly, Jay seemed to be keeping up with the lesson, as is revealed below:

S10 You could use ‘*n*’.

T Who believes I can use ‘*n*’? [All raised their hands, including Jay.]

T Could I put the ‘*n*’ on either side of the equals sign? [Reagan wrote on the whiteboard: ‘*n* x 3 + 1 = *n*’*)*

S [All of the students said, “No,” again, including Jay.]

T What about this? [Reagan drew a flower and heart.] What is the flower representing, and what is the heart representing?

Jay The heart is representing the number of flowers, no, the number of sticks.

T Could I put this? [Reagan wrote 'a x 3 + 1 = b' next to element two.] (T4WK2D3).

The students nodded, and she asked if she could use the letter '*b*' as a pronumeral. Craig called out, "No, because *b* is not the 7th letter" (T4WK2D3). Reagan did not hear Craig; however, through discussion he resolved his own misconception, as will be revealed.

Reagan asked the students if they could use the letters '*a*' and '*b*' as pronumerals, and all students raised their hands, excluding Craig. She then asked whether the '*a*' and '*b'* could be exchanged with flower and hearts, Craig was puzzled:

Craig No, '*a*' is the first letter. [Craig tried again to make his point.]

T Does it matter if I change the letters?

S11 No, because it is the thing [symbol] that stands for the number.

S12 How would they know what the flower means and what the heart means?

S13 Couldn't they just look at the equation?

S14 And I guess they could just look at the rule.

S15 It still represents the number of elements.

S16 But don't you have to use *a*, *b*, *c*, *d* for high school maths.

[There was a knock at the door and a student arrived with a message for Reagan which required her immediate attention.]

T Discuss this for five minutes. Will it matter if I use letters or hearts and flowers? (T4WK2D3)

The interactions were now moving beyond teacher-student interactions. The classroom atmosphere had altered. Generally, the discussion involved a dialogue between Reagan and individual students, but now the students seemed to be listening to each other's ideas, reflecting on those ideas and responding accordingly. However, Craig was still confused; he believed that the value of a letter which was being used as a pronumeral should correspond with its position in the alphabet. His confusion had not been noticed by Reagan or his peers. Another student drew on his existing perceptions about high school mathematics and algebra.

Whilst Reagan attended to the message, the students were all engaged in discussion. Craig took the opportunity to discuss his confusion with his peers. Craig was one of the five boys who often worked in a group together. I listened discreetly as they spoke:

Craig So it should be ‘*a*’ for one and then ‘*d*’ for four, not ‘a x 3 + 1 = b’ .

Tad Well, if you did that, then in step 20 you would have used up all the letters of the alphabet.

Craig Oh, we would run out of letters, there wouldn’t be any left to use.

Craig could see that the pronumeral was not part of a code. I also observed Jay work alone; this day she seemed particularly withdrawn and distracted. Without knowing Jay personally, any reasoning would be speculative. Maybe she was distracted when she felt extended mathematically, or the mode of delivery did not capture her attention. Or maybe Jay came to school that day with personal problems. Or it was possible that she was still thinking, even though her hands were busy because she appeared to know what was happening in the lesson.

Reagan returned to the students and asked each group to share their thoughts. Craig’s group was the first to share, and he spoke with a smile on his face as if to say, ‘Now I understand where I was confused.’ He said:

Craig I thought that ‘*a*’ should equal one, and ‘*b*’ should equal two, and ‘*c*’ should equal 3, but I would run out of letters.

T So you approached it like a code. [Craig nodded.] What is the other problem?

S17 We need to keep the symbols the same because the symbol represents the unknown value, so if it were a code, it would have a value, and the symbol is for the unknown value. [Reagan nodded]

S18 But I thought that ‘*a*’ could equal 25, or 36, or 3, or any number because, like a triangle, you see $a + b = c$.

T Yes, the symbol is for the unknown value, but Craig identified a problem; if it were a code, it would have a value. (T4WK2D3)

The student referred to the triangle because the *Maths at School* journal had used shapes as pronumerals, although clearly she had understood that a pronumeral represented an unknown value. Craig’s clarification of his own misconception confirmed the value of communicating one’s thoughts, as recommended by various researchers (Askew et al, 1999; Clarke et al., 2002; Hiebert et al, 2000; Van de Walle & Lovin, 2006). As Craig discussed his thinking with his peers, he was able to make sense of things for himself, which in turn assisted his own knowledge construction.

The discussion continued, and a student pointed out how the symbols that were used on the T-table (on the whiteboard) could be confusing. Reagan had used a flower and a heart as pronumerals in the equation for step one, and then the letters '*a*' and '*b*' for the step two equation. This girl said, "I thought it would be confusing for people if you used flowers and hearts because '*a*' and '*b*' represents something else" (T4WK2D3). She had recognised that the pronumeral needed to represent the same variable, which demonstrated deep mathematical thought. Another student then asked:

S19 How do you work it out if you don't have a table?

T What is the table doing for us?

S20 Showing us the number of sticks and the element number.

S21 It is giving us information.

T Yes, it is giving us information about the pattern; instead of always drawing the pattern to see the next step, we use the table to represent it. (T4WK2D3).

Reagan reminded the students that the T-table and equations replaced and represented the pictorial representation of the pattern. The reminder was timely because the students had worked on the equations so much that maybe this student (S19) had forgotten what the equations were representing.

Discussion helped Ryan to clarify his understandings. He had been slightly confused in the lesson where Reagan reversed the formula. He had been listening intently, and appeared deep in thought, reflective thought. He had an insightful moment in this discussion where sense was made, and excitedly interrupted Reagan:

Ryan What if you knew what the sticks are, but not the elements. Would you do a turnaround?

T Yes, [smiling] that is what we did yesterday, so we can do the reverse. (T4WK2D3).

Reagan glanced across at me at this point with a smile; she appeared pleased that the discussion had helped Ryan. As he listened to the students expressing their mathematical thinking, his own thinking became clarified.

7.5.1 Reflecting on Learning

However, Reagan was also aware that not every student had reached the same level of understanding. She positioned herself in the middle of the classroom, sat down at a desk where she was at the same level as the students, and asked, "Who feels lost?" (T4WK2D3). Reagan's positioning related to Groundwater-Smith and colleagues' (2003) advice about effective communication. They suggested that teachers need to pay attention to their tone and body language to establish an open rapport with their students. Clearly, she was aware of both tone and body language as she sat in a non-threatening position, level with the students, and spoke gently. Eight students raised their hands, of which seven were confused about the inclusion of the hearts and flowers. Maybe Reagan's attempt to be creative confused the students; however, further explanation clarified that the hearts and flowers were merely symbols she had used to represent the unknown value. The last student expressed his concern:

S26 Well I don't know where we can use this stuff outside the classroom.

T Okay, can anybody help out here? Where can we apply this knowledge outside of the classroom?

S27 To work out how much material we need to make a wall, or if we made a horse stable, how many elements and how many sticks, bits of wood would we need to build the stable.

S28 When you make clothes, how much material would you need.

S1 Carpet laying.

T How?

S1 I don't know how, but I know you would use it for that, I can't explain it.

T What if we knew the room was 15 x 25–what is the unknown value?

S1 Oh, how many square metres.

T Yes, that is algebraic thinking.

S2 Building an apartment. You need to work out how much glass you would need for each level.

T Yes, we will have to hold it there, our maths classes are getting longer.

S3 Yeah, we have been going for over one and a half hours. (T4WK2D3).

The time had extended way beyond the usual allocated time for mathematics lessons at the start of this research; hence, the last student's surprise when he realised how long they had been working mathematically. However, what was interesting in this last discussion was the students' view about the relevance of algebraic thinking and reasoning. Their thoughts indicated that they saw the relevance in the mathematics they were learning, even though Reagan had not made the learning relevant, so to speak. In other words, she had not tried to create a relevant, real world situation from which to springboard the learning experiences; rather the mathematics became relevant to the students when they reflected on where in their daily lives, or future lives, they could apply these mathematical understandings.

In summary, the students and Reagan were beginning to use language that signified that they were applying the mathematics practices, such as using 'if…, then…' statements, as well as concluding their investigative thinking with a 'so' statement (T4WK2D2). These statements were especially evident when Reagan asked them how to determine the number of objects if they knew the step number. They seemed to be more involved and interested in the learning process as they persevered and tried to figure out the 'reverse rule' and its application (T4WK2D2). She had challenged their thinking possibly because she was inspired when they started to discuss their ideas more freely. Interestingly as the students justified their ideas, Reagan appeared to pose more 'What if?' types of questions, which then encouraged them to pose questions and validate the truth of their own ideas. This justification process aligned with DETA (2008), who suggested that as students work mathematically, they develop the ability to pose questions, check, verify and communicate ideas. The students resembled 'little mathematicians,' as Reagan referred to them, feeling secure and licensed to contribute effectively to the discussion. The process also reflected Kilpatrick and colleagues' (2006) view that when students sense they are in a process of learning, and that their mathematical abilities and ideas are not fixed, they are less likely to fear failure and more likely to initiate their own mathematical inquiry.

Sense was also made when the students discussed ideas with each other. For example, Craig had been dependent on Reagan to clarify his misunderstanding about pronumerals, and through his own inquiry with a peer, he altered his misconception (T4WK2D3). Discussion and reflective thinking also helped Ryan make sense of reverse operations for himself (T4WK2D3). Ryan and Craig were developing a sense that they were capable of thinking and reasoning mathematically. The last lesson in Reagan's sequence involved an activity that required the students to apply all that they had learned so far, in order to determine the successes of their learning.

7.6 Assessment of the Learning

Reagan was aware of the students' 'developing' confidence, and to minimise fear, she did not let them know that they were being assessed in this lesson. Instead, she revised what they had learned up to this point to ensure that all of the students believed they understood what a pronumeral was, and where it could be applied. Reagan asked them to recall where they had used a pronumeral in the work leading up to this lesson and they were able to provide many examples. She explained that the next activity required them to use the knowledge they had learned over the last few weeks to help solve a pool builder's problem. As Reagan turned and wrote on the whiteboard, the students started to chatter; seemingly, this was nervous chatter. Even though she had not indicated that this was an assessment lesson, they sensed it was. They were required to draw the following pattern (Figure 7.5), which was on the whiteboard, and copy the instructions:

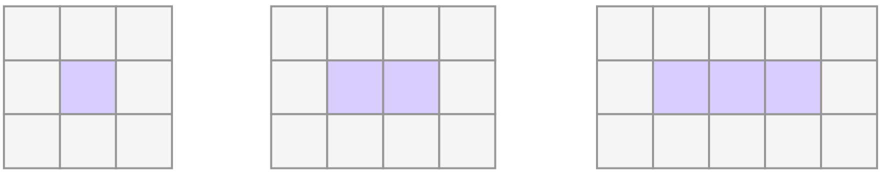


Figure 7.5. Building swimming pools pattern (Taken from Reys, Lindquist, Lambdin, Smith, & Suydam, 2004, p. 351).

- Use a table to record the number of tiles needed to build the pools.
- How many tiles are needed to build a pool that is 10 tiles long?
- How many tiles are needed to build a pool that is 20 tiles long?
- Find and record a general (proven) rule that describes a pool '*n*' tiles long. (T4WK4D1)

Apart from the rustling of paper and shuffling of pens, the classroom was quiet. Some students started writing immediately, and they appeared to be eager to solve the problem. Jay, however, sorted her coloured pencils into groups ready to draw the pattern. Reagan circulated the classroom, observing the students working, and said:

T Some people are working very well setting out their work neatly so as they can understand what to do. Please write down the instructions before you start. The reason is so as you can look back on your work, and know what you have done. When you have finished, think about how you would have approached this over the last few weeks.

[Five minutes passed.]

T Okay, look this way, and put your pencils down. You can use a T-table, or any table that has two columns like in your journal [*Maths at School* journal]; it has to be something that makes sense to you. What would I find on the table?

S1 Titles.

T Well done, what will the titles tell you?

S2 They tell you what the numbers mean.

T Yes, the headings tell you what information is in the table. When we did this with our worm, you came up with a rule, and then you 'proved' it worked. In this activity you also have to prove that your rule will work. What is the '*n*' in a number sentence?

S3 It is a pronumeral.

T So do a table and create a rule. Then answer the last question with an equation. Who needs more instruction?

One student commented that she did not get the 'general rule,' and Reagan asked her to provide an example of a rule that they had used previously. This student gave an example, indicating that she had understood. Another student asked:

S4 So, is this a test?

T You are testing yourself; showing what knowledge you have learned so far. (T4WK4D1)

Reagan moved around the classroom to ensure that all of the students understood what was required, and I remained seated to make certain that I did not interrupt them. She spoke quietly and gently to individual students. I noticed that Tad had completed the task within the first five minutes (see Figure 7.7). He sat at his desk watching the others work, and his smile suggested that he felt proud of what he had achieved. After a couple of minutes he took out his *Maths at School* journal to continue working through activities not yet completed. Reagan circulated the room for the whole twenty minutes; however, this was not in an authoritarian way. She seemed to encourage the students by posing thought provoking questions when they were stuck. When the time was up, she asked them to rule a line under their work to mark what they had done over this time period. Reagan did not gather up their books; instead, she selected various students to communicate their ideas to the classroom.

7.6.1 Communicating Ideas: Successes

The next two sections reveal the benefit and challenges inherent when students communicate their mathematical thoughts. Reagan wanted the students to discuss their mathematical thinking processes as a way to collaboratively solve the pool builder's problem. Her intention, it seemed, was to draw out and formatively assess the students' thinking and reasoning processes, rather than simply look over their written work to assess their knowledge. At this stage the syllabus document (QSA, 2004) was still encouraging the teacher to promote thinking, reasoning and working mathematically, and hence, Reagan was not formally required to assess their thinking and reasoning processes. However, clearly Reagan felt that the students' thinking and reasoning processes were an important aspect of the assessment task, and thus, she involved the students in whole class discussion to elicit their mathematical thinking, ideas and understandings. She observed intently.

Jed was the first student to discuss his ideas (see Figure 7.6):

T	What headings did you use Jed, and what was your rule?
Jed	My headings were 'middle' and 'outside' and my rule was, 'The number of pool [element number] times 2, plus 2, plus 4 equals the amount of tiles around the outside.'
T	What was the question asking you to find? [Reagan wrote Jed's generalisation on the whiteboard.]
Jed	How to work out what the 10^{th} element would be?
T	So how do I apply your rule for question 2? Can I write '$n2 + 2 + 4 = d$'? Everybody, in Jed's rule, what is the unknown part?
S5	The amount of tiles around the outside of the pool,
T	What else?
S6	The number of pool area. [Reagan nodded]
T	Can I represent it like this? [Reagan pointed to the generalisation.] What was the answer?
Jed	22, I did 10 times 2, plus 2, plus 4, equals 22.
T	Did I prove your rule? [Jed shook his head] Good try. (T4WK4D1)

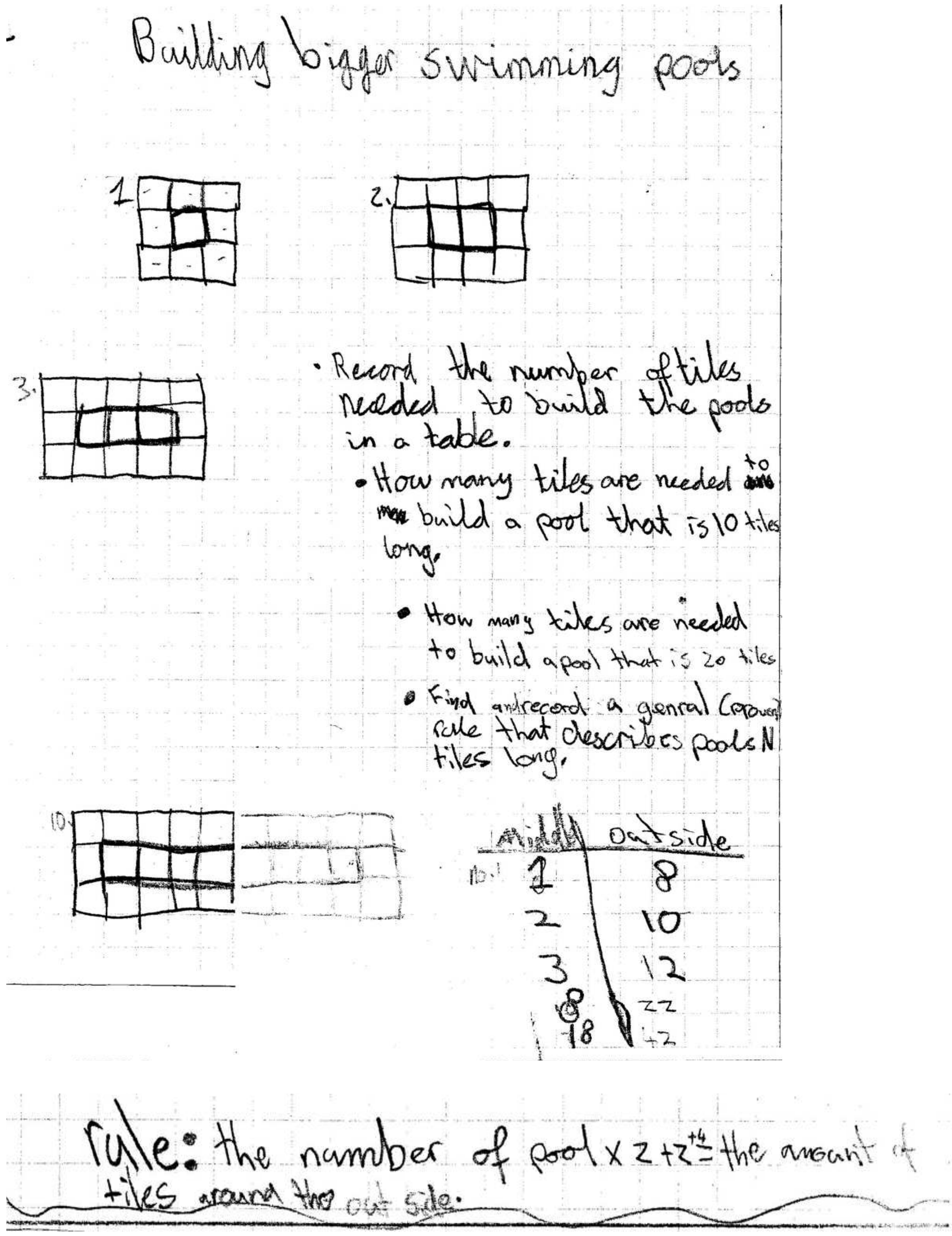


Figure 7.6. Jed's assessment task.

Jed looked puzzled; he was trying to justify his generalisation for the 10th element. However, he was using the answer he had written for the eighth element. As he communicated his ideas, it appeared that he was incorrect. Yet, when I looked at his book I saw that he had solved the generalisation, but had not reached the stage of applying the generalisation to work out the 10th or 20th element (see Figure 7.6). Jed had justified his ideas to himself as was evident by the inclusion of the little '+ 4' to make the generalisation work (see Figure 7.6). From looking at Jed's work, it was clear that he had recognised that he needed to add six each time, and thus, he included '+4' alongside his '+2' to alter his general rule. Of course, Reagan realised Jed's miscommunication when she looked over the students' work after class. She pointed out to me then that she was concerned that he may have felt that he was incorrect, when, in fact, his thinking was on the right track (T4WK4D1). Hence, as Reagan expressed in the interview (IV5), determining the students' thinking and reasoning processes was complex, especially in the moment. We spoke about using document readers. For instance, if Jed's work had been displayed to the class, then he could have used his written representations to justify and communicate his ideas, thereby enabling Reagan, Jed, or one of the other students to recognise his confusion.

Tad, who had completed his work within the first five minutes, was the next student to explain how he approached the task (see Figure 7.7). This time Reagan asked Tad to use the whiteboard to record his representation. I wondered whether this was an intuitive decision she had made after 'reflecting in action' on Jed's experience. Tad spoke as he wrote:

Tad	First I numbered the elements.
T	What do you mean by numbering the elements? [Reagan turned to the class.] This is what Tad did; he is looking at the number of steps.
Tad	I am recording elements and groups of tiles.
T	Why that way?
Tad	Because I looked at it as a group; 10 tiles long equals 36.
T	Tell us your general rule.
Tad	'Elements time 3 plus 6 = total number of tiles needed to build the pool'
S6	Did we count the pool?
T	Well, you could do it either way.
S7	As long as you can prove it without having to build the whole pattern to find it. Is that right?
T	Yes. Okay, Tad, prove your rule.
Tad	1 times 3 plus 6 equals 9, 2 times 3 plus 6 equals 12.
T	What are the things known and unknown in Tad's rule?
S8	The total number of tiles needed is unknown.
S9	The elements are unknown.
T	So how could you write it as an algebraic equation? Write it down to prove it.

Tad [Tad wrote the equation using the pronumerals, and then wrote the corresponding equation for each step as below.]

a x 3 + 6 = b

1 x 3 + 6 = 9

2 x 3 + 6 = 12 (T4WK4D1)

These students indicated that they understood how to determine which part of the numeric equation changed. Even though Tad had not recorded these equations in his book, he understood how to represent the generalisation with a formula.

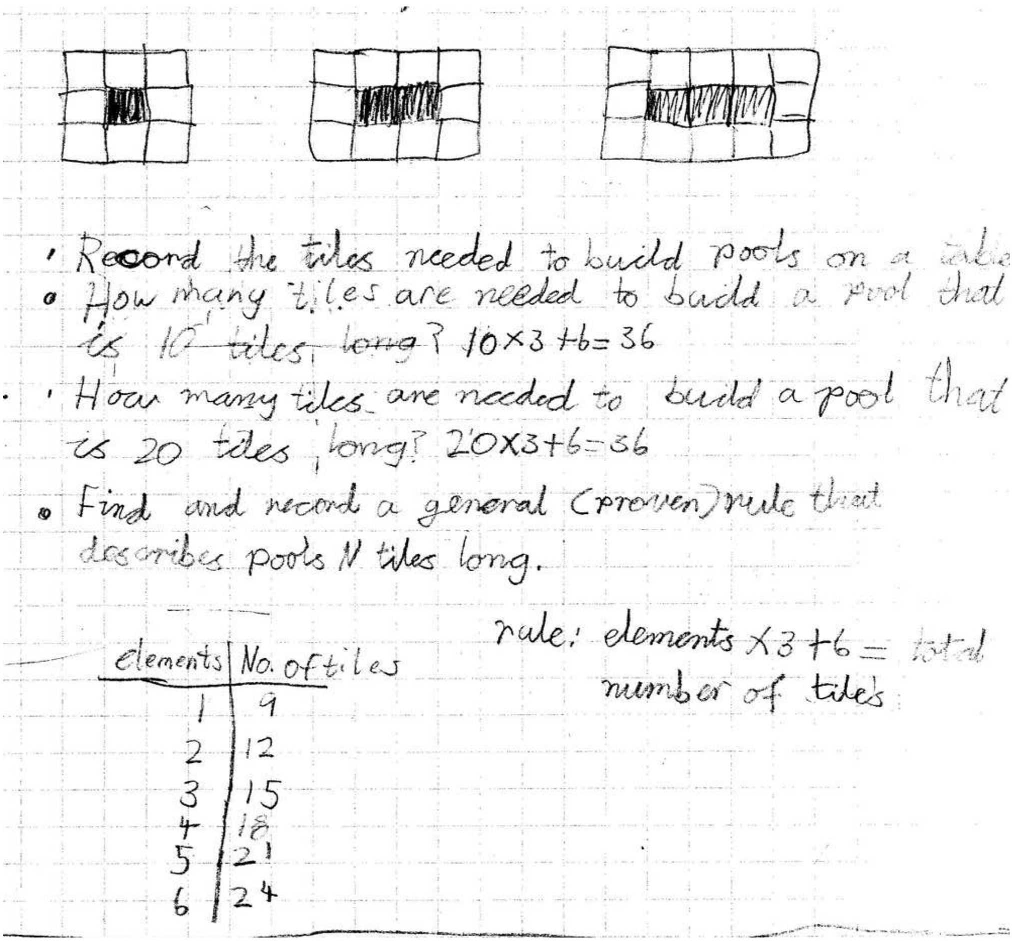


Figure 7.7. Tad’s assessment task.

Reagan then asked the students what part of the numeric equation stayed the same:

T What stays the same?

S10 The ‘times 3 plus 6’ part. [Many students agreed by nodding their heads, although Kay raised her hand to disagree.]

Kay	I had it different, I wrote: ‘Number of water plus number of water plus 6 equals the total number of tiles.’
T	Okay, now write it using pronumerals. [Reagan invited her to the whiteboard and she wrote, as follows.]
Kay	‘$j + j + 6 = s$’
S11	Or you could put number of water times 2.
Kay	[Responding to this student, Kay wrote:] ‘$j2 + 6 = s$’
T	How did you work that out?
Kay	If the pool is ten tiles long, then 10 + 10 + 6 = 26.
	[The lunch bell rang and Reagan congratulated the students on their efforts and let them go; she glanced at me with a big smile.] (T4WK4D1)

Kay had not included all of the tiles in her generalisation. She had separated the pool water from the bordering tiles; hence, her numeric equation differed from Tad’s.

The benefit of discussion was becoming apparent. For instance, what was evident in this last discussion, that had not been so obvious previously, was that Kay was comfortable to challenge Tad’s ideas. She trusted her own ideas, and could validate her reasoning with an ‘if… then…’ justification. It was at this point that the classroom was taking up yet another change. It was now assuming a communal conflict atmosphere, which reflected Palmer’s (1999) notion that students are more willing to challenge each other’s ideas when they sense that they are in a supportive environment. I wondered whether Kay may have felt more confident to challenge Tad’s ideas because she was able to validate the truth of her own ideas, through using a representation and justification.

Upon looking over the students’ work, it was clear that they were thinking and reasoning mathematically, although the work samples revealed varying levels of understanding. Reagan and I could make sense of what the students had written and what they were thinking about. For example, Shae, who Reagan explained often struggled with mathematics, was confused about the element number, yet her work indicated that she had been thinking mathematically (see Figure 7.8). Shae had not labeled her table with ‘element number.’ Rather, her headings read, ‘Number of tiles long’ and ‘The sum ‘*n*’ answer,’ although she had written a numeric equation for each step of her pattern. For the first pictorial representation, Shae had recorded three on her T-table to represent the number of tiles across the pool. Then she recorded eight for the next column, which represented the number of tiles around the border of the pool. She repeated this for each step of the pattern, and then wrote a correlating equation next to each step. Reagan and I could conclude that Shae's generalisation was '$n + 2 + (n - 1) + 1 = x$'. Thus, whilst Shae was confused about numbering the element number, her numeric equations were consistent for each step, which depicted complex mathematical thinking.

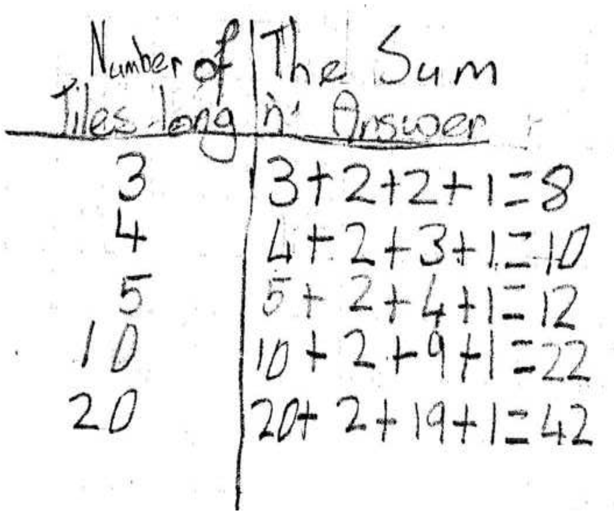


Figure 7.8. Shae's assessment task.

7.6.2 Communicating Ideas: Challenges

Orchestrating discussion and enticing investigative thought also presented challenges in this activity and previously. One challenge was to keep all of the students mentally engaged at all times. For example, when Nick called out 'circumnavigate,' clearly, he had not switched on to the mathematics lesson, despite being nearly halfway into the lesson. Also, as Reagan attended to Shaun, some students became restless, which meant that she had to ensure that both the whole class and Shaun were mentally stimulated at all times. Then in another lesson, Jay was quite distracted, although she seemed to know what was going on. For instance, she had correctly placed the brackets around the part of the pattern that had changed prior to the correct bracket placement being discussed. Then there was also the issue of knowing whether or not to extend students' thinking in the moment, and if so, how. For instance, Reagan had modeled how to prove Guy's generalisation by applying it to the pattern, yet he did not get the opportunity to investigate further, and see where he could have altered his generalisation. Then, in this assessment lesson, Jed's insightful mathematical thinking may have been marred because he did not have the opportunity to effectively use his representations to justify his general rule. Nevertheless, there were also benefits of discussion that were apparent in this last lesson. The students were able to collaboratively test generalisations, and apply this thinking to the work they had achieved. The discussion also helped reiterate how to determine which part of the equation was constant and which part changed, and thus reconfirmed how to formulate generalisations using variables. Despite the challenges of discussion, Reagan was pleased with how the students were progressing (IV5). She believed that the students were starting to develop a sense that they were capable mathematics users, and there was evidence in the data to qualify her thoughts, as will be explained below.

7.7 Ways of Being

Developing mathematical ways of being involves developing the confidence, practices and dispositions of successful mathematics users (Ball, 2003; Kilpatrick et al., 2006). As the data revealed above, the students and Reagan started to use the mathematics practices as a way to think and reason mathematically. However, this was a process that evolved gradually. As both Reagan and her students started to take up the essence of the practices, which was to represent and justify the truth of mathematical ideas to formulate a generalisation, their mathematical discussions reflected a more inquisitive tone than had been evident at the start of this research. For instance, in the initial stages of this research, even though Reagan invited the students to share their thinking about mathematical ideas, she appeared hesitant about how to deal with some of the complex ideas shared. Hence, she did not invite the students to explain their thoughts, nor did she draw on these ideas to conduct further inquiry. When two girls failed to be able to apply the long division procedure they had memorised, Reagan became more determined to promote opportunities for the students to think and reason about mathematics to develop deep understanding. Her beliefs about the importance of learning mathematics with understanding, to ensure successful application of mathematical ideas, were confirmed (IV5).

Thus Reagan's teaching practice changed to accommodate opportunities for mathematical inquiry. She used a combination of teaching methods. When she introduced new concepts she guided the students' learning by using direct and explicit instruction to demonstrate ideas. Then she provided opportunities for investigative thinking though discussion. This combination of explicit instruction and investigation appeared in most lessons. It seemed that the more the students discussed ideas and showed what they knew, the more sensitive Reagan became to their mathematical thinking. Each time a question was posed, she gave sufficient wait time for the students to think about a response, and then often she asked someone whose hand was not raised to respond. However, she was careful not to put anyone on the spot, as she expressed in the interviews. For instance, she did not select students who did not understand; rather she selected those whose ideas could be extended, in the hope that other students would clarify their thinking through the expression of their peers. This was evident when Ryan was able to clarify his understandings about the reverse formula though discussion. If Reagan had asked him to share his ideas while he was confused, he may have lost confidence. As Reagan pointed out in the interview, students can freeze if they are uncertain, and put on the spot (IV5).

In essence, the whole class discussion could be considered as cooperative thinking aloud (Killen, 2003). For instance, as the students discussed ideas, the classroom became a conglomerate of shared thoughts. Reagan often demonstrated an idea, and then asked questions to encourage a sense of “I wonder” type of curiosity about that idea. One example was when she asked them what to do if they knew the number of objects in the pattern, but not the element number. Reagan had sensed that the students were inspired by puzzling over the generalisation, and hence, she enticed more curiosity by reversing the general rule. A large part of each lesson came from the learners’ input as they actively exchanged ideas, which meant that the classroom was both teacher-directed and student-centered. Reagan’s strategy to combine explicit instruction with investigation aligned with findings of Masters’ (2009a) meta-analysis of research on highly effective teaching. He described that highly effective teachers promote student learning through direct instruction, and motivate student learning through curiosity.

The classroom atmosphere was promoting a more mathematical way of being. The interactions became more investigative in nature, and had broadened by the end of the research, to include student-student as well as teacher-student discussion. There were two distinct changes: firstly, instead of Reagan having one-to-one discussion whilst others listened on, she created opportunities where the students discussed ideas with each other. Secondly, the students started to discuss and challenge each other’s ideas throughout the classroom discussion, as was evident when Shae challenged the generalisation that Tad had formulated. It appeared that, as the students experienced success, their confidence grew, which encouraged them to inquire more deeply into ideas, as well as to accept further challenges, thus confirming other researchers' perspectives (e.g., Boaler, 2002; Killen, 2003; Kilpatrick et al., 2006; Palmer, 1999). Importantly, Reagan positioned her students as capable mathematics users, and encouraged them to view themselves in the same light. She said to the students things such as, "I know you can work this out" (T4WK2D2) or told them they were doing grade seven work. This positioning also related to Boaler's (2002) view that students need to be able to identify themselves as capable mathematics users. The next section looks at the students' dispositions towards mathematics and mathematics learning through their eyes.

7.8 Ways of Being: The Students' Voice

A goal of this research was to include the 'student's voice,' as Jenkins (2006) recommended, to identify effective features of teaching that may enhance quality learning or change levels of interest, disposition and engagement. Hence, I gave the students pre- and post-questionnaires to complete. The students' beliefs and attitudes about mathematics and mathematics learning have been generated over the past six years at school, and hence, to measure a change over a short period of time was unreasonable. For instance, when asked to indicate whether they enjoyed puzzling things out for themselves, one boy wrote that he did enjoy it on both the pre- and post-forms, yet, his example was: "I enjoyed puzzling in year two when I tried to make a picture" (S14). Thus to determine whether the students' thoughts related to this particular research period was difficult. Also, another student was concerned that if he was honest he might offend Reagan, and I am sure he was not the only student who felt that way. Nevertheless, whilst the ratings scale showed minimal change across all questions, there were some interesting comments made.

The students' responses indicated that Reagan's attempt to promote a community of inquiry seemed to have been successful. There was one statement in the survey response that stood out from the rest as revealing the most change. This statement was: 'In maths classes we ask lots of questions.' Over half of the students, fifteen out of the twenty six, who completed both forms, indicated that they were undecided on the pre-questionnaire, two disagreed, three strongly disagreed, and six agreed. Then in the post-response, seventeen agreed, three strongly agreed, four were undecided and two disagreed. Thus, clearly the students believed that their classroom had taken up a more inquisitive nature. Another statement related to inquiry, which was: “I enjoy puzzling things out for myself in maths lesson.” Even though the scale did not reveal change, some students gave examples of where they enjoyed puzzling over ideas. The comments on the pre-questionnaire related to tests, for instance, three students wrote that they enjoyed doing ‘times table tests’ (S1; S4; S14). On the post-questionnaire the responses were varied; three indicated that they enjoyed the growing worm activity (S8; S15; S17). Hence, these comments suggest that the students’ thoughts about what puzzling over ideas involved had changed from trying to answer an algorithmic procedure, to thinking and reasoning about mathematical ideas. However, Jay made an interesting comment, she wrote: “Sometimes, I wanna show everyone I can do maths.” Then, further on, she made another comment related to her confidence, which will be revealed below.

7.8.1 Developing Mathematical Confidence

Some students changed their view about their mathematical ability. For instance, when the students responded to the statement: “I am good at maths,” three students, Tad, and two others (S8 & S9), had marked ‘strongly agree’ in the pre-questionnaire and then marked ‘agree’ in the post-questionnaire. One student commented, “I just know I am good at maths,” on the first form, and then wrote, “Because I am good at some and not at others” (S9), on the second. Maybe she was sensing that mathematics learning involved investigative thinking and reasoning, whereas previously she had attributed her success to her ability to passively receive and regurgitate knowledge. Tad’s comments also revealed a change in attitude about mathematics learning, for example, on the first form he wrote: “I understand it very well,” whereas on the second form he wrote, “when I am asked to tell something to the class, I get most things right.” Reagan had extended Tad’s thinking in the classroom, which is possibly why he now viewed himself as ‘getting most things right’, as opposed to ‘understanding very well.’ The students’ comments related to the new mode of mathematics teaching and learning they were experiencing. They were now required to think and reason things through, which became even more evident when asked to name an ‘important piece of maths learned at school, and give a reason why you believe this is important.’ Tad wrote: “How to show working out because you may need to tell someone how you got the answer for something.” Hence, his beliefs about mathematics learning had changed.

Learning now involved more than learning facts and procedures, as was on his first form; now it was learning how to justify mathematical ideas.

Another student, Roxy, altered her perception about her mathematical ability. She indicated that she strongly disagreed that she was good at maths on the first form, and then wrote that she was undecided on the second. Roxy's comments were interesting, and revealed that she was altering the way she felt about mathematics as a discipline and her own mathematical ability. On the first form she wrote: "I let myself down all the time," and later when commenting about how she enjoyed mathematics learning in the classroom, she wrote: "People might laugh at me because I usually don't know what to do." However, on the second form, her confidence had changed slightly. On this form, in response to, 'I am good at maths,' she wrote: "Sometimes when I understand things." She was beginning to believe that she understood some mathematics, for instance, her example of a time when she enjoyed puzzling something out was when she did repeated subtraction. On the second form, she agreed that she enjoyed puzzling things out, whereas previously she had strongly disagreed. Roxy's change in attitude relates to some researchers' (Killen, 2003; Kilpatrick et al., 2006) view that successful learning promotes more successful learning. Also, Roxy's anxiousness appeared to be lessening, which relates to Skemp's (1986) notion that mathematical anxiety can be a result of not knowing, or not believing in, one's own mathematically ability.

Indeed, the students do want to experience success. It is probable that Jay's classroom behaviour may also be partially related to her beliefs about herself as a mathematics user. Possibly when she fidgets, Jay may be reacting to feelings of nervousness. As mentioned above, Jay wanted to show others that she can do mathematics. Then, when she responded to the statement: 'I enjoy what we do in our maths classes,' on the first form she wrote: "I feel smart when I get it right," and on the second, she wrote: "When I do repeated subtraction." Jay believed she understood repeated subtraction, and thus she enjoyed this lesson. She commented that, "Understanding mathematics makes you smart and no one will tease you." Thus, for Jay, enjoyment related to success, which again relates to Killen's (2003), Kilpatrick and colleagues' (2006) and Skemp's (1986) notion that success breeds success. The enjoyment for Jay and Roxy did not come from the activity itself; rather, it was the result of the learning and understanding which developed. In a sense, both Jay and Roxy were developing a "productive relationship with the discipline of mathematics" (Boaler, 2002, p. 47), because they were beginning to believe that they could think, reason and work mathematically.

7.8.2 Reagan's Interpretation of the Students' Voice

Reagan was also interested in what the students had expressed on the pre- and post-questionnaires. After reading these questionnaires, she expressed her delight that the students were now beginning to think for themselves. However, she thought that some of the students had reacted to the questions in response to the way they had been conditioned to think about mathematics and mathematics learning. She reflected:

> Just students thinking of themselves a bit was good, especially using words like 'proactive' in their surveys. I think in the surveys we saw we have got a long way to go with kids to be feeling good about maths, by their first five words that they think of, but I also think that that is really entrenched in them. They are year six and if they haven't had really engaging experiences, it is just going to be almost an automatic response. So we can say, 'What did you do at school today?' and [they reply] 'Nothing.' It is an automatic response. When they actually then break it down, that is not what they are thinking. I do feel generally the actual whole class atmosphere is a lot more positive because of the way they are approaching their activities. For example, when they had free time the other day, they actually got their maths journals out and worked on that. They wanted to finish it as well, but they were still motivated enough to do that, over their free time; they could have been doing anything. (IV5, p. 18)

In this last vignette, Reagan referred to the first five words that came into the students' minds when they thought about mathematics. Almost every student commented that mathematics was boring. However, in the second forms, two students elaborated. The first described mathematics as interesting when they learned new things (S4). She went on to say that:

> When we work with our friends, even when we work with ourselves, it can still be fun. When everyone gets it done and everyone listens, it's fun. We need to be proactive for our future (S4).

This student was referring to the whole class discussion that happened after each activity or during the lessons, when Reagan combined explicit instruction with student investigation. She obviously enjoyed the discussion. Possibly this was because it clarified her understandings, and hence, once again enjoyment was related to understanding. I drew this conclusion because of this student's last sentence, where she stated: "We need to be proactive for our future." She recognised that mathematics learning was an essential learning for her future.

The second student to elaborate on his five words related to the change of atmosphere in the classroom. He also included me in his comment, which indicated that the change had occurred over the course of this research project. He wrote:

> Maths is always fun with Miss [Reagan] and Miss Smith, because we learn maths in a funny way, and it makes me just want to learn maths. It's better to learn it when you are smaller, or you will struggle when you are older (S20).

His choice of words was interesting, suggesting that Reagan's new classroom approach to mathematics teaching and learning was a 'funny' approach. Possibly, he believed it was funny because it was vastly different from how he had been taught previously. However, he also perceived that it was important to learn with understanding so as to avoid difficulties in later learning.

Reagan was interested to read the students' thoughts about how they thought mathematics teaching and learning could be improved. She believed that it was important to let the students voice their ideas, as follows:

> Something I value was in our research; we asked the kids what they thought was important in maths, or what they liked in maths, or what they found exciting. Because, as a school, we often make decisions to go with a certain text; we say, 'This is how, as a whole school, we are going to teach our maths,' and when I've asked around, nobody asked the kids; nobody asked the kids whether it was actually working for them. So I guess with this research, because of the implementation of the actual surveys, they were given a voice, and I think some of them found that quite challenging. They seemed to be shocked, and they may have been thinking something like, 'Gee, someone is asking me what I think!' I think that is really important. (IV5, p. 18)

7.9 In Conclusion

In conclusion, the mathematics practices' framework assisted the students' cognitive development and more. As they justified the representations, they seemed to develop clearer perceptions about what to look for when determining both regularity and similarity in mathematical patterns. For example, they were able to examine what changed or what stayed the same in patterns, which helped them to formulate generalisations. This development of algebraic thinking has implications for future learning, as was also pointed out by Mulligan (2006). She believes that students' mathematical achievement may be improved if they are explicitly taught to recognise a variety of patterns and structure in mathematics. Hence, the students were being given the opportunity to construct mathematical knowledge, as well as knowledge about the discipline of mathematics itself. Through investigating mathematical patterns they were beginning to realise that mathematics is logical, and that it makes sense. In short, the sequence of activities helped them to make sense of mathematical ideas for themselves. The mathematics practices' framework supported a methodical mathematical inquiry. The framework helped the students to communicate their thinking processes, which evoked clarity, and thereby learning *with* understanding.

However, their development went beyond cognitive growth. The students were also developing a way of working mathematically through using representations and justifications. Consequently, they were assuming practices of successful mathematics users, as proposed by the RAND Study Panel (Ball, 2003). The students developed a sense that they were in a process of learning, as Kilpatrick and colleagues (2006) pointed out, and that the process was collaborative. Importantly, they recognised that their knowledge was not fixed, and thus the fear of making mistakes appeared to be on the wane. This was particularly evident in the last lesson, when some students challenged others' thoughts and ideas. It seemed that, as they believed in their own mathematical knowledge, understandings and abilities, they became more confident and competent to think and reason mathematically. This confidence in turn, advanced further cognitive development and ways of working mathematically. Significantly though, the students appeared to be developing positive mathematical dispositions, the goal of the curriculum ideals, as outlined by DETA (2008) and QSA (2004), and productive dispositions, which was recommended by the RAND Study Panel (Ball, 2003) and Kilpatrick and colleagues (2006). In other words, their interest levels towards mathematics, their willingness to participate in mathematical investigations and discussion and their desire to learn appeared to improve. Even though their self-reported data suggested otherwise, as Reagan explained, it would take more than a school term to disrupt their conditioned responses. It was their actions that revealed change.

This chapter has revealed that change is possible although it is a process in itself. For Reagan the process of change was gradual. She realised that change takes time and that she could not change everything at once, and hence, she was flexible, reflective and methodical in her approach. For instance, she discussed her reflective thoughts about the implemented mathematical activities to determine how and why they did or did not work, and then used the ideas gleaned from discussion to frame follow up lessons. Hence, through reflecting in action and on action Reagan made decisions about what understandings she wanted the students to develop and how. She was particularly focussed on helping the students to be able to communicate their ideas in ways that would enable sense-making through representing and justifying ideas both individually and collaboratively. For example, she used the whiteboard to assist the students' communication of ideas and she created opportunities for all students to listen and contribute to each group's justification processes when she utilised the outdoors space. Gradually the students started to communicate in ways that enabled them to validate the truth of ideas for themselves, as was evident by their use of representations and 'if…, then…' statements. The chapter revealed that as the students felt the success of their efforts, the classroom environment altered. The students started to initiate their own inquiry and appeared to be more willing to persist with trying to figure things out for themselves, and in their own way. Hence, as Reagan's teaching practice altered, so did the students' involvement in the learning, which in turn, enabled Reagan to test out other teaching strategies. The next chapter synthesises the data to answer the research questions.

CHAPTER EIGHT
THE COMPLEXITY OF CHANGE

"Good teachers join self, subject, and students in the fabric of life because they teach from an integral and undivided self… They are able to weave a complex web of connections between themselves, their subjects, and their students, so that students can learn to weave a world for themselves."

(Palmer, 1997, p. 2)

8.1 Introduction

This research makes clear that successful curriculum implementation is largely dependent upon a willing participant such as Reagan. She was inspired by the opportunity to investigate the inquiry-based process approach that she had studied at university, but had not yet experienced in practice. In addition, we were both fortunate that the students were also willing to participate and contribute, and that the principal was accommodating. As a researcher, and because curriculum implementation is a complex process affected by many interrelated contextual factors, I realise that this study may not have produced such favourable outcomes had the participants not been as willing. The teacher's knowledge and practice, the students' knowledge, practices and dispositions, and the classroom atmosphere were all implicated to some degree when more mathematical thinking and reasoning was encouraged. Many factors affected both Reagan's and the students' experiences throughout the implementation process, as the data have shown. This chapter analyses the teaching and implementation experience, and the subsequent learning experiences. The chapter delineates five contextual features that are inextricably linked; the processes of implementing change, constructing knowledge, developing practices, establishing communities of practice and producing productive dispositions. Each feature supports and influences the development of the others; however, for the purpose of clarity they have been separated. The following chapter evaluates the findings in response to the research questions.

8.2 The Process of Change

The process of implementing curriculum change requires empirical inquiry (Ball, 2003; BTR, 2005; Lloyd, 2008) as well as time and support (Bobis, 2004; Clandinin, 2008; DETA, 2005; Masters, 2009a). Indeed, as Farmer and colleagues (2003) suggested, without a sense of control over curriculum ideals, the teacher may be reluctant to change. However, this research has found that gaining a sense of control over curriculum change is an investigative process in itself. The process is influenced by internal factors such as the way the teacher understands and thinks about mathematics, the curriculum, his or her teaching practice, the students and the learning process. There are also the external communities of practice that influence the teacher and the subsequent process of change.

Indeed, research has found that the translation of curriculum incentives into the practice of teaching is limited (e.g., Anderson & Bobis, 2005; Cavanagh, 2006; Reys et al., 1997). In the past research has consistently pointed out that the teacher's pedagogical and content knowledge (Ball, 1988; Ball et al, 2005; Brown & Borko, 1992; Manouchehri & Goodman, 1998; Shulman, 1986) and curriculum knowhow (Cuban, 1993; Handal & Herrington, 2003; Stenhouse, 1978; Shulman & Shulman, 2004; Reys et al., 1997) are integral to the improvement of mathematics teaching and learning. Now curriculum documents (DETA, 2008; QSA, 2004) strongly emphasise that to improve mathematics education, teachers need to encourage thinking, reasoning and working mathematically. In fact, this emphasis is evident in documents worldwide (e.g., NCTM, 2000; 2007; OECD, 2006). The focus is now on helping students to learn mathematics *with* understanding to ensure successful and continued learning. Thinking, reasoning and working mathematically demand new ways of being for the students as well as the teacher. Hence, whilst the teacher's mathematical and pedagogical knowledge are important, he or she also needs to understand how to develop students' mathematical understandings, practices and dispositions. One cannot be developed without the other.

Thinking, reasoning and working mathematically demand a new classroom atmosphere, one that supports quality mathematics learning through communal inquiry. If the teacher has not experienced this before, then gaining control over the curriculum ideals will be difficult. As Walshaw and Anthony (2008, p. 517) pointed out, there is still an uncertainty about how to establish quality learning experiences that "promote communal production and validation of mathematical ideas." Teachers need the opportunity to investigate and discuss curriculum incentives, as in this research. Through active teacher inquiry and teacher-researcher collaboration, this research has highlighted some features of the learning environment that have the potential to support communal and individual sense-making of mathematical ideas. Even though this sample size of one cannot be generalised, the advantage of this small study was its scope for focussing on the practicalities (Ball, 2003; Masters, 2009a) of implementing thinking, reasoning and working mathematically.

First, a gradual process was necessary for the teacher to develop ownership over curriculum ideals and for quality teaching and learning to take place. As Piaget (1896-1980) explained, past experiences are used to bear meaning on new ideas which in turn cause internal disequilibrium. Once sense is made, equilibrium is restored and ideas are integrated with understanding. Hence, personal experience is paramount in any sort of learning endeavour. Without personal experience the teacher cannot be expected to develop a sense of control over the curriculum incentives, and similarly, without personal experience, the students cannot be expected to develop a sense of control over mathematical ideas. Thus, when the learning process was gradual, more time was available for existing ideas to be remodelled or replaced with understanding. For instance, in Reagan's case, as she experienced different methods of teaching, and then reflected on the benefits of those approaches for teaching and learning, she made sense of the ideas about learning she had gleaned at university and their relevance.

Second, this research highlights that to assume ownership over curriculum ideals the teacher needs to sense that he or she has the right to make informed teaching and learning decisions. Like the students, the teacher needs an opportunity to be active in her own knowledge construction. Without being positioned to take up that role, ownership will possibly be limited. For instance, despite believing that the QSA (2004) syllabus document positioned her as a responsible, 'thinking' teacher, Reagan did not sense the same acknowledgement in her workplace environment. This meant that prior to this research she did not sense the freedom to operationalise existing or new teaching ideas. As the initial interview data revealed, her school encouraged the use of the *Maths at School* program, which seemed to be very prescriptive. However, once permission was granted by the principal to conduct this research in her classroom, she appeared empowered to explore new teaching strategies. The principal's acceptance was important; it was as if she had been given a licence to trial ideas. This acceptance meant that she felt the freedom to plan based on the students' learning needs, as opposed to simply covering content.

Third, to implement thinking, reasoning and working mathematically requires an environment conducive to inquiry. Hence, a challenge when attempting to implement change may involve a significant change of focus, as in Reagan's situation. The students were learning mathematical ideas as isolated bits of knowledge, which leads to compartmentalising (Kilpatrick et al., 2006). Conversely, she valued the development of related webs of mathematical ideas that researchers and cognitive theorists recommend (e.g., Anderson et al., 1998; Hiebert et al., 2000; Hiebert & Carpenter, 1992; Skemp, 1986; Van de Walle, 2007; Van de Walle & Lovin, 2006). Reagan was drawn to constructivist epistemologies and inspired by the opportunity to promote personal sense-making through active inquiry, also recommended in research (AEC, 1991; Ball, 2003; DETA, 2008; Hiebert et al., 2000; Klein, 2000; Lovitt & Williams, 2004; NCTM, 2000; QSA, 2004; Savery & Duffy, 2001; Van de Walle 2007; Van de Walle & Lovin, 2006). However, because she perceived the school's program as reflecting worksheets, which provided minimal, if any, scope for investigative thought, she was in turmoil about how to go about promoting active knowledge construction through inquiry. On one hand she wanted to relinquish the program altogether, and on the other hand, she was concerned about the parents' and the school's financial investment in the program. Later, she did recognise value in the program as a reinforcement tool.

In this case, Reagan felt constrained by the way things were done at her school. Prior to this research, her views of mathematics and mathematics learning did not correspond with how she was teaching. For example, she believed that students needed to develop deep mathematical understandings based on developing conceptual understanding about how and why mathematical ideas work. Her aspiration was to help her students develop, in her words, 'functional' mathematics, especially because she believed she was not mathematically confident until her understandings evolved. This issue does pose potential for further widespread research to be conducted because, in Reagan's case, it had caused a significant barrier to implementing her interpretation of the QSA (2004) curriculum document. One suggestion is to conduct a comparative analysis between the school's recommended curriculum or program, the State or National curriculum documents, and interview data gathered from teachers.

Clearly, the implementation process is complex and multifaceted. Promoting active investigation can, as Hartland (2006, p. 6) pointed out, require a "massive paradigm shift," particularly if the teacher is accustomed to planning learning experiences to impart knowledge to students. In Reagan's case, initially, the students were accustomed to the teacher imparting knowledge, while they remained passive. However, she realised that her role and the students' experience were vital if she was to, as Grootenboer and Zevenbergen (2008) suggested, connect the student with the mathematics. The students could not be expected to suddenly switch from being passive recipients of knowledge to active mathematical inquirers. They needed some guidance about how to think mathematically, as stressed by Kilpatrick and colleagues (2006). In this case they also needed to re-develop foundational mathematical understandings. Hence, even when the teacher understands contemporary ideals, implementation is not as straightforward as it may seem.

8.3 Constructing Knowledge

For many years, those interested in mathematics education have been advocating for students to develop deep mathematical understandings, beyond solely mastering facts and procedures. For example, Skemp (1986), Stenhouse (1978), Schoenfeld, (1992), Romberg (1992) and the AEC (1991) asserted that a focus on procedural knowledge alone will not develop mathematical understandings that enable further success and confidence in mathematics learning. The same emphasis is present today. For instance, the *National Numeracy Review Report* (COAG, 2008, p. xii) stated that "the rush to apparent proficiency at the expense of sound conceptual development needed for sustained and ongoing mathematical proficiency must be rejected." The report went on to say that the focus should now be on the development of "understanding and thoughtful action that deep mathematical learning requires" (p. xii). Despite the abundance of insightful literature and research, the focus on developing procedural knowledge alone lingers, as was discovered early on in this research.

Reagan was alarmed when she realised that many of her students had not mastered basic number facts. Her concern was warranted because at year six, students should have mastered multiplication and division facts (DETA, 2008; Van de Walle, 2007). Before students can be expected to think and reason mathematically, they need to have developed foundational mathematical understandings. Students who are absorbed with trying to solve basic number facts exhaust their capacity for more complex thinking, according to researchers (Anthony & Knight, 1999; Battista, 1999; Booker et al., 1998; Madell, 1985; Sun & Zhang, 2001; Van de Walle, 2004; Van de Walle & Lovin 2006; Westwood, 2003; Willingham, 2003;) and mathematicians (Howe, 1997; Klein et al., 2005; MSSG, 2004; Quirk, 2005; Raimi, 2000; Wu, 1999). Hence, she was intent on helping the students to re-learn foundational mathematical concepts of place value and basic number facts before she was prepared to introduce investigative tasks.

The lack of conceptual understanding and procedural fluency of basic number facts and place value knowledge at this stage in the students' learning is not something to brush over. Why is it that at year six many of these students could not recall basic number facts, or apply thinking strategies to assist number fact recall? This is the issue that the media (Donnelly, 2005; 2005a; 2008; Maiden, 2005) frequently point out to the public. Recently the ABC News (Butler, 2009) alerted the public to a new vision to test teacher's mathematical knowledge. Hence, the teacher's knowledge is being blamed for the lack of conceptual understanding. However, as was evident in this research, the students' learning benefited when Reagan focussed on the simultaneous development of conceptual understanding and procedural fluency. As the BTR (2005) argued, it is what the teacher does in the classroom that can be the most important predictor of the students' numeracy learning. To reiterate, what the teacher knows is important, but what the teacher does is of equal importance.

Interestingly, and fortuitously for this research, Reagan's discovery of the students' lack of foundational knowledge intensified her desire to assist their development of mathematical competence and confidence. In her own learning experiences she had found that hands-on learning helped to broaden her mathematical understandings. Thus she envisaged that hands-on learning would lead to learning with understanding for her students as well. However, her inquiry into her own practice highlighted the fact that she had taken for granted the thinking and reasoning learning processes needed for abstraction and cognitive development. She assumed that when she demonstrated various diagrammatic, symbolic or concrete representations, or when the students drew representations, they would automatically engage in inner speech that cognitive theorists believe helps knowledge construction (Bruner, 1985; Piaget, 1896-1980; Skemp, 1986; Vygotsky, 1896-1934). Then in an experience where she was modeling her own thinking processes, she recognised the importance of communication and reflective thinking. Hence, this instance instigated the first change she made to her practice. She realised that her role required attending to the students' investigative thinking processes, and that these processes also needed to be structured in a way that would help them to construct meaningful abstract images in their minds. These abstract images would, in turn, support further mathematical thinking.

8.4 Developing Practices

When the teacher has never experienced approaches where the focus is on drawing out the students' mathematical thoughts, they may be uncertain how to elicit the students' thinking. As in Reagan's case, making sure that the students could communicate mathematical ideas in ways that made sense to themselves and others (DETA, 2008; QSA, 2004) was challenging. She turned to me, as the researcher, for support because she felt that she was floundering in her attempts to draw out, or extend upon, the students' thinking. Based on the literature reviewed about helping effective communication of ideas (e.g., Ball, 2003; Hiebert et al., 2000; Kilpatrick et al., 2006; Skemp, 1986), I saw potential in the mathematics practices to help scaffold students' communication of their thinking and reasoning. I suggested the whiteboard as a strategy because I envisaged that the students could use it to create representations that would assist them to justify their ideas. Hence, in this instance, collegial support was helpful in the sense that the understandings I had generated from the literature researched, and Reagan's practical experiences were coming together in meaningful ways.

The whiteboard was one teaching strategy that proved helpful in assisting whole class discussion. Slowly the whiteboard became an integral feature of the learning environment, where the students started to feel more confident to express their ideas. I noted a sense of directness as they spoke their ideas. For example, some used the marker in the same way that Reagan had which, I believe, indicated that they were sure about their ideas and possibly wanted to make their ideas clear to their peers. It could also have meant that they were merely mimicking the teacher because they had now stepped into her role of addressing the class. Nonetheless, their communication absolved Reagan from trying to express their thinking processes for them. As a teacher, she could then pose additional questions to probe the student's, or the whole group's, thinking either for clarification purposes or to expand on the mathematical ideas being investigated. Eventually, as the data portrayed, the students became less dependent on Reagan's clarification and more trusting of their own thinking processes.

The mathematics practices framework underscored the students' communication and sense-making. Interestingly, Reagan did not set out to explicitly teach these practices as such; rather, she used the practices to frame investigative, hands-on tasks. The initial activity was the outdoors measurement activity, and then as mentioned in the previous chapters, we planned activities indicating opportunities for the students to represent and justify their thoughts, or to make generalisations (see appendix). Reagan had expressed, and the data revealed, that the framework helped her to structure a methodical inquiry process. The way she utilised the mathematics practices framework drew the students in as more active participants in the process of 'doing' and investigating mathematics. For example, in her first attempt, despite her nervousness, the students became more actively involved in the learning, both physically and intellectually. They initiated inquiry by challenging each other to estimate measurements. Then they used the concrete representations to justify, and thereby, validate the truth of their ideas for themselves and each other.

Adopting a balanced pedagogical approach may also be helpful to teachers. For example, through combining explicit teaching with student investigation, Reagan seemed to be able to model a thinking process that the students would then have an opportunity to practise. After a short period of student investigation, she called them back in to discuss and justify their thoughts. By switching from explicit teaching and/or discussion to student investigation, the uptake of the practices seemed to evolve gradually, and purposefully. The students were beginning to use language that signified that they were applying the mathematics practices. They started to use 'If…, then…' statements to clarify their thoughts. The use of these terms was as a natural progression, something the students had devised themselves to justify their thoughts and representation. Consequently, as they justified their ideas, Reagan appeared to pose more 'What if?' types of questions, which in turn encouraged the students to pose their own questions. The thinking processes were methodical as the students represented, justified, and at times concluded with a 'so' statement, thereby signifying a generalisation. Thus, they were validating the truth of their own strategies, as well as the mathematical ideas being investigated.

To briefly summarise, successful curriculum implementation involved more than adjusting knowledge; it also involved the uptake of mathematical practices as a way to structure thinking and reasoning processes. It was when Reagan observed the students justify their ideas that she realised that they were making sense of ideas for themselves. She was inspired to capture this quality learning process, and therefore, continued using tasks that would evoke investigative thinking. In essence, it was the sense-making process that 'enabled' the students to begin constructing meaningful understandings, confirming that knowledge, thinking, communicating and doing are inextricably intertwined, as upheld by constructivist (Piaget, 1896-1980) and sociocultural theories of learning (Bruner, 1985; Cobb, 1994; Lincoln & Guba, 2000; Putman et al., 1990; Vygotsky, 1978). However, teaching and learning mathematics through inquiry also summoned changes to be made to the learning environment.

8.5 Establishing Communities of Practice

Supportive environments are necessary for successful learning. For students learning mathematics, the culture of the classroom needs to reflect the social and intellectual practices associated with working like a mathematician (Ball, 2003; Cobb, 1994; Goos et al., 2004; Lave & Wenger, 1991; Romberg, 1992). Similarly, it could be said that for teachers implementing inquiry based teaching approaches, the workplace environment should also reflect a culture of inquiry. Reagan, like Hartland (2006), believed that her own mathematical and pedagogical inquiry assisted her to become a better teacher. Similar to the teachers in Steinberg and colleagues' (2004) research, she acknowledged that the second pair of eyes that I as a researcher brought to the classroom helped. She found that through sharing her thoughts about the specifics of teaching enabled her to assume a more autonomous and critically reflective approach to her planning and teaching decisions.

Collegial support appears to be important for the teacher who is attempting to implement curriculum change. In this case, Reagan had not always felt supported. She experienced a type of professional isolation in her 'quest' to become a better teacher as the data drawn from the interviews portrayed. Without a colleague with whom to discuss and share teaching ideas, she perceived that her vision was not a shared vision. Fortunately for Reagan and this research, the notion of “researcher as sounding board” (Cavanagh, 2008, p. 123), helping teachers to generate and substantiate beliefs, proved to be beneficial. It became increasingly evident that her opportunity to co-reflect helped clarification and affirmation of the value of new methods of teaching.

In addition, like learning mathematics, implementing change is a continuous process requiring continued support. In this research, the researcher-teacher contact established a rapport that enabled Reagan to sense that the research was being carried out with her, rather than on her, as suggested by Glesne (2006). For example, she viewed the research to be contributing to her teaching practice as well as to the students’ learning, as reflected in this statement: “Something I value was in our research; we asked the kids what they thought was important in maths, or what they liked in maths, or what they found exciting” (IV5 p. 18). Clearly, she viewed the research as collaborative, which was a distinct advantage of the case study methodology. Yet she also viewed the research as something that would contribute to the students’ lives, as evident in the following statement:

> As a school, we often make decisions to go with a certain text; we say, ‘This is how, as a whole school, we are going to teach our maths,’ and when I’ve asked around, nobody asked the kids; nobody asked the kids whether it was actually working for them. (IV5, p. 18)

This statement summed up the essence of Reagan’s attitude towards teaching. She saw her practice as a way to connect with the students to ensure the best possible outcome. It seemed that in her view, her teaching and the students’ learning experiences were equally important and mutually supportive. This sensitivity to the students enabled her to establish an environment that was conducive to their mathematical knowledge development, practices and productive dispositions, as follows.

8.5.1 Establishing a Classroom Community of Practice

The learning environment includes both tangible and intangible environments. Both play a part in the establishment of positive learning experiences. In this case, Reagan was willing to make the environment more interesting for the students. She believed that her students viewed mathematics learning as boring, which was also evident in the initial self-report data, and thus to evoke interest she explored the notion of outdoors learning. This proved to be beneficial for many reasons. First, the students' interest in the learning, for some at least, increased, as was evident when two students had been removed from the outdoors measurement activity. The mere fact that one of the boys bothered to tell his parents about the incident indicated his disappointment about missing out. Second, the spaciousness seemed to be effective in a physical and cognitive sense. For instance, in the lesson where the students reconstructed their pictorial and numeric representations on the pavement, the outdoors openness provided opportunities for individual and collaborative investigative thinking. The space provided scope for creating large and clear visual images, and hence, each student was in clear view of each representation as the groups used these to justify their generalisations. The repetitiveness of this discussion appeared to promote greater clarity and sense making of the regularity and similarity in the mathematical patterns, a foundational aspect of algebraic reasoning (Mulligan, 2006).

Third, the spaciousness also helped the students to assume more control of the direction of the activity physically and mentally. This was especially evident in the measurement activity where they initiated inquiry, which in turn promoted personal sense-making. This third point relates to a notion reflected in the *Numeracy Position Paper* (Queensland School Curriculum Council [QSCC], 2001, p. 5). The paper stated that the "degree of personal meaning that students gain from particular tasks is proportional to the level of control each student has over the direction of the activity." However, clearly the confidence to challenge each other's ideas was also proportional to the students' ability to justify their mathematical thoughts and ideas, which became more apparent in the later stages of the research. Thus the outdoors learning enticed the students' interest, and gave them some control over the activity and a sense of ownership over the learning, which together promoted sense-making. They were developing knowledge, practising how to think and reason, and gradually expanding their mathematical confidence. This confidence however did not just occur; it involved sensitivity to the intangible environment, as below.

8.6 Developing Productive Dispositions

The opening quote of this paragraph stated that: "Good teachers join self, subject, and students in the fabric of life because they teach from an integral and undivided self" (Palmer, 1997, p. 2). The quote goes on to say that good teachers help students to "learn to weave a world for themselves.” This quote captures the communal spirit that Reagan brought to the classroom. Rather than drag or push the students along, she led them and showed them a way that helped in the establishment of their own sense-making processes and construction of meaningful mathematical understandings. Her aim was to help all of the students become, as she termed it, ‘mathematically smart.’ She wanted them to realise that they had a thinking process to follow and that there were a variety of ways to approach a problem, regardless of their level of understanding. Her goal was for them to realise that they simply needed to stop and think about what they did understand, and then work from that point, as she explained in the last interview. In essence, she wanted them to develop what Kilpatrick and colleagues (2006) described as a productive disposition.

As has been mentioned in the past, the learning setting needs to alter in ways that allow the students to feel a sense of belonging (Klein, 2000; Osterman, 2000). The setting thus needs to be about leading and guiding, as opposed to dictating, such that each student senses that their participation contributes to the learning within the community. Metaphorically speaking, the learning community is like being on a canoe, each person’s stroke contributes to the moving forward of the canoe, and the propulsion is dependent on effort, not just ability. As Kilpatrick and colleagues (2006) asserted, achievement should be viewed as a product of effort not ability. In this case, Reagan had an innate sense of the way to help her students sense that they were in a process of learning, regardless of their mathematical ability. She seemed to be able to help them to recognise that their present knowledge was not fixed and that it could be expanded upon. For example, she would either model her own thinking processes or ask them to help her solve a problem, suggesting at times that she was ‘stuck,’ rather than tell them, ‘This is how to solve the problem.’ She also pointed out that they were responsible for others’ learning as well as their own, as in the situation where one boy backed away from the group when it became clear that their pattern had not repeated itself.

Nonverbal cues also seemed to be helpful in whole class discussion. Inviting students into discussion is important because sense-making is dependent on students’ reflective thinking and communication of ideas, as pointed out by researchers (Battista, 1999; Clarke et al., 2002; Hiebert et al., 2000; Hunter, 2008; Steinberg et al., 2004), and as recommended in curriculum documents (DETA, 2008; QSA, 2004). However, there were two problems apparent when enticing students into discussion. First, self-confident students, if let go, can dominate discussion, as Baxter, Woodward and Olsen's (2001) study showed, and second, as Reagan explained, putting students on the spot when they are not ready can cause them to freeze and create further mathematical anxiety. Hence, promoting discussion is a sensitive issue, especially when communal conflict is necessary to promote deep mathematical inquiry (Palmer, 1999). Reagan was sensitive about the students' readiness to be probed further, and aware that some quieter students needed to speak their ideas. She used eye contact to invite students' ideas in discussion, and her strategy appeared to ease them into that discussion. Possibly because they could use non-verbal clues to indicate if they were not ready to speak, the subtle invitation seemed to coax them, as opposed to put them on the spot.

Holding high expectations of students has been pointed out as an attribute of effective mathematics teachers (Askew et al., 1999). There were subtleties in Reagan's interactional approach that showed the students that she held high expectations of them, but more than that, these subtleties also seemed to help the students sense that she thought they were capable, despite their ability. For example, she offered encouragement by saying that they were doing 'grade seven work,' and she set them in good stead for assuming practices reflecting successful mathematics users by referring to them as 'little mathematicians.' She also gave them time to think things through and demonstrated confidence in their mathematical abilities by making statements such as, 'I know you can work this out.' However, at times, this time delay meant she had to resist some students' ideas to give others in the class an opportunity to work though the process themselves. The point is that whilst she was encouraging, it did not come at the expense of enabling all students an opportunity to develop robust mathematical understandings and mathematics practices, as well as a sense that they all needed to be productive.

Learning is an individual and social process that occurs at different rates for different students within the one community. As Masters (2009a) reported, by year five the variation between the students' levels of understanding can span five years. Reagan was sensitive to this variation and was willing to accept all ideas regardless of how off the mark the ideas may have seemed. She also seemed to be able to resist interjecting, in the hope that the students would eventually figure things out for themselves. One example was when some of the students were not counting all of the objects in a pattern, and eventually, through the repeat discussion, they clarified this aspect for themselves.

Moreover, being a primary school classroom this learning community participated in different subjects, not just mathematics. Thus when one student called out the word ‘circumnavigate,’ which was totally unrelated to the mathematics lesson, Reagan did not ridicule him; she recognised that he was still making sense of his participation in a previous subject.

When addressing the varied levels of understanding, there is danger in any classroom that attention becomes focussed on the struggling learner, and the advanced learner may not always be extended to their full potential. However, as this research advanced, it seemed that the mathematics practices framework helped Reagan to extend some of the advanced learners, thereby making the learning more equitable for all. This was particularly noticeable in Tad’s comments on the self-report questionnaire. Tad was one of the advanced students, and when asked how he rated his mathematical ability, on the first form he indicated that he was ‘good at maths.’ Then on the second he described that he was ‘good at some things.’ Clearly, he was beginning to feel as challenged as his peers, yet at a different level. Furthermore, he had developed a sense that mathematics now involved more than ‘getting it right.’ In his response to the statement: ‘Describe an important piece of maths learned at school, and give a reason why you believe this is important,’ he wrote on the second form: “How to show working out because you may need to tell someone how you got the answer for something.” Mathematics for Tad, now included being able to justify his ideas.

8.7 In Summary

In summary, this chapter has highlighted that successful implementation, like learning is a process. Both require time, support and an element of sensitivity. The conceptual framework assisted in the organisation and analysis of the data because, as anticipated, successful learning was dependent on the interplay and interrelationships of ways of knowing, doing, thinking, being and interacting. For instance, the students needed some prior knowledge to bring to the learning experience, and then they needed to know how to work with, think about and communicate that knowledge confidently and productively. Their confidence was in turn dependent on their trust that they had some prior understandings to help make sense of the learning experience, as well as a belief in their ability to be able to apply that knowledge in thoughtful and productive ways. Importantly though, Reagan helped the learning process by establishing a classroom atmosphere that was built on trust, support and teamwork, rather than competition and dominance. The students were made aware that when working as part of a team, they were all responsible for their own, as well as each other's learning. Even though the students at the beginning of this research were unfamiliar with challenging each other's ideas in productive ways, they were beginning to recognise that communal inquiry and conflict were legitimate parts of the learning process.

The case study methodology was successful in the sense that change could be supported at a foundational level, within the lived experience of the classroom context, as suggested by researchers (Glesne, 2006; Lankshear & Knobel, 2005; Merriam, 1998; Stake, 2005; Stenhouse 1978; Yin, 1994). This attention then helped to build upon those experiences to make teaching decisions based on circumstantial evidence about how and what the students were learning. The pre- and post-questionnaires, teacher interviews and classroom observations were illuminating, as were the students' work samples. However, I recommend further research to also interview the students to draw out what contextual features they believe might enhance or inhibit their learning. The next chapter makes further recommendations as it responds to the research questions.

CHAPTER NINE
IN CONCLUSION

"A teacher affects eternity; he can never tell where his influence stops"
(Henry Brooks Adams, US Historian 1838-1918)

9.1 Introduction

This research has highlighted that teaching and learning occur through complex interactional relationships between the teacher, students and the curriculum. It revealed that developing meaningful mathematical understanding involves personal sense-making experiences, as does developing understandings about the proposed curriculum intentions and how to implement those intentions into daily teaching practices. The previous chapters make apparent that there are no quick fixes when implementing new approaches, as Lankshear and Knobel (2005) also pointed out; nevertheless, change in this case was possible. Even within the relatively short period of time in this research, changes were made to the teacher's practice. Whilst this single case study cannot be generalised to a whole generation of teachers (Burns, 2000; Shank, 2005), it has revealed characteristics of the implementation process that have the potential to guide further research about 'how' and 'why' decisions related to curriculum implementation. The chapter responds to the research questions:

1. What do teachers need to know and do to incorporate thinking, reasoning and working mathematically in their practice?
2. What effect will thinking, reasoning and working mathematically have on students' engagement and disposition towards mathematics learning?

The term 'responds' was deliberately chosen because to 'answer' the questions would involve generalised conclusions which was not the intention of this case study. Qualitative research is subject to change, thus making generalised conclusions impossible (Burns, 2000). For instance, the participants' and the researcher's understandings will continue to evolve, which would make this study difficult to replicate. A different setting, or the same setting at a different time, would inevitably highlight other contextual features. Hence, the questions will be addressed as applicable to the uniqueness of this case (Merriam, 1998). The hope is that as other teachers relate this case to their own experiences, they may draw their own conclusions, and then extrapolate ideas to trial and test themselves. In addition, the hope is to contribute insights about the process of implementation to inform those who make decisions regarding policy in mathematics education. The first research question has been separated, as follows.

9.2 What do Teachers Need to Know?

What do teachers need to know to incorporate thinking, reasoning and working mathematically in their practice? Educators have stressed that the teacher needs to help students understand the meanings or reasons for mathematical procedures based on recognising mathematical relationships (Ball, 1988; Ball, 2003; Hiebert et al., 2000; Kilpatrick et al., 2006; Mulligan, 2006; Skemp, 1986; Van de Walle & Lovin, 2006). However, the demise of mathematical thinking and intellectual rigour in mathematics classrooms (Hollingsworth et al., 2003; Masters, 2009a) indicates that teaching and learning is still predominantly procedure based. Hence, in some classrooms students are not being given the opportunity to investigate the how and why of mathematical procedures or the connections between mathematical ideas. The gap between curriculum intention and implementation therefore lingers. Indeed, the teacher's mathematical and pedagogical understandings and orientations are important; however, the findings of this research also add other dimensions.

First of all the teacher needs to recognise that there is a problem. If the teacher does not recognise this, the likelihood of change is limited. As Cavanagh (2006, p. 117) found, curriculum incentives often require teachers to "reconceptualise their views on the process of learning mathematics." Hence, implementing change is demanding; the teacher's constituted way of knowing and being will be at stake initially. Barriers created by internal thoughts and external influences may be difficult to overcome without guidance, support and confidence. Thus, without a sense of dissatisfaction about one's own practice, it is possible that teaching practices may never change. Reagan was not satisfied with the way she learned mathematics, nor was she satisfied with the way the students were currently learning mathematics in her school and classroom. Consequently, she was willing to implement the revitalised curriculum in the hope of promoting active inquiry for developing, in her words, 'functional' mathematics. Upon reflection, I realise that Reagan's willingness enabled collaborative inquiry between us, teacher and researcher, that helped us both reconceptualise our views about the process of learning mathematics.

Change can be unsettling, although it does not necessarily mean total abandonment of existing ideas, beliefs, or programs. For instance, in this research, at first Reagan feared that she may need to relinquish the program the school encouraged teachers to use. Programs, as in the one this school employed, often require financial outlay by the parents and the school; thus, to abandon them would be unreasonable. In addition, schools have vested interests in such programs, namely to benefit the children's learning outcomes. It became evident that the teacher does not need to abandon the school incentives to incorporate another approach; rather, in this case, she just needed to alter her view of the program, which was to integrate the program with her approach as a reinforcement tool. However, as became evident in this research, change can also be daunting because the teacher's desire to change has the potential to cause disequilibrium in the way things are done at the school, or in what the school values. For example, in this case Reagan was challenging entrenched values about the very meaning of what it meant to 'teach' and 'learn' mathematics.

Nevertheless, change does not need to disrupt the flow of things completely if it is implemented gradually. For instance, if the teacher is accustomed to direct teaching, then he or she may not need to abandon that method entirely. As in Reagan's case, she was able to combine approaches. The combination of approaches also appeared to support the learners and ease them into more investigative thinking practices. Reagan's students were unfamiliar with other teaching approaches; thus, to switch pedagogical styles suddenly could have been too disruptive to the classroom and possibly the school community. It seemed that when new teaching strategies were integrated into the classroom environment as necessary, the change appeared to be more enduring, at least while this research took place. Hence, implementing curriculum change is a process that takes time, as does learning.

The interpretation of the data highlights that learning is a complex and reciprocal process. Quality teaching and learning involve more than imparting and developing mathematical knowledge, and thus require more than simply focussing on the teacher's mathematical knowledge. The learning process is integral to the construction of robust mathematical understandings, idiosyncratic thinking and reasoning processes, and positive mathematical identities, and all three aspects are necessary to support further learning. As Grootenboer and Zevenbergen (2008) suggested, the students' past experiences add dimensions to their mathematical identities which are integral to further mathematics learning. Yet, as this research found, it is when the teacher knows that each and every student is capable of thinking, reasoning and working mathematically, and positions them as such, that positive experiences can be added to their mathematical identities.

The quality Reagan portrayed in this research was her ability to build on the students' pre-determined identities in a way that they could start to recognise themselves as capable mathematics learners. As indicated on the students' initial questionnaire responses, many of them believed they were 'not good at maths,' yet the second questionnaire, along with the observations, revealed that the students were beginning to recognise that they were in a process of learning. They were sensing that their mathematical knowledge was not fixed, which Kilpatrick and colleagues (2006) stressed was important for further learning. Yet, above that, they were sensing that their mathematical *identities* were not fixed. This innate quality was something that Reagan brought to the classroom environment prior to the commencement of this research, yet it became more prevalent as the research progressed. As both Reagan and the students were experiencing successes, for Reagan with the curriculum ideals and for the students with the process of learning, their competence and confidence expanded.

This research highlights that learning ultimately resides in the space between the learner and the teacher, and that sense-making is generated through internal and external interactional experiences. For example, Reagan herself found that when she communicated her reflective thinking about certain mathematical ideas, in her mind she recognised mathematical relationships. As she expressed in an interview, if this recognition was just occurring for her, then her students needed explicit guidance to draw out their mathematical thinking processes. Similarly, as she reflected upon and communicated about hers and the students' practical experiences she generated new understandings about both teaching and learning. In particular, she recognised pedagogical value in the mathematics practices framework as an effective scaffolding strategy to expand mathematics thinking and learning, as below.

Reagan believed, and I observed, that the three mathematics practices supported thinking and reasoning, communication and reflection. As the students used representations to test and justify their ideas, they seemed to be able to communicate their thinking effectively. Through discussion, when speaking as well as listening, they were also engaged in reflective thinking that further advanced sense-making. As Battista (1999) and Hiebert and colleagues (2000) suggested, when the students replayed experiences, either as they listened or spoke, they seemed to be able to connect ideas in their minds. It was this linking of ideas that Battista and Hiebert and colleagues, as well as Skemp (1986), believe supports conceptual/relational understanding, and thereby cognitive development and knowledge construction. Further, Reagan could more easily determine the students' mathematical thinking, which meant she believed she was starting to develop more effective questioning techniques to scaffold, clarify and expand mathematical thoughts.

Sense-making was integral to the development of mathematical competence, confidence and inquiry. For instance, as the students represented and justified their ideas they were able to validate the truth of ideas for themselves. This self validation meant that they were less reliant on the teacher and more equipped to engage in mathematical inquiry. However, reasoned argument only started to appear in the last lesson, thus reiterating that learning is a process that takes time. The students needed to learn how to think and reason mathematically, to substantiate their own ideas, before they could effectively challenge and debate each other's ideas. Conversely, once they started to recognise that they could think and reason effectively, they appeared to become more engaged physically, socially and intellectually in the learning process.

Thus what does the teacher need to know? In this case it became apparent that recognising the intention of the curriculum to address a real problem was important to promote a willingness to implement change. Secondly, the teacher's patience, investigative and reflective nature towards the implementation process and subsequent learning experiences helped to yield results. In addition, the fact that Reagan valued the students' sense making processes helped her to persevere, trial, and discuss ideas until she believed that they were making sense of things for themselves. In her own experience, she found that sense-making helped her to see sense in mathematics, and thus she wanted the students to experience the same empowerment in the hope that they would recognise that mathematics was logical, useful and worthwhile. Importantly, it was when she felt that she could structure the thinking and reasoning, communicating and reflective thinking processes, through employing the mathematics practices, that substantial changes were made to the learning process. However, promoting the mathematics practices framework to scaffold robust mathematical thinking and reasoning required altering the environment in some deliberate ways.

9.3 What do Teachers Need to Do?

The learning environment needs to support active mathematical inquiry. Whilst the practices of both teaching and learning are idiosyncratic, this research makes clear that these practices are bound by a system of complex and interrelated classroom features and relationships. First, the environment needed to be stimulating to evoke mental and physical engagement. Hands-on learning and outdoors learning seemed to stimulate the learners, and at times, motivated self initiated, mathematical inquiry. Second, the activities needed to promote investigative thinking. The collaborative teacher-researcher effort enabled tasks to be modified to indicate potential for the three mathematics practices to be utilised and practised. Hence, collegiality or a mentor may be helpful for teachers in their quest to plan inquiry-based activities.

However, it was Reagan's ability to establish a culture of neutrality that appeared to effectively support engagement, positive dispositions and thereby productive learning. She valued each learner's ability, not just the individual as a person, but also his or her mathematical ability, despite their level of understanding. This was evident in the way she encouraged the students by showing them that she had high expectations of them, such as, 'I know you can do this,' and, 'You are doing grade seven work.' Her choice of phrasing also demonstrated that she valued their ideas, such as: 'Good thinking,' as opposed to: 'Good boy' or 'Good girl.' Hence, rather than focus on developing a 'positive' disposition as such, she was focussed on developing a 'productive' disposition. Her aim was to help each student recognise his or herself as "an effective learner and doer of mathematics" (Kilpatrick et al., 2006, p. 131). She wanted them all to believe, as Kilpatrick and colleagues and the RAND Study Panel (Ball, 2003) also made clear, that steady effort contributed to successful learning.

Thus, what does the teacher need to do? In this case valuing the stages of learning, and each learner's capabilities helped Reagan to value the gradual process of implementation. She realised that the environment needed to change, yet she was not pushy about instigating change. Rather, she appeared to have an innate ability to show the students 'ways of being' that they could then take up. First, she showed them how to interact in positive, supportive and productive ways, and second, she showed them ways to think and reason, communicate and reflect mathematically. Instead of directly teaching the mathematics practices, she modeled the practices as she spoke her own thinking processes. This showing of 'ways of being' seemed to enable a natural uptake of the practices, as was evident by the students use of 'If…then' statements. Furthermore, importantly, it helped to enable the uptake of interactional approaches indicative of a hospitable classroom atmosphere, as recommended by Palmer (1999).

9.4 Changes to the Students' Engagement and Disposition

In the later stages of the research, the students resembled 'little mathematicians,' as Reagan referred to them, feeling secure and licensed to contribute effectively to the discussion. They appeared to relate mathematical confidence to successful learning, which related to some researchers' perspectives (Killen, 2003; Kilpatrick et al., 2006; Skemp, 1986). The data drawn from the pre- and post-questionnaires suggested that over the course of the research the students altered their view of mathematics as a discipline. Initially they perceived mathematics to be a whole host of isolated facts, evidenced by the way they described mathematics as symbols such as: "+, -, x, ÷". The symbolic representations indicated that they viewed mathematics as operations, rather than as a body of knowledge and applications that assist one to make sense of the world. Their restricted view was evident in the observed data as well, such as not knowing how or why certain procedures worked.

There was a distinct difference between the students' pre- and post-statements related to the importance of mathematics and mathematics learning. Many students were beginning to recognise the relevance of learning mathematics, beyond the fact that it was a requirement of their schooling. Their change in attitude was also noted in the classroom. For example, towards the end of the research, many responded to a student's confusion about the purpose of developing algebraic thinking with varying real-world examples. Seeing the relevance of mathematical applications is an integral aspect of becoming numerate, and clearly these students were able to bridge the gap between in-school mathematics and beyond school mathematics, as recommended by educators (Coben, 2000; Thornton & Hogan, 2004; Willis, 1998) and policy documents (DETA, 2008; NCB, 2008; QSA, 2004).

Disengagement and mathematical anxiety have been long standing problems (Battista, 1999; Ellerton & Clements, 1989; Grootenboer & Zevenbergen, 2008; Wilson & Thornton, 2006; Skemp, 1986). Clearly, before this research began the students were disengaged. For instance, in my orientation visits to the classroom I was greeted by groans from the students when I entered. The groans were not directed at me personally; rather, at the thought of an upcoming mathematics lesson. By contrast, as the research progressed, an obvious change in the students' thinking about mathematics learning was evident in their attitude about what was involved in the learning process. This change was evident by the unintentional yet expanded length of mathematics lessons and their willingness to persevere. It seemed that the students' motivation to puzzle over ideas reprieved their preconceived attitudes that mathematics was boring. It is interesting to note that the self-report data did not reveal such enthusiasm. However, as Reagan pointed out, maybe the accumulation of the students' past experiences had conditioned them to dislike mathematics in ways that would take more than two school terms to disrupt.

9.5 Where to from Here?

The gap between curriculum intention and implementation is a serious issue, especially in respect to the endeavour to improve the quality of mathematics teaching and learning. This study has revealed that implementation of new teaching practices can be worthwhile, and that changes can be made even in short spaces of time, as also confirmed by Barber and Mourshed (2007). However, this research has also highlighted potential barriers to change. First might be the misalignment of the teacher's, the school community's and the curriculum writers' pedagogical orientation and values. Further widespread research is needed to investigate what is required to ensure all communities of practice are working towards the same vision. Whilst collaborative efforts have been recommended by the Deputy Prime Minister, Julia Gillard (2008), Freebody (2005), and Masters (2009) to investigate and share stories about what works, suggestions about how to create such efforts are limited. Aligning the school's, the teacher's and policy values about what is important in mathematics teaching and learning requires effort, which leads on to the next point.

Implementing change can require preconceived ideas about teaching and learning, and possibly mathematics, to be interrupted. If the teacher is expected to gain control over curriculum ideals he or she needs to personally experience and explore the ideas (Lloyd, 2008). However, if he or she is not dissatisfied with the ways things are, then potentially there will be minimal motivation to change. Curriculum documents (DETA, 2008; QSA, 2004) encourage constructivist theories of learning for students, yet teachers often participate in professional programs where they are positioned as passive recipients of information. Ultimately, the teacher, too, is a capable and intelligent learner too, who will bring prior experiences to make sense of new learning, as students do. If the teacher has never experienced teaching as inquiry, then this is not an experience he or she will bring to the teaching and learning environment, or to his or her sense-making of the curriculum incentives. Thus, the teacher may also need collaborative inquiry-based opportunities to construct new knowledge or remodel existing pedagogical and/or mathematical knowledge and understandings.

One way to assist teachers and research may be to generate other detailed studies focussing on the specifics of teaching and learning, as in this study. The teacher-researcher collaboration could assist both the teacher and the researcher to reconceptualise their views, as in this research, yet in other ways. Then, as has been recommended (Burns, 2000; Lankshear & Knobel, 2005; Merriam, 1998; Stenhouse, 1978), an accumulation of other unique case studies could begin to form a data base for future comparisons to be made for theory building. However, upon reflection, it may be fruitful to conduct student interviews, as well as teacher interviews, to draw out their thoughts about what inspires them when learning mathematics. The aim would be to gain further insight into what it takes to develop productive mathematical dispositions and identities needed for successful and lifelong numeracy learning. However, such detailed studies can be time consuming and resource dependent, and possibly too localised to make enduring changes to a wider spectrum of teaching and learning environments.

Another recommendation to help teachers disrupt and remodel existing pedagogical and/or mathematical knowledge and understandings could involve a collegial mentoring program. For example, educational bodies may need to allocate time for professional learning days where groups of teachers come together to share experiences. As in Cavanagh's (2006) research, and as Clandinin (2008) expressed, creating opportunities for teachers to share stories can help to shed light on what works and how. Teachers need to be respected for the experiences they bring because the implementation occurs within their classroom. It is the teachers themselves who experience the potential barriers or successes of practical implementation. They need to be given a presence in the research community and an opportunity to share some of the positive teaching and learning events they have experienced with each other. Criticising teachers for their perceived lack of knowledge is unproductive. Rather, teachers need to be recognised as capable 'thinking' teachers, as Reagan suggested. They need to be given opportunities, like students learning mathematics, to investigate teaching and learning ideas in ways that assist meaningful knowledge construction of contemporary curriculum goals.

As a suggestion, a teaching 'community of practice' could be established through bringing groups of teachers together to mentor each other at professional learning days. These days could involve teachers collaboratively planning investigative tasks, composed of thinking and reasoning, communicating and reflecting learning processes. One way to scaffold such learning processes may be to indicate opportunities to promote the mathematics practices. I suggest that this is 'one way' because other teachers may bring other ideas. Then, after implementation, the teachers could bring their experiences back to the collegial mentoring group to point out challenges or successes of implementation and of the students' subsequent learning experiences. Student work samples could also be analysed and evaluated to highlight successes of implementation and to work towards improving these learning experiences. Such a program could work in collaboration with input from universities or curriculum bodies. Sense-making is generated through interactions and thus the opportunity to reflect on and communicate experiences with other professionals may enhance the depth of pedagogical inquiry.

Without engaging teachers in collaborative and personal inquiry, the proposed directions for teachers may never be actualised. Curriculum bodies are committed to improving the situation in mathematics teaching and learning and continue to propose broad directions for teachers as in the recent the *National Mathematics Curriculum: Framing paper* (NCB, 2009b). If teachers are to develop productive professional dispositions towards improving mathematics teaching and learning, then they need personal sense making experiences, like students, to substantiate the value of proposed ideals for themselves. I make this point because the paper, *The Shape of the National Curriculum: A Proposal for Discussion* (NCB, 2009a) has suggested providing work samples to enhance teachers' learning about achievement standards set out by the curriculum documents. The intended examples will outline the task and the student's response, and then indicate annotations setting out the basis for assessment (NCB, 2009a). However, when teachers are being encouraged to assess students' thinking and reasoning processes, the question remains as to how these learning processes can be demonstrated accurately on given work samples. As Reagan pointed out, often what is recorded on the page does not clearly depict the thinking and reasoning, communicating or reflection processes that occurred. Each teacher's experience will be context specific and thus generic examples do not give the teachers opportunities for personal sense-making, nor to develop ownership or control, at a personal level, of the curriculum intentions.

Furthermore, the notion of inquiry has been re-emphasised in the *National Mathematics Curriculum: Framing paper* (NCB, 2009b). The board intends for teachers to frame learning around the development of *content* and *proficiency* strands (pp. 2-3). The inclusion of proficiency strands demands increased emphasis on being able to think and act in mathematically proficient ways. These actions include the development of understanding, fluency, problem solving and reasoning. Understanding involves the development of 'conceptual understanding,' building robust and transferable understandings based on recognising mathematical relationships, and knowing how and why ideas work. Fluency means developing procedural fluency, being able to readily recall, choose and use procedures flexibly. Problem solving requires the development of strategic competence, an ability to investigate problems and effectively communicate thoughts and mathematical ideas. Last, reasoning includes the capacity to think and reason mathematically in ways that students can justify and prove ideas for themselves and formulate mathematical generalisations. The NCB (2009b) has based the proficiency strands on Kilpatrick and colleagues' (2006) ideas. However, if the teacher has never experienced inquiry-based teaching to develop these proficiency strands, they will need external support, as well as personal experience and possibly collaborative exploration.

Hence, establishing collegial mentoring groups has the potential to support and guide generative change in similar ways to Lave and Wenger's (1991) 'community of practice' supports collaborative classroom mathematics learning. Just as students learn through social interaction, when they communicate their thinking about ideas, the constant collegial contact may enable new ideas to filter into the teachers' existing repertoires in meaningful and manageable ways. Learning is a reciprocal and complex process and new strategies, like new mathematical ideas, need to be integrated with understanding. Thus the immersion may also impact on the teacher who is not yet dissatisfied with their practice. In other words, the content teacher, whose practice does not alter from year-to-year may, by virtue of discussion, listening or observation, may gradually see potential in trialling some new or different approaches. Without feeling pushed or pulled to take-up a new way, this gentle immersion in collegial discussion could create potential for the content teacher to recognise benefits of new ideas by what others have been reporting.

Another advantage of establishing a collegial community of practice is the development of productive dispositions towards mathematics teaching and learning. If the teacher feels productive, then this attitude will hopefully be translated into the classroom to provide more positive and productive experiences for the students. Hence, both the teacher and the students will be learning through inquiry-based modes as they challenge their own and each other's ideas in productive ways. Also, as the teachers support each other's inquiry, they may take away ideas from colleagues to trial and investigate, and shed further light on the successes of different approaches or tasks, or refute claims. By testing and discussing ideas, the teacher will have the opportunity to develop idiosyncratic pedagogical strategies based on their own substantiated professional inquiry. Importantly though, each and every teacher will have the opportunity to reflect on his or her practice and the students' learning to take up, in their own way, revitalised curriculum ideals. The sharing of these stories has the potential to enhance learning for all, teachers, students, schools, teacher-educators and policy makers. If these practical experiences were documented they could also be added to the aforementioned data base, and thereby significantly contribute to theory building in mathematics education literature.

In essence, there is not one right way of knowing or doing for mathematics teachers who are attempting to implement curriculum change. Quick solutions to improve mathematics teaching and learning are non-existent. However, change can be possible when teachers, like their students, are given support, when they can discuss and inquire into both the teaching and learning processes, and when they can then base further decisions on this inquiry. As the literature review revealed, mathematics teaching and learning requires a shift in gears to drive change. Now investigative thinking must underpin mathematics learning, and thus, teachers need to engage students actively, physically, mentally and socially in the learning process. The classroom context thereby, needs to be transformed to evoke supportive and effective communal inquiry and where attention is given to simultaneously developing the students' ways of knowing, doing, thinking and being; one cannot be developed without the other. Hence, if a context could be established where mathematics teachers, teacher-educators, and curriculum writers were engaged in similar levels of communal inquiry, then it is possible that the support may assist teachers to revitalise their programs in productive and structured ways. The landscape for teaching will continue to change with rapid technological advances and changes to society, thus teaching and learning practices will need to continue to evolve. A collaboration of exchanges based on experiential inquiry may assist in continued expansion and growth of teaching and learning ideas.

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Growing Patterns (Sequences)

Activity 1 – Extend and Explain (Adapted from Van de Walle & Lovin, 2006, p. 266)

CLO PA3.1 Core content: Patterns -rules based on previous term -missing term

Show students the first three or four steps of a pattern.

Provide them with appropriate materials, e.g. workstations

1. Grid paper and coloured pencils
2. Lego blocks
3. Counters
4. Numbers and letters with paper
5. Various shapes – triangles, circles, squares
6. Match sticks

Students work in groups or pairs at each station. They create (represent) and discuss patterns and should try to determine how each step in the pattern differs from the preceding step. Look to see if each new step can be built onto or changing the previous step, the discussion should include how this can be done (justification). Students should record their growing patterns (to use in the next activity when they begin to look at the numeric component of the patterns). Their discovery of the next step in the sequence is the generalisation. They conclude that if the pattern grows because ... then the next step of the sequences will look like...

Key idea:

Represent, justify and generalise how patterns grow.

Reflection:

Growing Patterns (Sequences)

Activity 2 – Growing Patterns: The Numeric Component

(Adapted from Van de Walle & Lovin, 2006, p. 267)

CLO PA3.1
Core content: Patterns -rules based on previous term -missing term Functions -representations of relationships

Growing patterns also have a numeric component which is the number of objects in each step. A table can be made for any growing pattern. One row in the table is for the number of steps in the sequence and the other is for recording the number of objects in each step. Patterns grow quickly and therefore it is only reasonable to build the first five steps, because of the amount of blocks needed or the amount of space needed to draw the pattern.
In this activity students record their patterns in a table. They start to look at the relationship between the number of steps and number of objects in each step. This leads to being able to predict the next step.

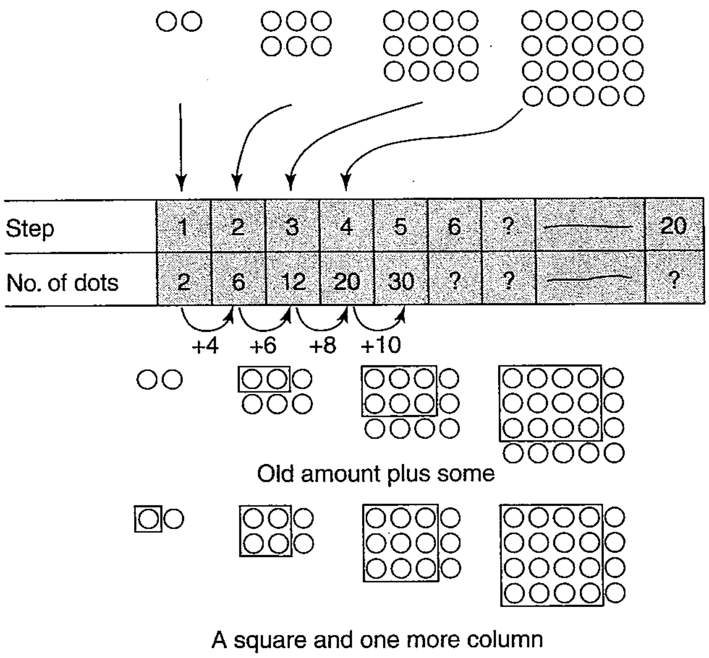


FIGURE 9.2

Two different relationships in a visual pattern.

(Van de Walle and Lovin, 2006, p. 267)

Key idea: A relationship exists between the step number and the corresponding number of objects.

Reflection:

Growing Patterns (Sequences)

Activity 3 – Predict How Many (Adapted from Van de Walle & Lovin, 2006, p. 267)

CLO PA3.1 & PA4.1	CLO N3.2 & N4.2
Core content: **Patterns** -rules based on position of terms -ordered pairs (tables) **Functions** – rules relating to two sets of data -representations of relationships -ordered pairs (tables) -number sentences	Core content: **Addition and subtraction** -whole numbers Connections -inverse (backtracking) **Computation methods** -written recordings - student generated

Students investigate and extend a simple (addition sequence) growing pattern either one that you provide, as in the following examples, or one that they have created. The students make a table showing how many objects/items are needed to make each step of the pattern. They predict the number of items in the twentieth step. The challenge will be to predict the number of items without filling in the first 19 steps/entries of the table. The students will:

- Create a table, manipulate objects or draw the objects to *represent* the pattern.
- *Justify* their reasoning, for example, how many items would be in step 20 or step 6 etc.? How much does the pattern grow? What is the relationship between the step number and the number of items in the pattern?
- Conclude with a *generalisation* (rule) which will enable a prediction of the number of items in any step of the sequence.

Key idea: It is important for the students to see the **relationship** of the two representations of the patterns – the table and the drawing or objects.
The table is the numeric version and the created pattern is the physical version. Students must be given time to discover that when the pattern changes physically the numeric representation also changes, thus whatever relationships they discover actually exist in both forms of representation. Students can be challenged by finding a relationship in a table and investigating how it would play out in the physical version.

Definitions
The description that tells how a pattern changes from step to step is a *recursive relationship.*
Arithmetic sequences are created by adding or subtracting the same number each time.
Geometric sequences are created by multiplying the same number each time.
Functional relationship is the rule that determines the number of elements in each step from the step number

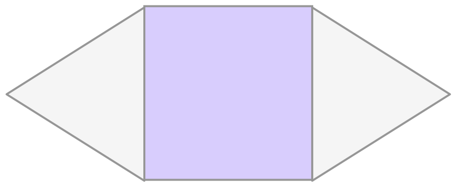
Reflection:

Growing Patterns (Sequences)

Activity 3: The Growing Worm (Adapted from an activity cited in Klein, 2006)

Procedure:

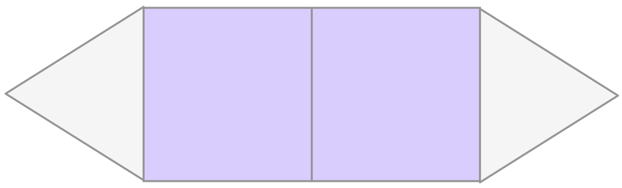
1. Using an overhead projector, display three pattern blocks (as in the following diagram) to represent the worm. Tell the class that this is a worm, and this worm eats a lot and grows a lot every day.



Representation

2. Ask students how many blocks were used for the first day. Make a T-table to record the days and number of blocks used.
3. On Day 2 the worm gets larger, and now is made up of four pattern blocks, as in the diagram here. Record these data in the T-table.

Day	Blocks
1	3
2	
3	
4	



Justification

4. The worm's body gets one square longer every day. Model how the worm grows by adding one square in the body and record the data in the T-table.
5. Have the children use pattern blocks to continue to build the figures and draw them on a recording sheet, recording the numbers in the T-table each time.

Generalisation

6. As a class, discuss the pattern and write a numeric equation for each step.
e.g. 1+2
2+2
3+2
4+2

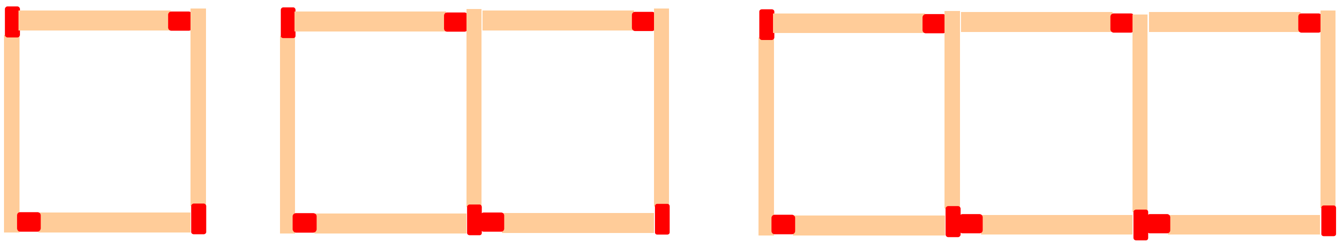
Growing Patterns (Sequences)

Activity 3: Growing Matchsticks Pattern

(Adapted from an activity cited in Klein, 2006)

Representation

Students make this pattern with matchsticks and record the pattern on a T table



No. of Element	Matchsticks
1	4
2	
3	
___	
___	
n	

Justification

Students will:

- Describe in words how many sticks must be added to make each new element in the pattern.
- Explain what changes, how does the pattern grow.
- Write their reasoning.

Generalisation

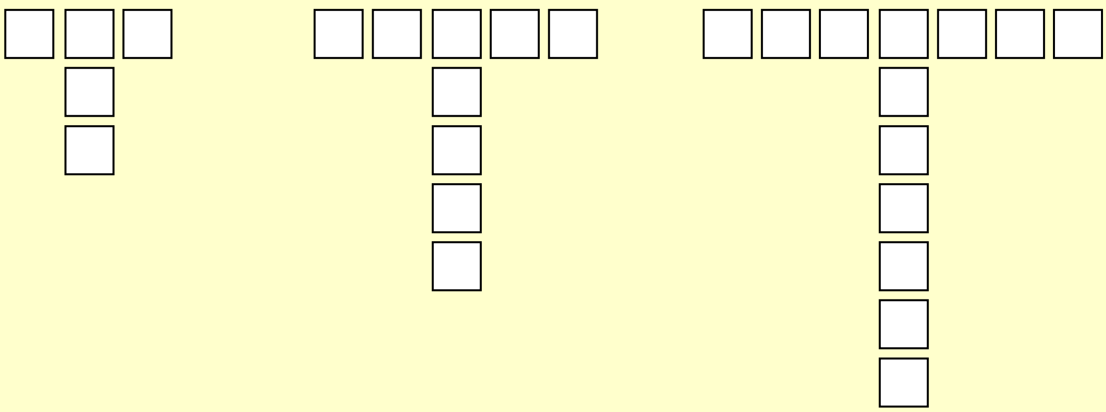
Write a numeric equation for each step (next to the corresponding step on the table)

Growing Patterns (Sequences)

Activity 3: Growing Tiles Pattern (Adapted from an activity cited in Klein, 2006)

TASK

Here are the first 3 steps of a pattern. Build the pattern with small squares, extend the pattern and find a way to show how the extension follows the pattern.



Students should comment on such things as the increase along the horizontal and vertical bars, and the amount of increase needed to construct each new shape.

The students will:

Represent

- Build the pattern, draw it, and create a table to record the number of tiles in each step.

Justification

- Describe shapes further along the sequence.
- Guess and check what shape after the next (fifth shape) will look like.
- Investigate if a shape will ever have 12 pieces along the top. Explain why
- Investigate if a shape will have 12 pieces below the top section. Explain why.
- Describe the twentieth shape in the sequence. Determine how many pieces will be needed to make it?

Generalisation

- Write a numeric expression for each step. Make a concluding statement about the growing pattern and the relationship of the number of elements to the step number.

Functional Relationships

The recursive step-to-step pattern

(Adapted from Van de Walle & Lovin, 2006, p. 268)

CLO PA3.1 PA3.2 & PA4.1 PA4.2	**CLO N3.2 N3.3 & N4.2 N4.3**
Core content: **Patterns** -rules based on position of terms (combination of operations -ordered pairs (tables, discreet data) **Functions** -rules relating to two sets of data -backtracking -representations of relationships -ordered pairs (tables) -number sentences **Equivalence** -guess and check -methods for solving equations **Representations** -equations (number sentences) -symbols for unknowns	Core content: **Addition and subtraction** -whole numbers Connections -inverse (backtracking) **Multiplication** -combinations of whole numbers **Connections** -inverse (backtracking) -missing factor **Computation methods** -written recordings - student generated

A *Functional relationship* is the rule that determines the number of elements in each step from the step number.

Van de Walle and Lovin (2006) suggest that there is not a single best method for finding the functional relationship between step number and step. Students need to 'play around', investigate with the numbers and ask questions such as, "What do I need to do to the number of the step to get the corresponding number in the table?"; "What relationship exists between this part of the pattern and the step number?"

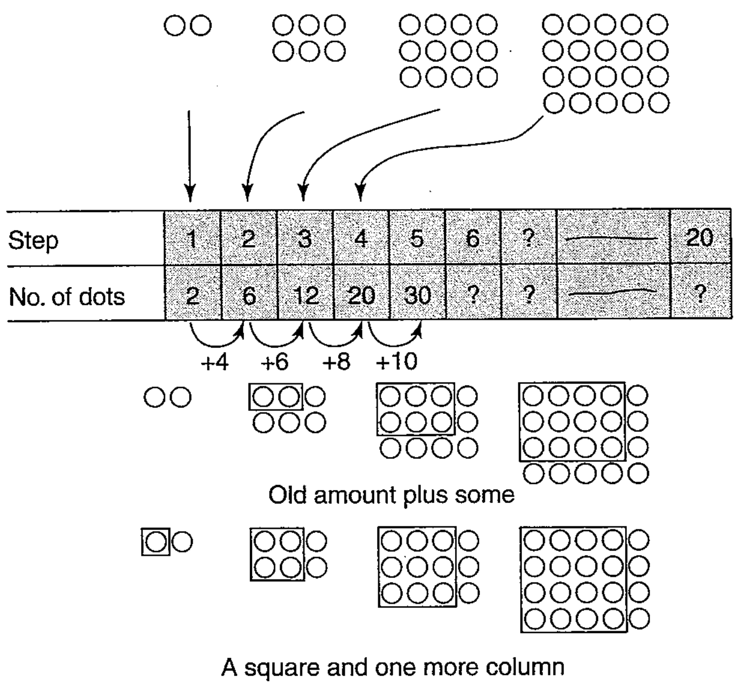


FIGURE 9.2

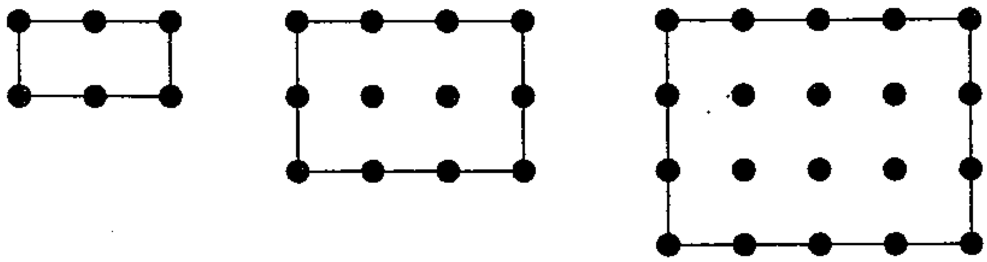
Two different relationships in a visual pattern.

Functional Relationships

(Adapted from Van de Walle & Lovin, 2006, p. 269)

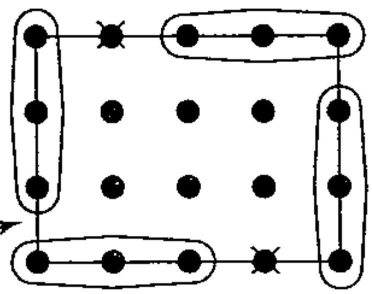
Moving from Patterns to function and variable

- Now that students have discovered a numeric expression for each pattern it is time to introduce the function variable expression.
- The students should be able to draw a bracket around the step number as this will be the only part of the numeric expression that changes from step to step. The other numbers in the expression should stay the same.
- The bracketed numbers can then be replaced with a variable. Thus they have just written a 'general' formula which explains the functional relationship between the step numbers and the step values.



Frame	1	2	3	4	
Dots on border	6	10	14		

Notice: Each long side has one more dot than the short side. Take these away and ×4 helps tell how many dots.



[1] × 4 + 2
[2] × 4 + 2
[3] × 4 + 2
[4] × 4 + 2

General formula
$(n \times 4) + 2$

Figure 9.3 (Van de Walle & Lovin, 2006, p. 269)

Functional Relationships

Activity 4 – Return to the *growing worm, matchstick* and *tile patterns.*

(Adapted from Van de Walle and Lovin, 2006, and activities cited in Klein, 2006)

The students will:

Represent

- Represent the pattern as a numeric pattern (table) and as the physical pattern (either draw it or manipulate objects. Then represent each step with a numeric expression.

Justification

- Investigate the relationship that exists between this part of the pattern and the step number
- Find as many patterns as possible in both representations – the table and the physical pattern
- Explore and explain the relationship that exists between the number of elements and the step number (the functional relationship)
- Create a numeric expression that is similar for each step of the physical pattern and of the numeric pattern (table). For example in figure 9.2 the first four steps are $1^2 + 1$, $2^2 + 2$, $3^2 +1$, $4^2 + 1$. (Be sure that students see that the physical pattern and the table are related, what occurs in the physical pattern also occurs in the table)
- Look for the part that stays the same and the part that changes in each numeric expression and put a bracket around the part that changes
- Insert a variable where the bracketed number is and explain why this can be done.
- Use the formula to predict the next entry on the table and check this with constructing the actual pattern
- Predict the twentieth step – the twentieth entry in the table

Generalisation

- Write a proven variable expression (the formula) to enable prediction of further steps.

Key idea: the search for the relationship, Van de Walle and Lovin suggest this may take time however; it is the most significant step in algebraic reasoning.

Reflection:

Functional Relationships

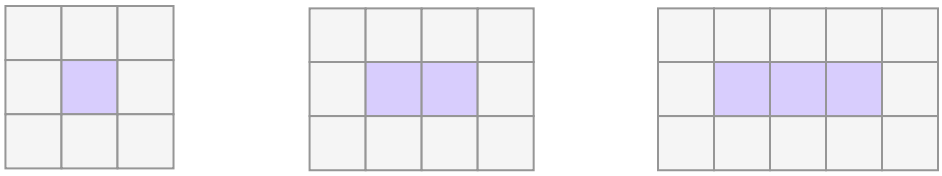
Building Bigger Swimming Pools (Assessment possibility)

[Adapted from Reys, Lindquist, Lambdin, Smith & Suydam, 2004, p. 351).

Key idea: Describe the functional relationship that determines the number of elements in each step from the step number. Students may use begin to use variables expressions.

Representation

Provide students with square tiles and instruct them to model swimming pools of different sizes, where the tiles surround the pool.



Record the number of tiles needed to build the pools in this table.

Size of Pool	Number of Cement Blocks Needed
1	8
2	10
3	12
4	14
?	
?	
10	
?	
?	
20	
?	
?	
N	?

Justification

- How many tiles are needed to build a pool that is 10 tiles long?
- How many tiles are needed to build a pool that is 20 tiles long?

Generalisation

- Find a general (proven) rule that describes pools *N* tiles long.

Functional Relationships

Extension activity: Number Patterns

(Adapted from Van de Walle & Lovin, 2006, p. 269)

The challenge in the number patterns or sequences is to determine the next number and to determine a general rule for the *n*th number in the sequence.

2, 4, 6, 8, 10, ...	Even numbers; add 2 each time
1, 4, 7, 10, 13, ...	Start with one and add 3 each time
1, 4, 9, 16, ...	Squares; 1^2, 2^2, 3^2
0, 1, 5, 14, 30, ...	add the next squared number
2, 5, 11, 23, ...	double the number and add 1
2, 6, 12, 20, 30, ...	multiply the successive pairs of counting numbers 2x3=6,3x4=12, 4x5=20, 5x6=30
3, 3, 6, 9, 15, 24, ...	Fibonacci sequence, add the two preceding numbers.

Functional Relationships

Activity 5 – graphing the patterns

(Adapted from Van de Walle & Lovin, 2006, p. 270)

CLOs: PA3.1 PA3.2 & PA4.1 PA4.2	**CLOs: N3.2 N3.3 & N4.2 N4.3**	**CD4.2**
Core content: **Patterns** -rules based on position of terms (combination of operations -ordered pairs (tables, discreet data) **Functions** -rules relating to two sets of data -backtracking -representations of relationships -ordered pairs (tables) -number sentences **Equivalence** -guess and check -methods for solving equations **Representations** -equations (number sentences) -symbols for unknowns	Core content: **Addition and subtraction** -whole numbers Connections -inverse (backtracking) **Multiplication** -combinations of whole numbers **Connections** -inverse (backtracking) -missing factor **Computation methods** -written recordings - student generated	Core content **Collecting and handling data** -Continuous data **Exploring and presenting data** -Line graphs

Growing worm, matchstick, tile patterns and *building a swimming pool*

With the four activities students plot the results on a graph. (If feeling brave ask the students to come up with their own way to represent the data on a graph.) or ...
The *x* axis can be the step number and the *y* axis can be the number of elements in each step. The students can then investigate other graphs and predict what the formula may be.

x axis is the step number
y axis is the number of elements in the "dots on a border pattern" (*n*x4 + 2) or 4*n* + 2

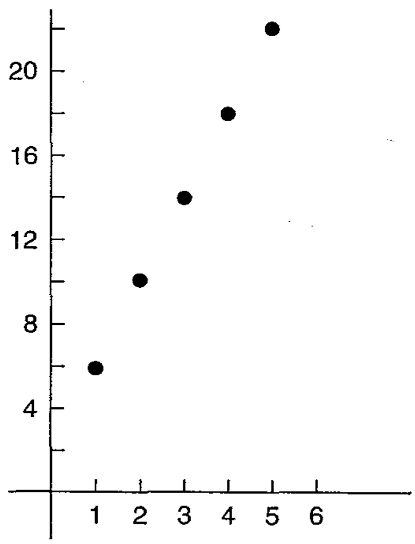
Therefore: $y = 4x + 2$

Key idea: to make the connection that data on a line graph can be a representation of a pattern or an algebraic formula.

Reflection:

FIGURE 9.4

Graphs of growing patterns.



"Dots on border" pattern from Figure 9.3

Activity 6 – Guess my rule

(Adapted from Van de Walle & Lovin, 2006, p. 272).

Key idea: To have fun and to practice algebraic formulas (functional relationships)

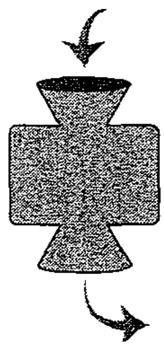
CLO PA3.1 PA3.2 & PA4.1 PA4.2	**CLO N3.2 N3.3 & N4.2 N4.3**
Core content: **Patterns** -rules based on position of terms (combination of operations -ordered pairs (tables, discreet data) **Functions** -rules relating to two sets of data -backtracking -representations of relationships -ordered pairs (tables) -number sentences **Equivalence** -guess and check -methods for solving equations **Representations** -equations (number sentences) -symbols for unknowns	Core content: **Addition and subtraction** -whole numbers Connections -inverse (backtracking) **Multiplication** -combinations of whole numbers **Connections** -inverse (backtracking) -missing factor **Computation methods** -written recordings - student generated

ACTIVITY 9.4

Guess My Rule

A simple in-out box is drawn on the board as in Figure 9.5. The machine "operator" knows the secret that is "stored" in the machine. For example, a rule might be $2n + 1$ (double the input number and add 1). Students try to guess the rule by putting numbers into the machine and observing what comes out. Students tell the operator what number to put in. The operator tells what number comes out. A list of "in-out" pairs are kept on the board. Students who think they have guessed the rule raise their hands. As more numbers are "put into the machine," those who think they know the rule tell what comes out. Continue until most have guessed the rule.

(Van de Walle and Lovin, 2006, p. 272)



In	Out
3	7
4	9
10	21

Represent

Students use symbols or drawings to represent their mathematical ideas

Justification

Students explain their thinking and reasoning – how they created the rule.

Generalisation

Students create a general formula

Reflection

Activity 7 – Another Growing Pattern (Adapted from Van de Walle & Lovin, 2006, p. 273)

'The Fibonacci Sequence'

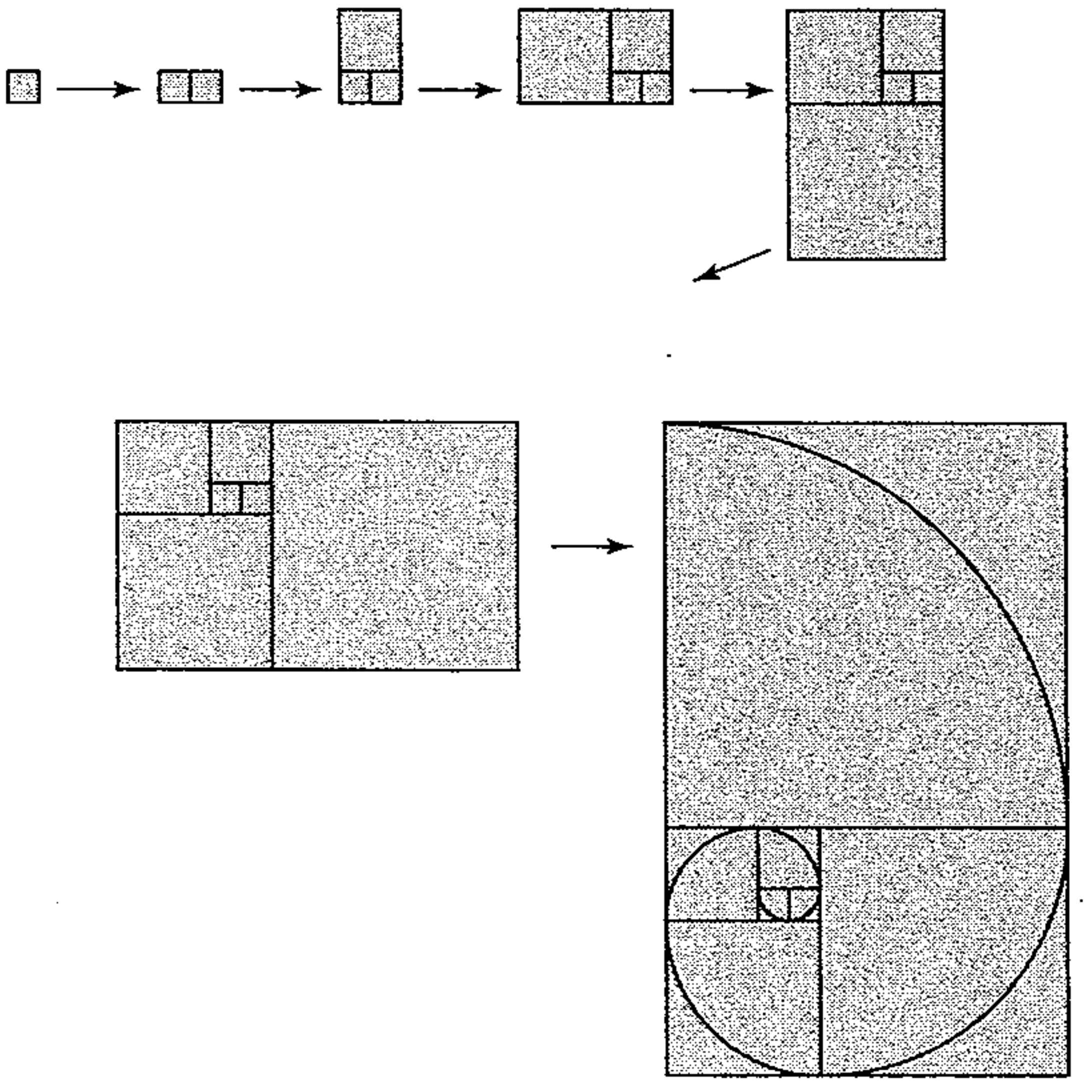


FIGURE 9.6

A growing pattern of squares: Each new rectangle is a little closer to a "golden rectangle."

Each successive step of this growing pattern is built by adding a new and larger square to the design. What would the side of the next square be?

1,1,2,3,5,8,13

The ratio for the adjacent numbers in the sequence gets closer and closer to the single fixed number known as the golden ratio, that is close to 1.618 or Phi.

Activity 8 – Variables and Equations (Adapted from Van de Walle & Lovin, 2006, p. 275)

Story Translations

Read a simple story problem to students but omit the question. Their task is to write an equation that means the same thing. For example: *There are 3 full boxes of pencils and 5 extra pencils. There are 41 pencils in all.* (3 × ▯ + 5 = 41). Be sure to include situations for all four operations. The activity can be reversed by providing an equation with an unknown and letting students make up a story to go with it. Once equations are agreed on, students should use whatever means they wish to find values that make the sentences true. Trial and error is a reasonable first strategy.

(Van de Walle & Lovin, 2006, p. 275)

Van de Walle and Lovin suggest that sometimes students write equations that may look different. For example, in the above equation one student may write 3 x n + 5 = 41, while another may write 3 x n = 41 – 5. Thus, students must discuss the similarities/differences between various equations. Therefore, they must be able to justify their own representation that has led them to conclude (generalise) that in the above scenario n = 9.

Extension Activity (Adapted from Van de Walle & Lovin, 2006, p. 276)

Number Tricks

Have students do the following sequence of operations:

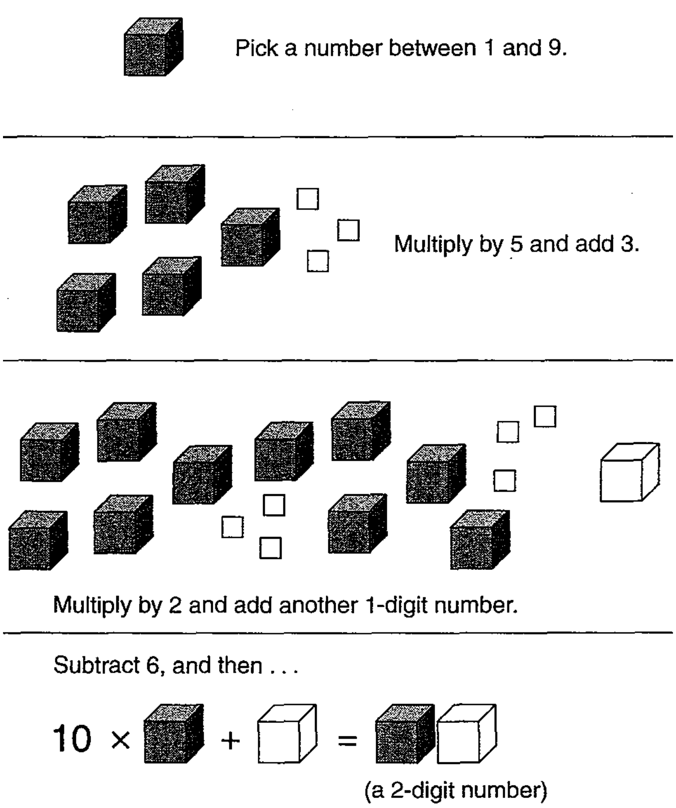
Write down any number.
Add to it the number that comes after it.
Add 9.
Divide by 2.
Subtract the number you began with.

Now you can "magically" read their minds. Everyone ended up with 5!
The challenge is to see if students can discover how the trick works. If students need a hint, suggest that instead of using an actual number, they use a box or a letter to begin with. Start with *N*. Add the next number: $N + (N + 1) = 2N + 1$. Adding 9 gives $2N + 10$. Dividing by 2 leaves $N + 5$. Now subtract the number you began with, leaving 5.
(Van de Walle and Lovin, 2006, p. 276)

- Pick a number, multiply by 6, add 12, take half of the result, subtract 6, divide by 3. What happens?
- Pick a number between 1 and 9, multiply by 5, add 3, multiply by 2, add another number between 1 and 9, subtract 6. What do you see?

Models of small cubes have been used to explore the number trick above.
An understanding of the place value component is required to enable the students to understand the result.

Additional numbers are shown here by using other counters or base ten blocks. For example, in the step to add 3 there are three base ten blocks added.



Extension Activity (Adapted from Van de Walle & Lovin, 2006, p. 277)

What's True for All Numbers?

Ask students how they know that 465 + 137 = 137 + 465 without doing the computation. Students' explanations should show evidence of understanding the commutative property for addition, although the name of the property is not important. How can this be written to show that it's a rule that is true for every number, even fractions and decimals? If students do not suggest it, offer the idea that letters or shapes could be used like this:

$$\Delta + \square = \square + \Delta \quad \text{or} \quad n + m = m + n$$

Be sure students understand that the choice of letter or shape is totally arbitrary as long as it is understood that each stands for any number and that when the same letter or shape appears in the same equation, it must represent the same value.

With this introduction, challenge students to find other statements that are true for all numbers.

The distributive principle can be shown using the rectangle, however, it is best to let students come up with the ideas of how and why themselves.
E.g.
(a x b) + (a x c) = a(b + c)

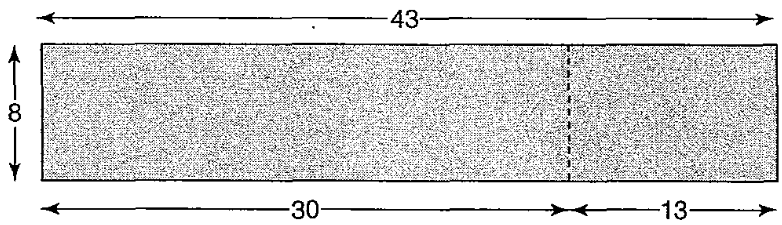


FIGURE 9.8

The distributive property is just one of many ideas that can be generalized using variables.

Special Quantities

What numeric expression would tell the number of chair legs on 376 chairs? (376 × 4) What about 195 chairs? (195 × 4) How would you write the number of legs on any number of chairs? (*N* × 4) Using this as an example, challenge students to write expressions for other types of quantities: fingers on students, eggs in cartons, crayons in boxes, wheels on tractor trailers, hours in days, inches in feet, quarts in gallons, and so on. Similarly, use variables to express these special numbers: any odd number, any even number, any multiple of 7, a multiple of 3 plus a different multiple of 5, any two-digit number, any power of 2. Once students get the idea, have them make up their own special quantities and see if others can describe them verbally.

(*Van de Walle and Lovin, 2006, p. 277*)

Activity 9 – Equations and Inequalities

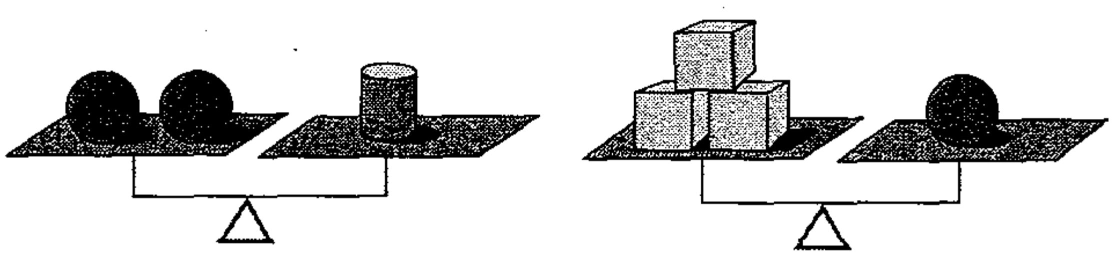
(Adapted from Van de Walle & Lovin, 2006, p. 278)

The equals sign '=' is often interpreted by students as an answer must follow as opposed to meaning that what is on the right side of the sign is equivalent to the left side of the sing. In other words the '=' sign shows a balance of objects or quantities. For example, 9 is another way of representing 4+5.

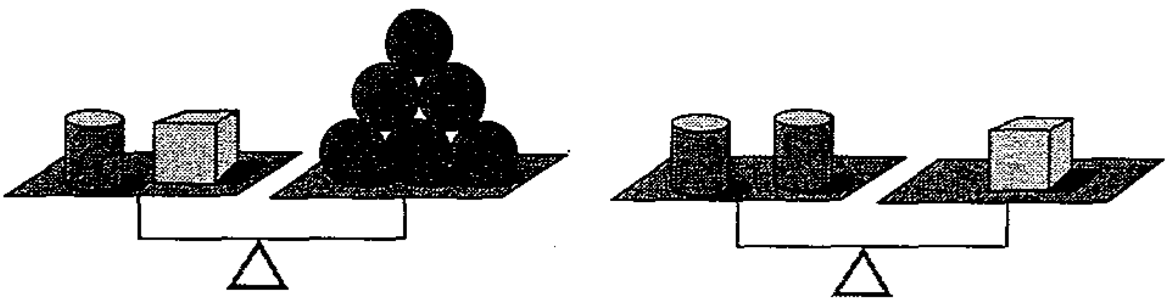
Activity 1: give students a number and ask them how many ways they can represent that number. E.g. 20 could be represented by 2x10, or 31-11, or 4x5

Activity 2: draw balancing scales on the board. Write a numeric expression in each pan and ask the students which way the pan will tilt. Then write a numeric expression (either on the left or the right pan) and ask the students to write a numeric expression on the other side to make the scales balance. The 'greater than' or 'less than' sign could be incorporated in this activity.

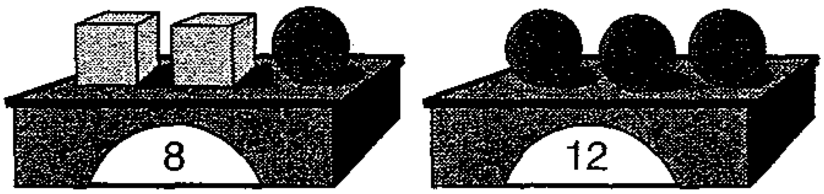
Then add variables to the activity
As students solve the equations below they are doing formal algebra!



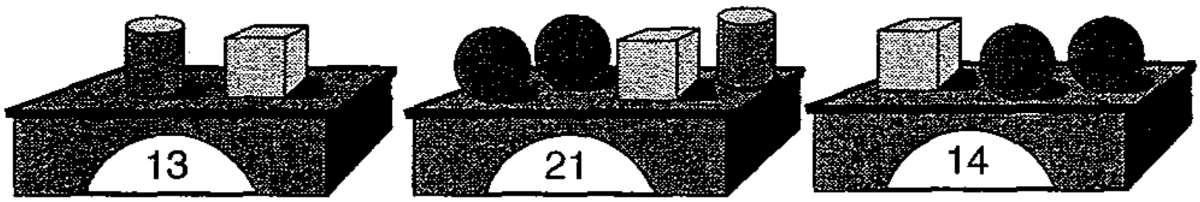
Which shape weighs the most? Explain.
Which shape weighs the least? Explain.



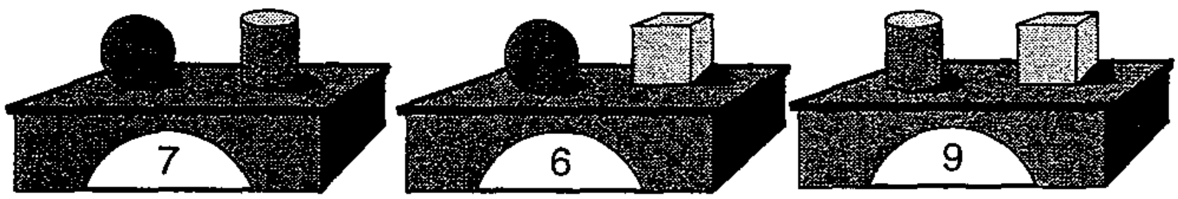
What will balance 2 spheres? Explain.



How much does each shape weigh? Explain.



How much does each shape weigh? Explain.



How much does each shape weigh? Explain.