Modified Richards equation and its exact solutions for soil water dynamics on eroding hillslopes

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Received 24 May 2000; revised 5 February 2002; accepted 22 February 2002; published 12 June 2002.

[1] The modified Richards equation (MRE) is presented by using a rotated coordinate system to accommodate the geometry of a hillslope and a moving boundary to represent an eroding surface on a hillslope. Exact analytical solutions of MRE are developed subject to *Fujita*'s [1952] diffusivity and *Sander et al.*'s [1988] unsaturated hydraulic conductivity. The mathematical analysis presented here for soil water dynamics and infiltration in particular on an eroding hillslope deviates from the traditional way in which infiltration has been investigated since *Green and Ampt*'s [1911] pioneering work. The MRE clearly improves mathematical representation of physical reality. Field data are used to derive parameters in a solution of MRE to illustrate the effect of erosion rates on soil moisture profiles in a moving boundary. *INDEX TERMS:* 1655 Global Change: Water cycles (1836); 1815 Hydrology: Erosion and sedimentation; 1866 Hydrology: Soil moisture; 3210 Mathematical Geophysics: Modeling; *KEYWORDS:* modified Richards equation, erosion, soil water flow

1. Introduction

[2] Numerous studies have been reported on quantifying soil water movement and infiltration since *Green and Ampt* [1911] proposed the first infiltration model. These studies regard the infiltration surface as a stable surface on which infiltration takes place, which is reasonable only for infiltration on flat surfaces. On a slope, however, runoff formed during rainfall erodes and entrains soil particles and solutes from the slope surface and moves them downslope, and soil erosion processes remove some of the topsoil, thus forming a new surface for the same processes to be repeated. In this case, the traditional infiltration theories do not apply.

[3] In this paper, we investigate soil water dynamics on a dynamic eroding hillslope which implies that two more concepts are introduced: one is a dynamic eroding hillslope, and the other is soil water movement on the dynamic slope. With these two concepts developed for soil water dynamics on eroding hillslopes, it is clear that the methodologies proposed in this presentation are more realistic and more reasonably represent natural hydraulic phenomena and improve mathematical representation of the physical processes.

2. Richards Equation for Soil Water Physics on an Eroding Hillslope

[4] Starting from Buckingham's concept of potentials, a nonlinear Fokker-Planck equation became the law for the study of flow in unsaturated soils since *Richards* [1931], which can be written [*Philip*, 1991]

$$\frac{\partial\theta}{\partial t} = \nabla \bullet (D(\theta)\nabla\theta) - \frac{dK}{d\theta}\frac{\partial\theta}{\partial z}, \qquad (1)$$

where

θ the soil moisture content;
$D(\theta)$ the diffusion coefficient;
K the hydraulic conductivity;
z the depth of soil;
t time;
∇ Laplacian operator.

[5] Equation (1) has been extensively investigated in soil physics and related areas in the last seven decades since *Richards* [1931] and has been used for modeling soil water movement including infiltration subject to different conditions.

[6] As far as infiltration is concerned, it is obvious from the literature that the vast majority of mathematical formulations and related various solutions published so far were developed for infiltration on a flat surface only. Various solutions for infiltration subject to different boundary conditions and different forms of the diffusion coefficient and hydraulic conductivity have been developed. For a limited number of reviews on this topic the reader is referred to *Philip* [1969, 1991], *Parlange et al.* [1980], and *Sposito* [1995].

[7] In order to incorporate the effect of hillslope in Richards equation for infiltration, *Philip* [1991] presented a modified Richards equation and related solutions for infiltration on planar slopes. However, Philip's approach is applicable to stable slopes only. As hillslopes are often susceptible to erosion, infiltration and related processes cannot be correctly interpreted using Philip's approach when erosion occurs.

[8] In this paper, we present procedures for analyzing soil water relationships on eroding hillslopes. These methodologies and procedures are based on the Richards equation and are linked to soil erosion processes.

[9] In order to clarify the definition to be used in this paper, Figure 1 draws a schematic illustration of the definitions used in the analysis. Figure 1 also highlights

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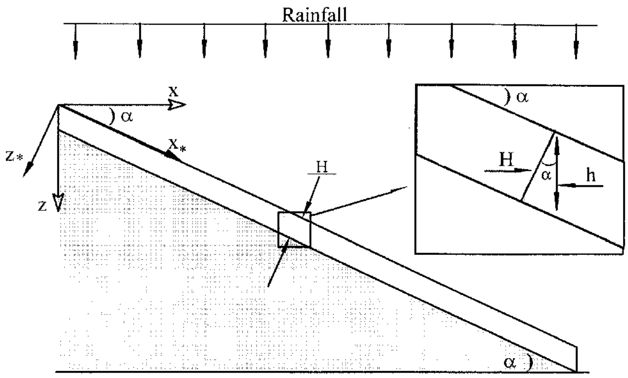

Figure 1. Definitions and schematic diagram of soil water movement on an eroding hillslope.

the key points of departure from the traditional theories on soil erosion and soil water flow.

[10] In Figure 1, the following nomenclature is defined. H is the depth of soil layer to be eroded during a rainfall event; h is the vertical depth of the same soil layer inclined at an angle of α; and x and z define the Cartesian coordinate, which after a rotation by an angle of α, is defined by (x_*, z_*).

[11] In Figure 1 the relationship between H and h is given by

$$H = h \cos \alpha \tag{2}$$

Figure 1 is a combination of the definitions developed in this paper and the schematic illustration of *Philip* [1991, Figure 1]. Figure 1 of this paper and Figure 1 of *Philip* [1991] are similar in geometry except for a moving boundary used in the present paper.

[12] With the aid of Figure 1 in this paper the coordinates (x_*, z_*) rotated by an angle of α are given by *Philip* [1991],

$$x_* = x \cos \alpha + z \sin \alpha \tag{3}$$

$$z_* = -x \sin \alpha + z \cos \alpha. \tag{4}$$

[13] With equations (3) and (4), equation (1) is transformed to

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z_*}\left(D(\theta)\frac{\partial \theta}{\partial z_*}\right) - \frac{dK}{d\theta}\left(\frac{\partial \theta}{\partial x_*}\sin \alpha + \frac{\partial \theta}{\partial z_*}\cos \alpha\right). \tag{5}$$

[14] *Philip* [1991] argued that the relevant solution of equation (5) is essentially independent of x_*, and dependent only on z_* and t. In these circumstances, with the aid of equations (3) and (4), equation (5) reduces to

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z_*}\left(D(\theta)\frac{\partial \theta}{\partial z_*}\right) - \frac{dK}{d\theta}\frac{\partial \theta}{\partial z_*}\cos \alpha. \tag{6}$$

[15] *Philip* [1991] presented solutions of equation (6) for hillslope infiltration and related various flow components for planar slopes. His formulations and solutions are applicable only for hillslope infiltration on a stable surface without material loss from the surface.

[16] In the following sections, we investigate soil moisture and hillslope infiltration on an eroding surface subject to the following initial and boundary conditions:

$$\theta = 0 \qquad z_* > 0 \quad t = 0 \tag{7}$$

$$\theta = \theta_S \qquad z_* > 0 \quad t > 0 \tag{8}$$

where θ_s is the saturated soil water content at the surface.

3. Analytical Solution for Soil Moisture Status on an Eroding Hillslope

[17] For analyzing water movement within and on an eroding hillslope we introduce the average surface soil erosion rate S,

$$S = H/T \tag{9}$$

From equation (2) we have

$$S = (h \cos a)/T. \tag{10}$$

Now we have introduced two expressions for S. If the depth of soil eroded in a storm is very small, one could approximately regard H and h as equal.

[18] We further use a moving coordinate system with a new variable ξ. If time is measured from the start of rainfall and runoff and soil erosion start t_0 after rainfall starts, then we have

$$\xi = z_* - S(t - t_0), \tag{11}$$

where t_0 is the time to ponding. Equation (11) implies that at the erosion rate S after each rainfall event, a new soil surface forms with its origin starting at ξ.

[19] With the aid of equation (11), equation (6) can be rewritten as

$$\frac{\partial \theta}{\partial \xi} = -\frac{1}{S}\frac{\partial}{\partial \xi}\left[D(\theta)\frac{\partial \theta}{\partial \xi}\right] + \frac{dK(\theta)}{d\theta}\frac{\partial \theta}{\partial \xi}\frac{\cos \alpha}{S}. \tag{12}$$

Equation (12) is the modified Richards equation (MRE), and it generalizes the partial differential equation governing soil water dynamics on a hillslope for both sloping and flat geometries and for both stable and eroding surfaces. Obviously, the MRE reduces to different forms subject to different conditions.

[20] In the present investigation, equation (12) can be written as an ordinary differential equation,

$$\frac{d\theta}{d\xi} = -\frac{1}{S}\frac{d}{d\xi}\left[D(\theta)\frac{d\theta}{d\xi}\right] + \frac{dK(\theta)}{d\xi}\frac{\cos\alpha}{S}, \tag{13}$$

and the initial and boundary conditions equations (7) and (8) now become

$$\theta = 0 \qquad \xi > 0 \tag{14}$$

$$\theta = \theta_S \qquad \xi = 0 \tag{15}$$

A first integral of equation (13) with respect to ξ is

$$\theta = -\frac{D(\theta)}{S}\frac{d\theta}{d\xi} + K(\theta)\frac{\cos\alpha}{S} + C_1, \tag{16}$$

where C_1 is a constant of integration.

[21] It is obvious that the form of solutions of equation (16) depends on the forms of $D(\theta)$ and $K(\theta)$. In the rest of the section, we further investigate the solutions of equation (16) subject to one set of functions for the unsaturated hydraulic conductivity and diffusivity.

[22] We use *Sander et al.*'s [1988] unsaturated hydraulic conductivity,

$$K(\theta) = \frac{K_1 + K_2\theta + K_3\theta^2}{1 - \nu\theta} \tag{17}$$

and *Fujita*'s [1952] diffusivity

$$D(\theta) = \frac{D_0}{(1-\nu\theta)^2}, \tag{18}$$

where K_1, K_2, K_3, D_0, and ν are constants determined from soil properties.

[23] Equations (17) and (18) were used by *Sander et al.* [1988] to derive exact solutions of the nonlinear Richards equation for constant flux infiltration (a boundary condition of the third type). In this paper, we use equations (17) and (18) for $K(\theta)$ and $D(\theta)$, respectively, in equation (16) to derive a set of exact analytical solutions for infiltration on an eroding hillslope subject to the concentration boundary condition (a boundary condition of the first type).

[24] In equations (16) and (17), if we define $K_1 = 0$ for $\theta = 0$ and $d\theta/d\xi|_{\theta=0} = 0$, we have $C_1 = 0$ for $\theta = 0$. Then, substitution of equations (17) and (18) into equation (16) gives

$$\frac{d\theta}{d\xi} + \frac{1}{D_0}(S - K_2\cos\alpha)\theta + \frac{1}{D_0}[(K_2\nu - K_3)\cos\alpha - 2\nu S]\theta^2 + \frac{1}{D_0}\left(S\nu^2 + \nu K_3\cos\alpha\right)\theta^3 = 0. \tag{19}$$

Equation (19) can be solved analytically. First, we rewrite equation (19) as

$$\frac{d\theta}{d\xi} = \theta R, \tag{20}$$

where

$$R = A + B\theta + C\theta^2 \tag{21}$$

with

$$A = \frac{1}{D_0}(K_2\cos\alpha - S) \tag{22}$$

$$B = \frac{1}{D_0}[(K_3 - K_2\nu)\cos\alpha + 2\nu S], \tag{23}$$

$$C = \frac{1}{D_0}\left(-S\nu^2 - \nu K_3\cos\alpha\right). \tag{24}$$

Then we integrate equation (20),

$$\int\frac{d\theta}{R\theta} = \int d\xi + C_2, \tag{25}$$

where C_2 is a constant of integration.

[25] Following *Gradshteyn and Ryzhik* [1994, equations (2.177-1) and (2.172), pp. 81, 83], equation (25) is integrated to give

$$\frac{1}{2A}\ln\left(\frac{\theta^2}{R}\right) - \left(\frac{B}{2A}\right)\frac{1}{\sqrt{-\Delta}}\ln\left[\frac{\sqrt{-\Delta} - (B + 2C\theta)}{\sqrt{-\Delta} + (B + 2C\theta)}\right] = \xi + C_2, \tag{26}$$

where

$$\Delta = 4AC - B^2. \tag{27}$$

Equation (26) takes three different forms depending on the values of Δ [*Gradshteyn and Ryzhik*, 1994, equation (2.177), p. 83].

[26] Case 1 is for $\Delta < 0$. In this case, equation (26) gives

$$\frac{1}{2A}\ln\left(\frac{\theta^2}{R}\right) + \frac{B}{A\sqrt{-\Delta}}Arth\left(\frac{B + 2C\theta}{\sqrt{-\Delta}}\right) = \xi + C_2 \qquad \Delta < 0, \tag{28}$$

where $\mathrm{Arth}\left(B + 2C\theta_1/\sqrt{-\Delta}\right)$ is the inverse hyperbolic tangent function. Applying the boundary condition to equation (28) yields

$$C_2 = \frac{1}{2A}\ln\left(\frac{\theta_s^2}{A + B\theta_s + C\theta_s^2}\right) + \frac{B}{A\sqrt{-\Delta}}Arth\left(\frac{B + 2C\theta_s}{\sqrt{-\Delta}}\right). \tag{29}$$

Substitution of (29) in (28) gives

$$\xi = \frac{1}{2A}\ln\left[\left(\frac{\theta^2}{\theta_s^2}\right)\left(\frac{A + B\theta_s + C\theta_s^2}{A + B\theta + C\theta^2}\right)\right] + \frac{B}{A\sqrt{-\Delta}} \cdot\left[\mathrm{Arth}\left(\frac{B + 2C\theta}{\sqrt{-\Delta}}\right) - \mathrm{Arth}\left(\frac{B + 2C\theta_s}{\sqrt{-\Delta}}\right)\right] \qquad \Delta < 0, \tag{30}$$

where A, B, and C are given by equations (22), (23), and (24), respectively. Equation (30) represents the relationship between the depth of soil in a moving coordinate ξ,

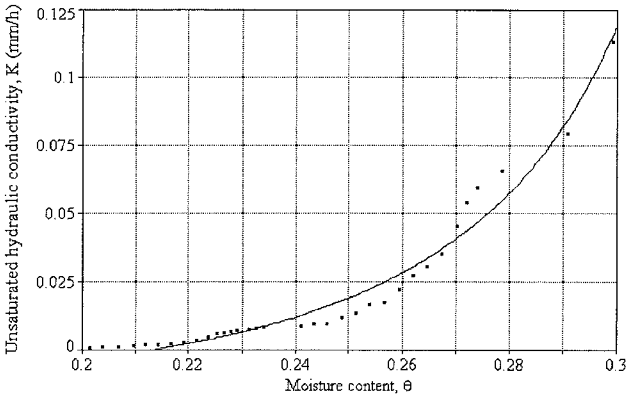

Figure 2. *Sander et al.*'s [1988] unsaturated hydraulic conductivity as a function of moisture content.

moisture content θ, erosion rate S, and other parameters incorporated in A, B, C, and Δ. With equation (11), equation (30) can be written for the fixed coordinate systems,

$$z_* = S(t - t_0) + \frac{1}{2A}\ln\left[\left(\frac{\theta^2}{\theta_s^2}\right)\left(\frac{A + B\theta_s + C\theta_s^2}{A + B\theta + C\theta^2}\right)\right] + \frac{B}{A\sqrt{-\Delta}} \cdot \left[\mathrm{Arth}\left(\frac{B+2C\theta}{\sqrt{-\Delta}}\right) - \mathrm{Arth}\left(\frac{B+2C\theta_s}{\sqrt{-\Delta}}\right)\right] \qquad \Delta < 0. \tag{31}$$

[27] Case 2 is for $\Delta = 0$. In this case, equation (26) gives

$$\frac{1}{2A}\ln\left(\frac{\theta^2}{R}\right) + \frac{B}{A(B+2C\theta)} = \xi + C_2 \quad \Delta = 0, \tag{32}$$

and applying the boundary condition yields

$$C_2 = \frac{1}{2A}\ln\left(\frac{\theta_s^2}{A + B\theta_s + C\theta_s^2}\right) + \frac{B}{A(B+2C\theta_s)}, \tag{33}$$

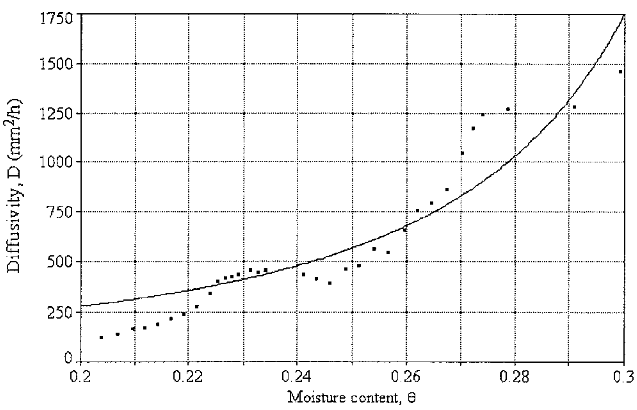

Figure 3. *Fujita*'s [1952] diffusivity as a function of moisture content.

and substitution of equation (33) in equation (32) gives

$$\xi = \frac{1}{2A}\ln\left[\left(\frac{\theta^2}{\theta_s^2}\right)\left(\frac{A + B\theta_s + C\theta_s^2}{A + B\theta + C\theta^2}\right)\right] + \frac{B}{A}\left[\frac{1}{(B+2C\theta)} - \frac{1}{(B+2C\theta_s)}\right] \qquad \Delta = 0 \tag{34}$$

for a moving coordinate system, or with equations (11) and (34), we have

$$z_* = S(t - t_0) + \frac{1}{2A}\ln\left[\left(\frac{\theta^2}{\theta_s^2}\right)\left(\frac{A + B\theta_s + C\theta_s^2}{A + B\theta + C\theta^2}\right)\right] + \frac{B}{A}\left[\frac{1}{(B+2C\theta)} - \frac{1}{(B+2C\theta_s)}\right] \qquad \Delta = 0 \tag{35}$$

for a fixed coordinate system.

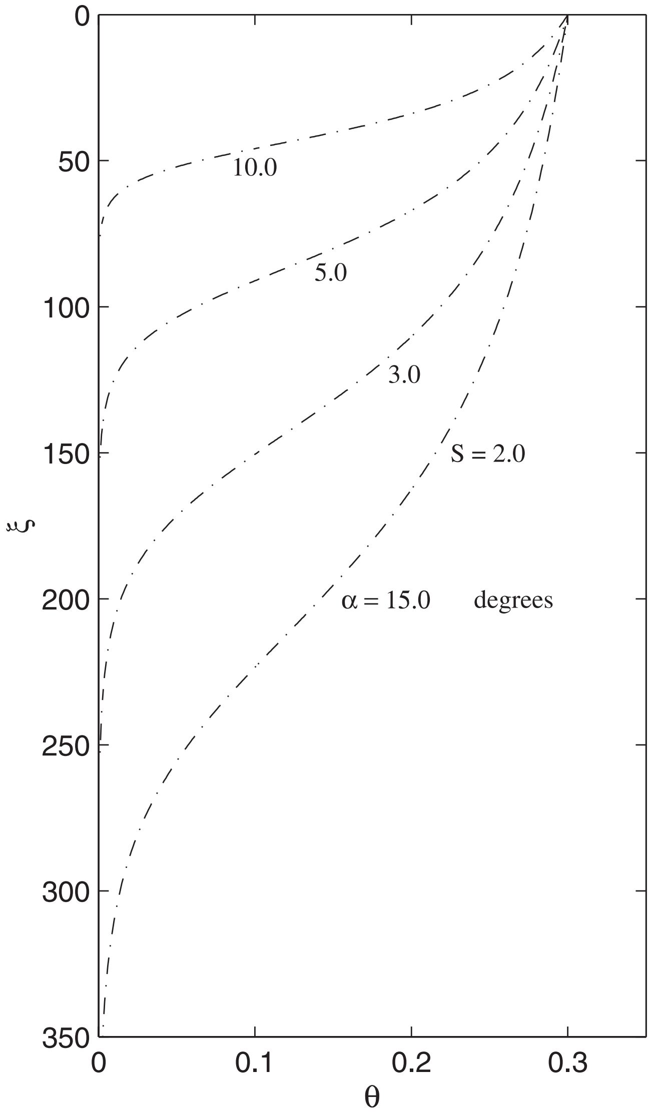

Figure 4. Effect of erosion rate on moisture profiles in a soil on an eroding hillslope.

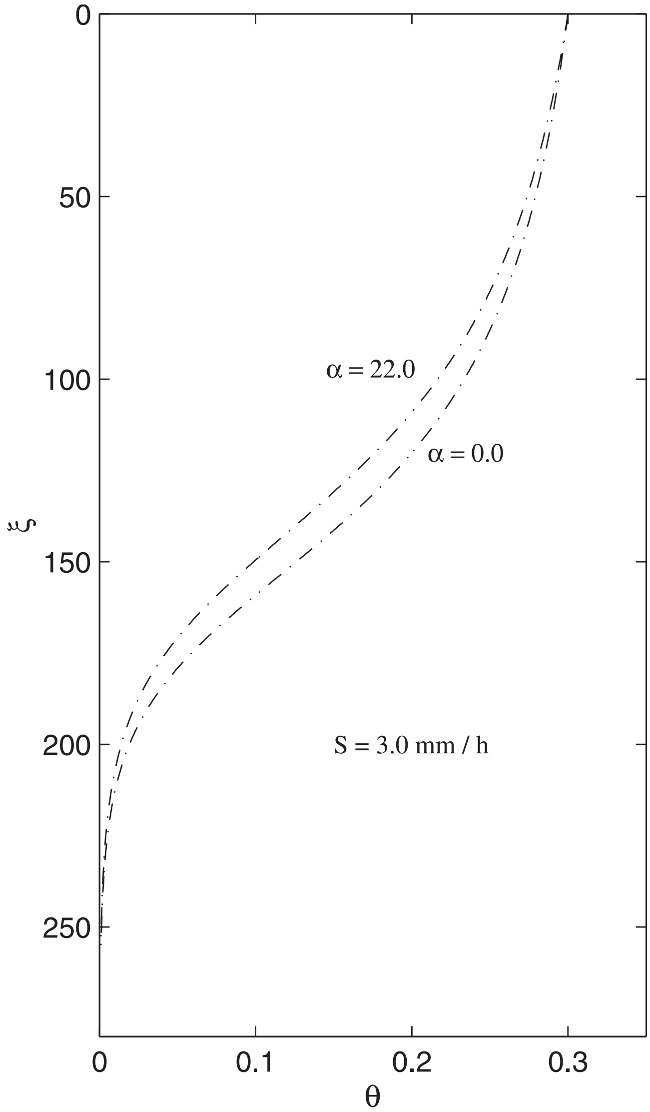

Figure 5. Effect of slope on moisture profiles in a soil on an eroding hillslope.

[28] Case 3 is for $\Delta > 0$. In this case, equation (26) gives

$$\frac{1}{2A}\ln\left(\frac{\theta^2}{R}\right) - \frac{B}{A\sqrt{\Delta}} arctg\left(\frac{B+2C\theta}{\sqrt{\Delta}}\right) = \xi + C_2 \quad \Delta > 0. \quad (36)$$

Applying the boundary condition yields

$$C_2 = \frac{1}{2A}\ln\left(\frac{\theta_1^2}{A+B\theta_s+C\theta_s^2}\right) - \frac{B}{A\sqrt{\Delta}} arctg\left(\frac{B+2C\theta_s}{\sqrt{\Delta}}\right) \quad (37)$$

Substitution of equation (37) in equation (36) gives

$$\xi = \frac{1}{2A}\ln\left[\left(\frac{\theta^2}{\theta_s^2}\right)\left(\frac{A+B\theta_s+C\theta_s^2}{A+B\theta+C\theta^2}\right)\right] - \frac{B}{A\sqrt{\Delta}}\left[\mathrm{arctg}\left(\frac{B+2C\theta}{\sqrt{\Delta}}\right) - \mathrm{arctg}\left(\frac{B+2C\theta_s}{\sqrt{\Delta}}\right)\right] \quad \Delta > 0 \quad (38)$$

for a moving coordinate system, or with equations (11) and (38) we have

$$z_* = S(t-t_0) + \frac{1}{2A}\ln\left[\left(\frac{\theta^2}{\theta_s^2}\right)\left(\frac{A+B\theta_s+C\theta_s^2}{A+B\theta+C\theta^2}\right)\right] - \frac{B}{A\sqrt{\Delta}}\left[\mathrm{arctg}\left(\frac{B+2C\theta}{\sqrt{\Delta}}\right) - \mathrm{arctg}\left(\frac{B+2C\theta_s}{\sqrt{\Delta}}\right)\right] \quad \Delta > 0 \quad (39)$$

for a fixed coordinate system.

[29] The analytical solutions of the generalized Richards equation presented above are subject to the first type boundary condition (or concentration boundary condition). With the diffusivity and unsaturated hydraulic conductivity functions described by *Fujita*'s [1952] and *Sander et al.*'s [1988] functions, respectively, these solutions given in both moving and fixed coordinate systems describe the relationships among soil moisture and other physical parameters on an eroding hillslope.

4. Illustrative Examples

[30] Now we illustrate the analytical solutions with data from the field (C. H. Roth, personal communication, 2000). The details of the data used in the analysis are given by *Roth et al.* [1995].

[31] The data on volumetric moisture content, θ (%), and hydraulic conductivity, K (mm/hr) are used to determine the parameters in the hydraulic conductivity function in equation (17). Then diffusivity, as defined by

$$D(\theta) = K(\theta)d\psi/d\theta \quad (40)$$

is computed on the basis of data for the moisture potential ψ (mm) and content θ (%).

[32] When the data on θ and $K(\theta)$ are fitted to equation (17), the parameters appearing in the expression are derived, namely, $K_1 = 0$, $K_2 = -0.1158$, $K_3 = 0.5424$, and $v = 2.93744$. Once $D(\theta)$ is computed using equation (40), fitting the data to equation (18) gives the values for $D_0 = 55.8290$ and $v = 2.73725$. The results are shown in Figures 2 and 3.

[33] It should be noted that the two curves fitted automatically using a computer program shown in Figures 2 and 3 have different values of v. In the subsequent simulations presented in Figures 4 and 5, an average value of $v = 2.855$ is used.

[34] With this set of parameters, it is found that $\Delta < 0$ in equation (27), thereby by making equations (30) and (31) applicable to this soil. Equation (30) is used to generate Figures 4 and 5 to illustrate the effects of erosion rate and slope on moisture distribution in the soil profile during simulated storm events.

[35] The curves with different erosion rates S (mm/hr) and slopes α (degrees) are moisture profiles below the moving surface in the moving coordinate ξ (mm). In other words, the ξ versus θ relationship defines the distribution of soil moisture content below the moving surface for different erosion rates and a given slope.

5. Discussion

[36] In the preceding analysis, we have presented the modified Richards equation (MRE), exact solutions of the

MRE for water movement on an eroding hillslope, and an example illustrating these solutions applied to a real soil. The following issues have been addressed in the presentation

1. The analysis presented in this paper establishes a realistic model for soil moisture physics on an eroding hillslope. With the aid of a transformation into a rotated coordinate and by introducing a new variable, ξ, the Richards equation, which is a nonlinear Fokker-Planck equation, reduces to an ordinary differential equation. The MRE generalizes the partial differential equation governing soil water dynamics on both sloping and flat geometries and for both stable and eroding surfaces. The following is a summary of the different forms of MREs: (1) $\alpha \neq 0$ in moving and rotated coordinates, MRE (i.e., equation (12) in this paper); (2) $\alpha = 0$ in moving coordinates, MRE (i.e., equation (12) if $\alpha = 0$ in this paper); (3) $\alpha \neq 0$ in fixed rotated coordinates, MRE [i.e., *Philip*, 1991]; and (4) $\alpha = 0$ in ordinary coordinates, RE [*Richards*, 1931].

2. With *Fujita*'s [1952] diffusivity and *Sander et al.*'s [1988] unsaturated hydraulic conductivity functions, exact analytical solutions of the MRE have been derived subject to a first type boundary condition. With these solutions, various properties of soil water dynamics on an eroding surface can be conveniently investigated. When the coordinate system is fixed, the analytical solutions are complementary to those presented by *Sander et al.* [1988] whose solutions are for the boundary condition of the third type or flux boundary condition.

3. Deviating from traditional ways in which infiltration has been investigated since *Green and Ampt* [1911] put forward the first infiltration model, this presentation establishes a model for two realistic physical phenomena taking place in nature, i.e., infiltration on a hillslope with soil erosion developing on the surface. The approach clearly improves the mathematical representation of the natural processes by taking into account a moving infiltration surface. The analysis implies that some of the present well-accepted methodologies for quantifying soil water movement, solute transport in soils on hillslopes, and sediment transport on eroding hillslopes may have to be modified.

4. In the preceding analysis, we used a spatially averaged erosion rate S over a rainfall event T in order to simplify the presentation and highlight the major technical points. To apply the approach initiated in this presentation to a more complex system consisting of a variable erosion rate, variable hydraulic conductivity of the soil, etc., appropriate numerical methods have to be used, which would be a more realistic implementation of the MRE. Further analysis should also consider the distinction between rill erosion and sheet erosion. It is clear that when more processes are included such as heterogeneity, hysteresis, multiphase transport etc., the analysis will certainly become sophisticated. This paper is not intended to address all these issues.

[37] **Acknowledgments.** This research was partly supported by the New Zealand Foundation for Research, Science and Technology, and partly by the Australian Research Council and the Chinese Academy of Sciences project KZCXZ-SW-317. The author wishes to acknowledge the suggestions of G. Salvucci, the Associate Editor of *Water Resources Research*, for plotting the solutions using field data and wishes to thank K. Verburg and C. H. Roth of CSIRO Land and Water for locating and providing data used for generating the graphs. The author also appreciates the reviewers for their comments and A. Parshotam of Landcare Research for his comment on the early manuscript of the paper.

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